Computational Approach to the Artinian Conjecture

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Generic Representation Theory

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Examples

Examples of such functors are:

- The n-fold tensor product $T^n : V \mapsto T^s(V) = V^{\otimes s}$, $s \in \mathbb{N}$
- The symmetric powers
  $\Lambda^s : V \mapsto \Lambda^n(V) = T^s(V)/(v \otimes v)$, $n \in \mathbb{N}$
- The $\mathbb{F}_q$-dual $\mathrm{Hom}(-, \mathbb{F}_q) : V \mapsto \mathrm{Hom}(-, \mathbb{F}_q)(V) = V^*$

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A functor F has only finitely many composition factors iff the function

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Theorem (L. 2012)

$\phi(\ker([f], -), n)$ is of closed form for all representable morphisms $([f], -)$ in the category $\mathcal{F}_q$.

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Let us now look at the closed form of $\phi(H', n)$, $\phi(H', n) = \sum_{i=0}^{s'} p_i^{H'}(n) q^{in}$. Because of the choice of $(U', -)$ and the structure of $(W, -)$ we must have $s' < s$.

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The End

Thank you for your attention!

Announcement: A Summer School of the Birep group will be held in Bad Driburg from 26th to 30th of August 2013.