From ring epimorphisms to universal localisations

joint with Jorge Vitoria

I. Ring epimorphisms

II. Universal localisations

III. Recollements

Throughout, A will denote a ring (with unit) and $\mathbb{K}$ a field.

I. Ring epimorphisms

Ring epimorphisms are epimorphisms in the category of rings, i.e., $f : A \to B$ is a ring epimorphism, if for all $g_1, g_2 : B \to C$ with

$$g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2$$

Example.

- *Surjective ring homomorphisms*
- $\mathbb{Z} \hookrightarrow \mathbb{Q}$
- *Ore localisations*

Proposition (Stenström'75)**.** *$f : A \to B$ a ring homomorphism. Then the following are equivalent.*

1. *f is a ring epimorphism.*
2. *the restriction functor $f_* : B\text{-}Mod \longrightarrow A\text{-}Mod$ is fully faithful.*
3. *$B \otimes_A coker(f) = 0$.*

A ring epimorphism $f : A \to B$ is called homological, if

$$Tor_i^A(B,B) = 0 \qquad \forall i > 0.$$

Proposition (Geigle-Lenzing'91)**.** *$f : A \to B$ a ring homomorphism. Then the following are equivalent.*

1. *f is a homological ring epimorphism.*
2. *the derived restriction functor*

   $$\mathcal{D}(f_*) : \mathcal{D}(B) := \mathcal{D}(B\text{-}Mod) \longrightarrow \mathcal{D}(A) := \mathcal{D}(A\text{-}Mod)$$

   is fully faithful.
3. *$B \otimes_A^{\mathbb{L}} C_f = 0$, where C_f denotes the cone of f in $\mathcal{D}(A)$.*

II. Universal localisations

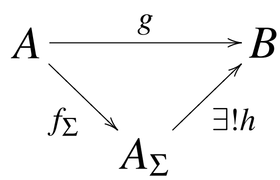
Definition/Theorem (Schofield'85)**.** *Let Σ be a set of maps in A-$proj$. Then there is a ring A_Σ*

– the universal localisation of A at Σ –

and a ring homomorphism $f_\Sigma : A \to A_\Sigma$ such that

i) $A_\Sigma \otimes_A \sigma$ is an isomorphism for all $\sigma \in \Sigma$.

ii) For all ring homomorphisms $g : A \to B$ fulfilling i) there is



Moreover, f_Σ is a ring epimorphism and $Tor_1^A(A_\Sigma, A_\Sigma) = 0$.

Theorem (Krause-Stovicek'10)**.** *Let A be a hereditary ring and $f : A \to B$ be a ring epimorphism. Then*

$$f \text{ is homological } \Leftrightarrow f \text{ is a universal localisation.}$$

Theorem (M.-Vitoria'12, Chen-Xi'12)**.** *Let $f : A \to B$ be a ring epimorphism such that*

- *${}_AB$ is finitely presented*
- *$pd_A B \leq 1$*

Then f is homological if and only if f is a universal localisation.

idea of the proof of "$\Rightarrow$":

I) $C_f \cong (\, P_f^{-1} \xrightarrow{g} P_f^0) =: P_f$ in $\mathcal{D}(A)$ with $P_f \in \mathcal{K}^b(A\text{-}proj)$.
Since $B \otimes_A^{\mathbb{L}} C_f = 0$, it follows that $B \otimes_A g$ is an isomorphism.

II) check universal property.

Corollary. *Let A be a finite dimensional $\mathbb{K}$-algebra of the form*

- *a group algebra of a finite group*
- *a self-injective and representation-finite algebra*

Then finite dimensional homological ring epimorphisms $f : A \to B$ are universal localisations, turning ${}_AB$ into a projective A-module.

III. Recollements

A homological ring epimorphism $A \xrightarrow{f} B$ yields a "semiorthogonal decomposition" of $\mathcal{D}(A)$ into smaller triangulated categories.

(recollement)

$$\mathcal{D}(B) \underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\mathcal{D}(f_*)}}} \mathcal{D}(A) \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} Tria C_f$$

Theorem (M.-Vitoria'12)**.** *Let $f : A \to B$ be a homological ring epimorphism such that*

- ${}_AB$ *is finitely presented*
- $pd_A B \leq 1$
- $Hom_A(coker(f), ker(f)) = 0$

Then there is a recollement

$$\mathcal{D}(B) \underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\mathcal{D}(f_*)}}} \mathcal{D}(A) \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} \mathcal{D}(End_{\mathcal{D}(A)}(C_f)).$$

Furthermore, if ${}_AB$ is projective, we have an isomorphism of rings

$$End_{\mathcal{D}(A)}(C_f) \cong A/\tau_B(A),$$

where $\tau_B(A)$ denotes the trace of ${}_AB$ in A.