What do cluster structures tell you about an algebra?

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Universität Bielefeld

Bad Boll, March 25, 2013

Overview

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In this talk, we wish to discuss the effect of cluster structures on a given commutative algebra.

In particular, we wish to address the following questions:

1. When is a cluster algebra a unique factorization domain?
2. What are its irreducible elements?
3. What is its divisor class group?

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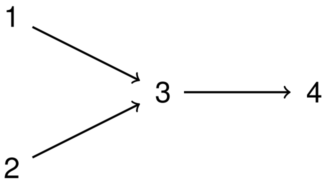

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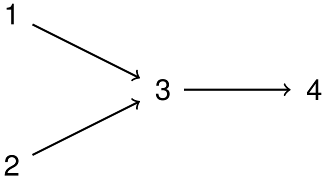

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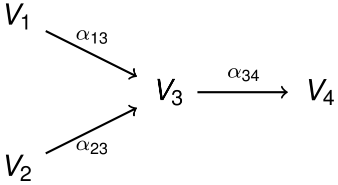

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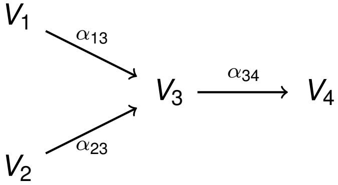

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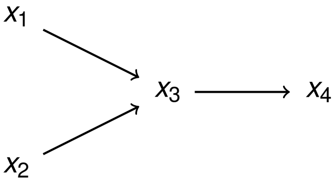

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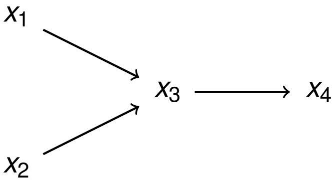

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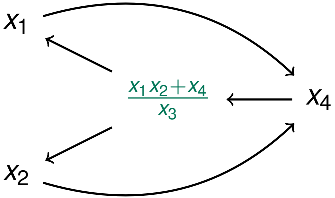

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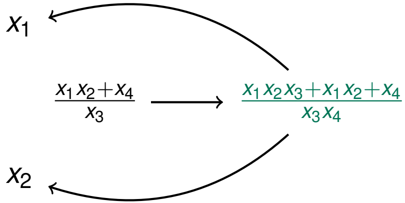

A method to study quiver representations are Fomin-Zelevinsky's cluster algebras:

1. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a sequence of algebr. independent variables over a base field K which constitute an initial cluster.
2. All variables obtained from the initial cluster by a sequence of mutations generate a cluster algebra

$$\mathcal{A}(\mathbf{x}, Q) \subseteq K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}].$$

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Dual semi-canonical basis	Attached to unipotent cells in Lie groups	Geiß-Leclerc-Schröer
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In contrast, our approach seeks to study more explicitly the algebraic properties of a given cluster algebra. This question may be of interest to representation theorists, because in the categorification of cluster algebras via Buan-Marsh-Reineke-Reiten-Todorov's cluster categories, such algebraic properties play a crucial role.

Basic notions

1. Freezing. Sometimes we freeze certain vertices of the quiver to obtain two kinds of vertices – mutable and frozen vertices. Sequences of mutations at mutable vertices yield a smaller set $\mathcal{C}$ of cluster variables. In this case, we define the cluster algebra without (or with) invertible coefficients to be generated by $\mathcal{C}$ and the set of frozen variables (plus their inverses, respectively).

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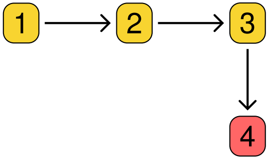
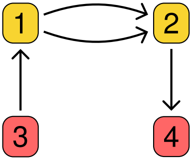
3. Cluster algebras of finite type. The cluster algebras with only finitely many cluster variables have been classified by finite type root systems. More precisely, a cluster algebra is of finite type if and only if the mutation class of Q contains an orientation of a Dynkin diagram of type A, D, E.

Example: Two cluster structures on the same algebra

Consider the acyclic quivers Q and Q'. Mutable vertices are colored yellow, frozen vertices are colored red.





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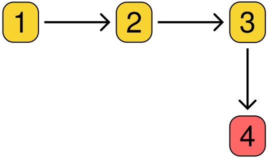

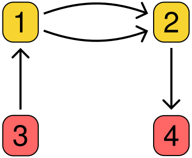

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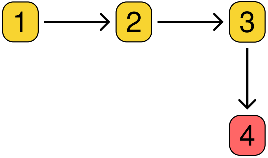

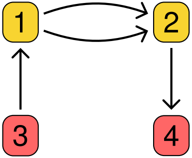

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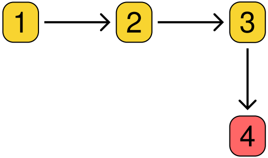

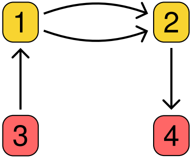

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So $\mathcal{A}(\mathbf{x}, Q) \cong \mathcal{A}(\mathbf{x}, Q')$ are both isomorphic to a polynomial algebra in four independent variables. The very same algebra is at the same time

- a cluster algebra of rank 3 and a cluster algebra of rank 2.
- a cluster algebra of infinite type and a cluster algebra of finite type.

Necessary conditions for a cluster algebra to be UFD

The main players (associated with mutable vertices i):

- initial exchange polynomials

$$f_i = x_i x_i' = \prod_{j \to i} x_j + \prod_{i \to k} x_k \in K[\mathbf{x}],$$

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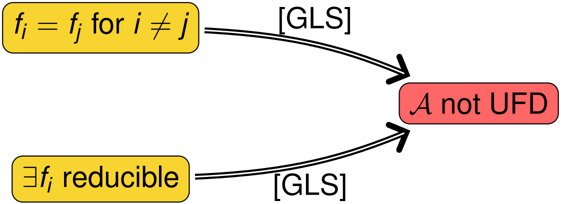
Theorem ([GLS]). Two necessary conditions for a cluster algebra to be a unique factorization domain are as follows:

- If an initial exchange polynomial f_i is reducible over the field K, then the cluster algebra $\mathcal{A}(\mathbf{x}, Q)$ is not a unique factorization domain.
- If two initial exchange polynomials f_i and f_j with $i \neq j$ are equal, then the cluster algebra $\mathcal{A}(\mathbf{x}, Q)$ is not a unique factorization domain.

A ring theoretic perspective on cluster algebras



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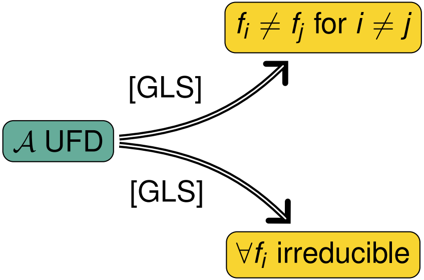

$f_i \neq f_j$ for $i \neq j$

[GLS]

$\mathcal{A}$ UFD

[GLS]

$\forall f_i$ irreducible

Thm.

- $f_i \neq f_j$ for $i \neq j \wedge \forall f_i$ irreducible
- $\forall \mathbf{a} \in \mathbb{N}^n$: $I_1^{a_1} I_2^{a_2} \cdots I_n^{a_n}$
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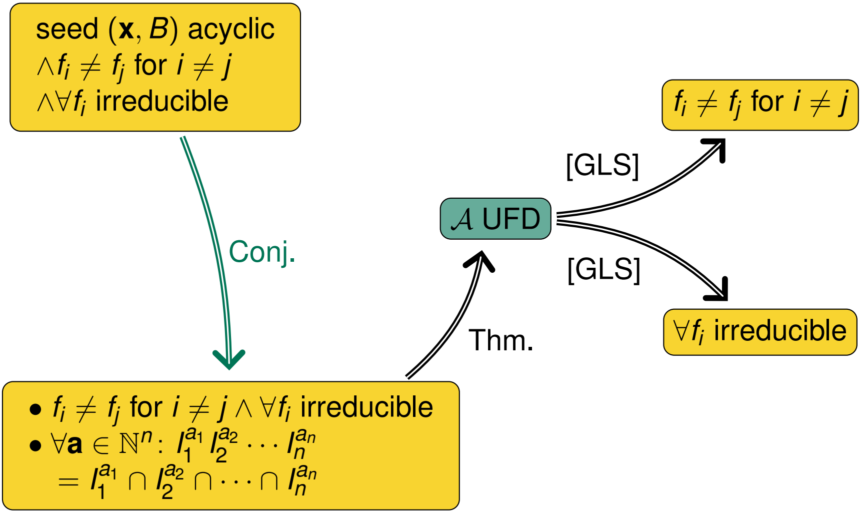

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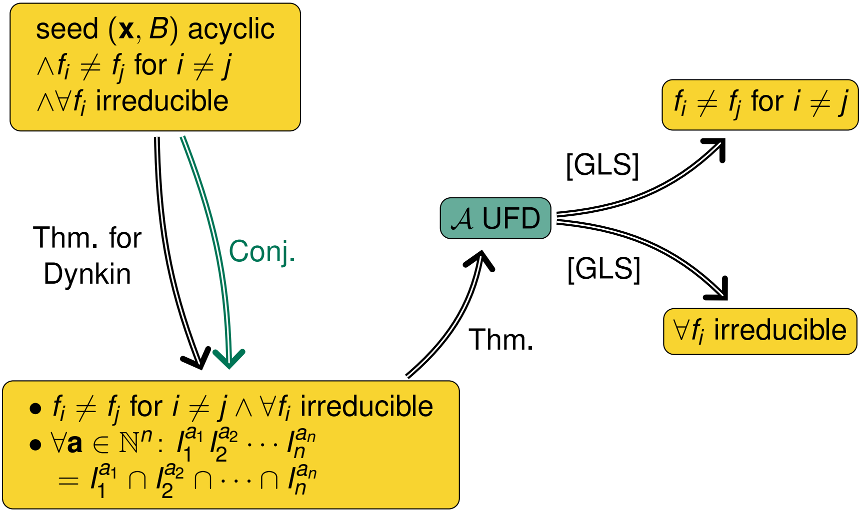

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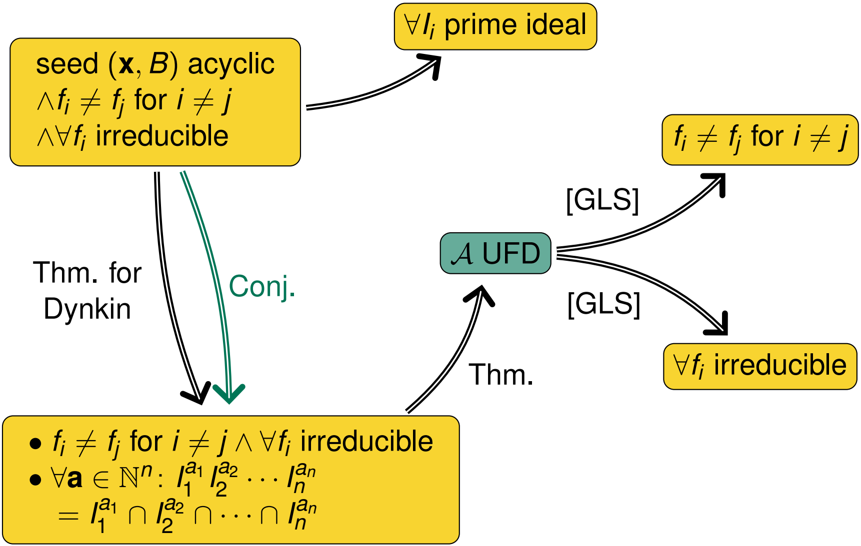

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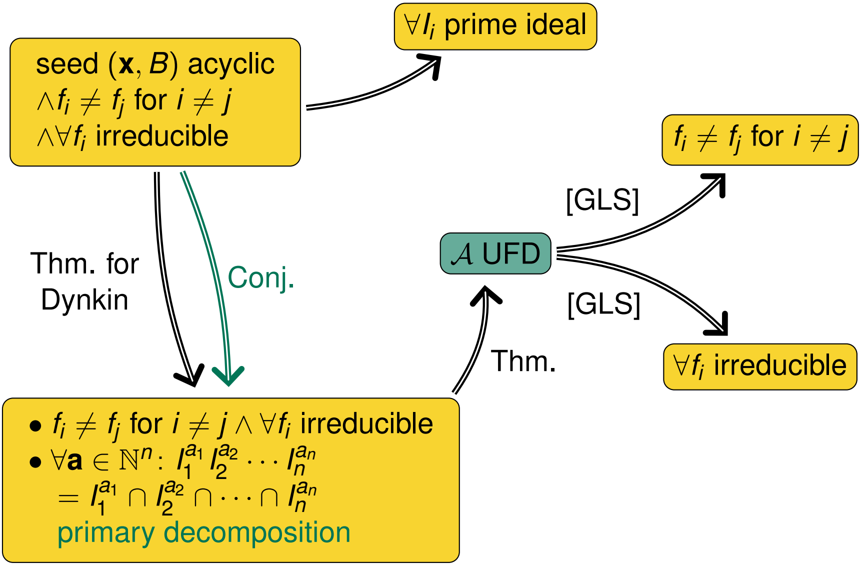
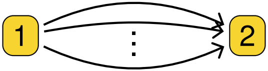


Application: Cluster algebras of Kronecker type

Let Q be the b-Kronecker quiver with $b \geq 1$ arrows between two vertices 1 and 2.



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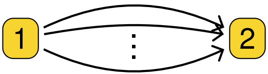

The initial exchange polynomials $f_1 = 1 + x_2^b$ and $f_2 = 1 + x_1^b$ are irreducible over the base field

- $K = \mathbb{Q}$ if and only if $b = 1, 2, 4, 8, \ldots$ is a power of 2.
- $K = \mathbb{C}$ if and only if $b = 1$.

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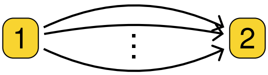

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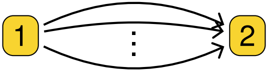

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Proof: The ideals $I_1 = (x_1, f_1)$ and $I_1 = (x_2, f_2)$ are coprime: the inclusions $x_1 \in I_1$ and $1 + x_1^b \in I_2$ yield $I_1 + I_2 = (1)$. We conclude that $I_1 I_2 = I_1 \cap I_2$. A similar argument implies that $I_1^{a_1} I_2^{a_2} = I_1^{a_1} \cap I_2^{a_2}$ for all $a_1, a_2 \geq 0$. $\square$

Application: Cluster algebras of Dynkin type

The cluster algebra of type A_3 is not a unique factorization domain: the initial exchange polynomials f_1 and f_3 are equal, so $x_1 y_1 = x_3 y_3$ are two non-unique factorizations of the element $f_1 = f_3 = 1 + x_2$ into irreducible elements.

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On the other hand, using the sufficient criterion we can prove that a cluster algebra of type A_n with $n \neq 3$ is a unique factorization domain. More generally, a cluster algebra of finite type is a unique factorization domain if and only if:

Type	A_n	D_n	E_n
UFD	$n \neq 3$	-	$n = 6, 7, 8$
Not UFD	$n = 3$	$n \geq 4$	-

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Theorem. Suppose that Q is a quiver with a mutable vertex i. We construct a new quiver Q' by freezing the vertex i. If the cluster variable x_i is a prime element in the cluster algebra $\mathcal{A}(\mathbf{x}, Q)$, then

$$\mathrm{Cl}(\mathcal{A}(\mathbf{x}, Q)) \cong \mathrm{Cl}(\mathcal{A}(\mathbf{x}, Q')),$$

where we consider cluster algebras with invertible coefficients.

Thank you all for listening!