Relative Singularity Categories

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Bad Boll
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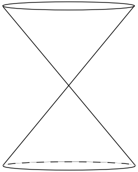
Motivation and Overview

Singularity



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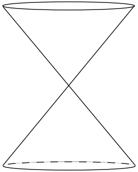

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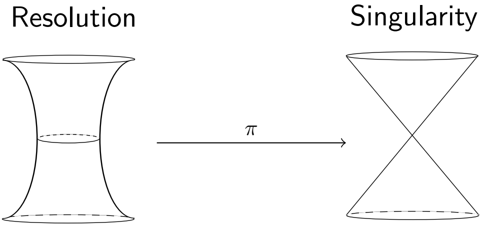

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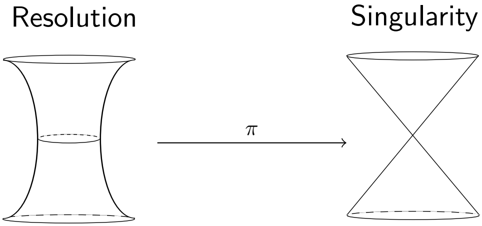

Resolution

Singularity

π

$\mathsf{Perf}(X)$

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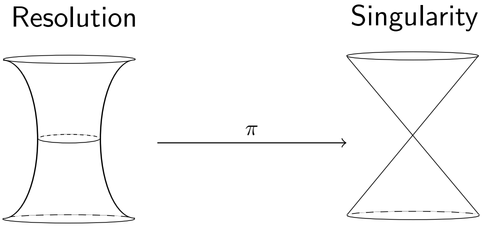

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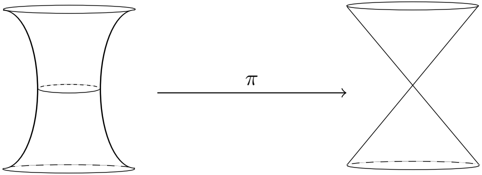

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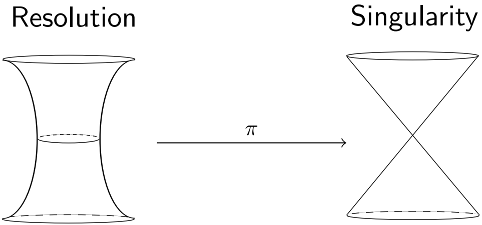

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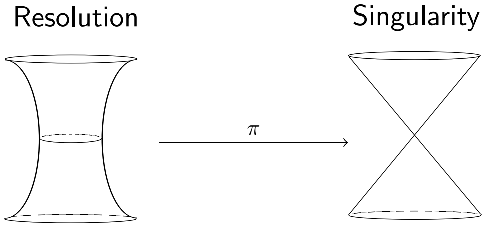

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Aim

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relative singularity category

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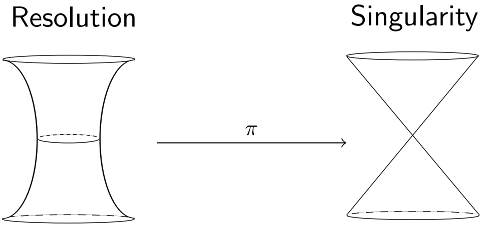

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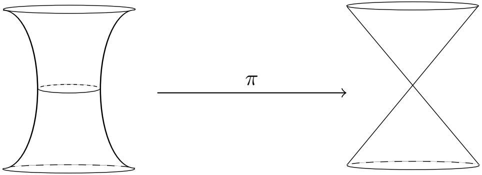

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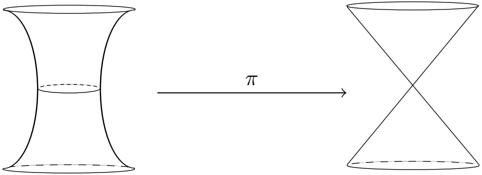

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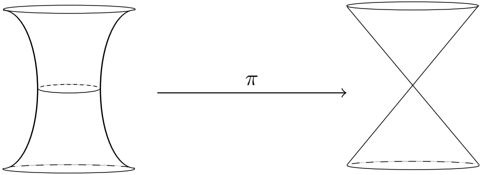

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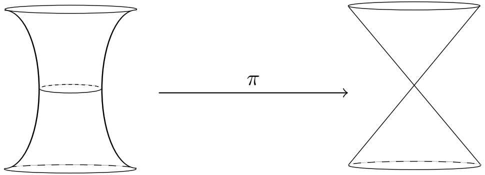

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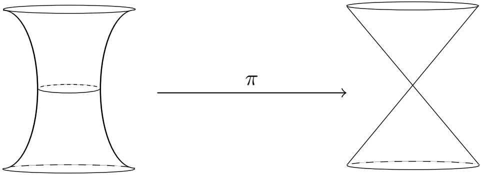

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Lemma: $\mathrm{gl.dim}(A) < \infty \Leftrightarrow A$ is semi-simple.

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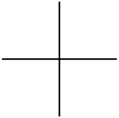
Some curve singularities



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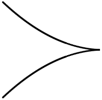

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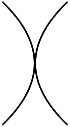

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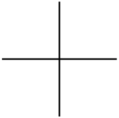

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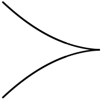

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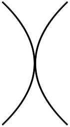

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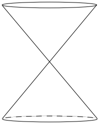

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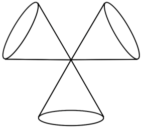

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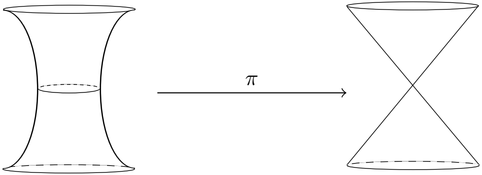
Overview

③ Resolutions ① Singularities



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2. Singularity categories

Definition/Theorem

Definition *of* $K^{-,b}(\mathrm{proj}-R)$*:*

Objects: *Right bounded complexes* P^* *of projective* R*-modules, i.e.*

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with **bounded cohomology**, *i.e.* $H^i(P^*) := \ker d_i / \mathrm{im}\, d_{i+1} = 0$ *for* $i \gg 0$.
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(v) A **triangulated subcategory** $\mathcal{U}$ of $\mathcal{D}^b(R)$ is a full subcategory, s.th. $\Sigma\colon \mathcal{U} \xrightarrow{\sim} \mathcal{U}$ and $C(f) \in \mathcal{U}$ for every morphism $f\colon X \to Y$ in $\mathcal{U}$.

2. Singularity categories

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Let R be a Noetherian ring. The **singularity category** of R is the quotient category

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- Moreover, this suggests to view $K^b(\mathrm{proj}-R) \subseteq \mathcal{D}^b(R)$ as the **'smooth part'** and $\mathcal{D}_{sg}(R)$ as a **measure for the complexity of the singularities** of $\mathrm{Spec}(R)$.

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Example

Let $R = k[x]/(x^2)$. We determine the indecomposables in $\mathcal{D}_{sg}(R)$.

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Let $R = k[x]/(x^2)$. We determine the indecomposables in $\mathcal{D}_{sg}(R)$. Recall,

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\cdots \arrow[r, "x\cdot"] & R \arrow[r, "x\cdot"] & R \arrow[r, "x\cdot"] & R \arrow[r, "x\cdot"] & \cdots \arrow[r, "x\cdot"] & R \arrow[r, "x\cdot"] & R \arrow[r] & 0 \arrow[r] & \cdots \in K^{-,b}(\operatorname{proj} -R)
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are the indecomposables in $\mathcal{D}^b(R)$ (up to shift).

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are the indecomposables in $\mathcal{D}^b(R)$ (up to shift). The bounded complexes vanish in the singularity category.

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This yields the triangle $X \xrightarrow{i} Y \xrightarrow{p} \Sigma^n(Y) \to \Sigma(X)$,

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$$\begin{tikzcd}
X = \cdots \arrow[r] \arrow[d, "i"'] & 0 \arrow[r] \arrow[d, "0"'] & 0 \arrow[r] \arrow[d, "0"'] & R \arrow[r, "x\cdot"] \arrow[d, "\mathrm{id}"'] \arrow[rrr, phantom, "\overbrace{\hspace{6cm}}^{n \text{ terms}}", yshift=4ex] & R \arrow[r, "x\cdot"] \arrow[d, "\mathrm{id}"'] & \cdots \arrow[r, "x\cdot"] & R \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \arrow[r] \arrow[d, "0"'] & \cdots \\
Y = \cdots \arrow[r, "x\cdot"] \arrow[d, "p"'] & R \arrow[r, "x\cdot"] \arrow[d, "\mathrm{id}"'] & R \arrow[r, "x\cdot"] \arrow[d, "\mathrm{id}"'] & R \arrow[r, "x\cdot"] \arrow[d, "0"'] & R \arrow[r, "x\cdot"] \arrow[d, "0"'] & \cdots \arrow[r, "x\cdot"] & R \arrow[r] \arrow[d, "0"'] & 0 \arrow[r] \arrow[d, "0"'] & \cdots \\
\Sigma^n(Y) = \cdots \arrow[r, "x\cdot"] & R \arrow[r, "x\cdot"] & R \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

This yields the triangle $X \xrightarrow{i} Y \xrightarrow{p} \Sigma^n(Y) \to \Sigma(X)$, which we may shift to obtain

$$Y \xrightarrow{p} \Sigma^n(Y) \to \Sigma(X) \xrightarrow{\Sigma(i)} \Sigma(Y)$$

2. Singularity categories

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- Moreover, the endomorphism algebra of Y in $\mathcal{D}_{sg}(R)$ is isomorphic to k. This gives an equivalence of additive categories

$$\mathcal{D}_{sg}(R) \cong \mathrm{mod}-k$$

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We can also consider the **stable category**

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(iv) **Hypersurface** rings $R = k[\![x_0, \ldots, x_d]\!]/(f)$ are Gorenstein.

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The composition of functors

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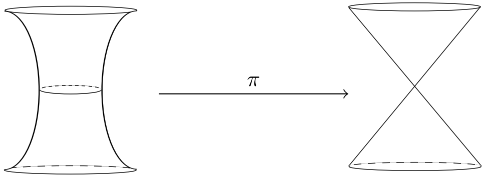
Overview

③ Resolutions ① Singularities



$$\begin{tikzcd} \mathcal{D}^b(\mathrm{Coh}\, Y) & \mathrm{Perf}(X) \arrow[l, hook', "\mathbb{L}\pi^*", "\text{‘categorical resolution’}"'] & \mathcal{D}^b(\mathrm{Coh}\, X) \arrow[l, phantom, "\underset{\text{‘smooth’}}{\supseteq}"] \end{tikzcd}$$

Aim

Use representation theory to describe the triangulated quotients

⑤ *Relations*

$$\Delta_X(Y) := \frac{\mathcal{D}^b(\mathrm{Coh}\, Y)}{\mathrm{Perf}(X)} \longleftrightarrow \mathcal{D}_{sg}(X) := \frac{\mathcal{D}^b(\mathrm{Coh}\, X)}{\mathrm{Perf}(X)}$$

④ *Relative singularity categories* ② *Singularity categories*

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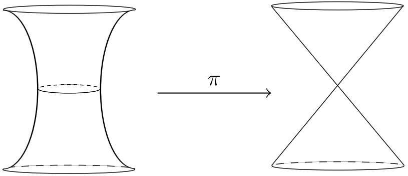

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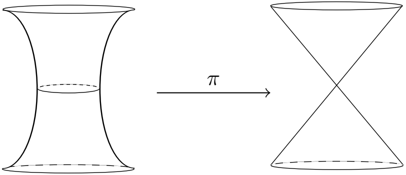

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Kapranov & Vasserot

Derived McKay Correspondence

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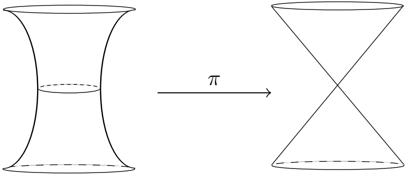

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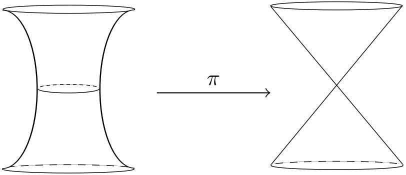

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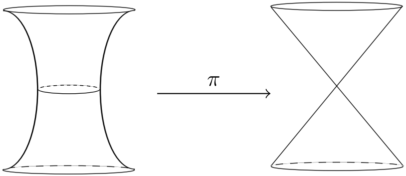

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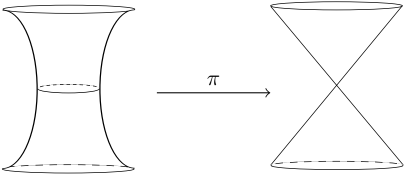

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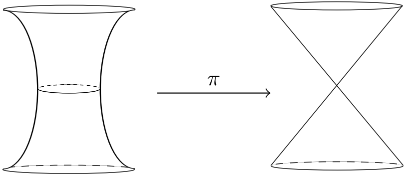

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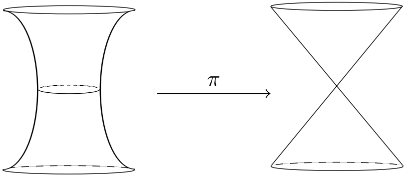

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3. Resolutions

Definition (see Dao, Iyama, Takahashi & Vial)

Let R be an Iwanaga–Gorenstein ring. Let $M \in \mathrm{MCM}(R) := \left\{ N \in \mathrm{mod}-R \,|\, \mathrm{Ext}^i_R(N, R) = 0 \text{ for all } i > 0 \right\}$ be a maximal Cohen–Macaulay module and $A := \mathrm{End}_R(R \oplus M)$. If $\mathrm{gl.\,dim}(A) < \infty$, then A is a **non-commutative resolution** (NCR) and $\mathcal{D}^b(A)$ is a **categorical resolution** of R.

Example

Let $R = k[x]/(x^2)$. Then the **Auslander algebra** of R

$$\mathsf{Aus}(R) = \mathrm{End}_R(R \oplus k) \cong k\Big(1 \underset{b}{\overset{a}{\rightleftarrows}} 2 \Big)/(ab)$$

has global dimension 2. In particular, $\mathsf{Aus}(R)$ is an NCR of R.

- For $k = \mathbb{C}$ there is an equivalence of categories

$$\mathrm{mod} - \mathsf{Aus}(R) \cong \mathcal{O}_0\big(\mathfrak{sl}_2(\mathbb{C})\big) \quad (\textbf{principal block of category } \mathcal{O})$$

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- More generally, if $\mathrm{char}\, k = p$, then the Schur algebra $S_k(p,p)$ gives an NCR of the group algebra $k[\mathfrak{S}_p]$.

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Theorem (Auslander, Auslander & Roggenkamp, Iyama)

Let R be a commutative complete Gorenstein k-algebra of Krull dimension d, with $k \cong R/\mathfrak{m}$ algebraically closed.

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$$\mathsf{Aus}(R) := \operatorname{End}_R \left(\bigoplus_{M \in \operatorname{ind} \operatorname{MCM}(R)} M \right)$$

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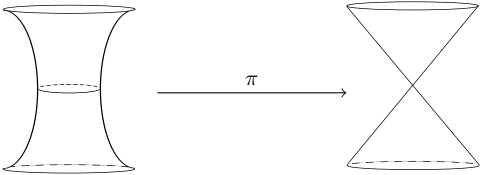
Remark

All ADE-singularities satisfy the conditions of this theorem, e.g. type (A_n):

$$k[\![x_0, \ldots, x_d]\!]/(x_0^{n+1} + x_1^2 + \ldots + x_d^2).$$

Overview

③ Resolutions ① Singularities



$$\begin{tikzcd}
\mathcal{D}^b(\mathsf{Coh}\,Y) & \mathrm{Perf}(X) \arrow[l, hook', "\mathbb{L}\pi^*", "\text{`categorical resolution'}"'] \arrow[r, phantom, "\subseteq", "\text{`smooth'}"'] & \mathcal{D}^b(\mathsf{Coh}\,X)
\end{tikzcd}$$

Aim

Use representation theory to describe the triangulated quotients

$$\Delta_X(Y) := \frac{\mathcal{D}^b(\mathsf{Coh}\,Y)}{\mathrm{Perf}(X)} \xleftrightarrow{\textcircled{5}\ \textit{Relations}} \mathcal{D}_{sg}(X) := \frac{\mathcal{D}^b(\mathsf{Coh}\,X)}{\mathrm{Perf}(X)}$$

④ *Relative singularity categories* ② *Singularity categories*

4. Relative singularity categories

Let A be an NCR of R, i.e.

$$A = \operatorname{End}_R(R \oplus M) \cong \begin{pmatrix} R & M \\ \operatorname{Hom}_R(R, M) & \operatorname{End}_R(M) \end{pmatrix}$$

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Then there is an **embedding of triangulated categories**

$$\begin{tikzcd}
K^b(\mathrm{proj} - R) \arrow[r, "="] & K^b(\mathrm{proj} - eAe) \arrow[r, hook, "- \otimes_{eAe} eA"] & \mathcal{D}^b(A) \\
R \arrow[r, mapsto] & eAe \arrow[r, mapsto] & eA
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Definition

The **relative singularity category** of the NCR A of R is the triangulated quotient category

$$\Delta_R(A) := \frac{\mathcal{D}^b(A)}{K^b(\operatorname{proj} -R)} = \frac{\mathcal{D}^b(A)}{\text{thick}(eA)}$$

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$\Delta_R(A)$ may be viewed as a **measure** for the **size** of the **resolution** $\mathcal{D}^b(A)$ relative to the 'smooth part' $K^b(\mathrm{proj} - R) \subseteq \mathcal{D}^b(A)$.

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Remark

Relative singularity categories were also studied by X.-W. Chen and Thanhoffer de Völcsey & Van den Bergh.

4. Relative singularity categories

Example

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$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \overbrace{P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1}^{\mathbb{N}_0 \ni l \text{ terms}} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \qquad \in K^b(\mathrm{proj} - R)$$

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$$\begin{tikzcd} K^b(\mathrm{proj}-R) \arrow[d, hook] \\ \mathcal{D}^b(A) \end{tikzcd}$$

$$\left.\begin{array}{l}
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\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1 \xrightarrow{a\cdot} P_2 \longrightarrow 0 \longrightarrow \cdots \\
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\end{array}\right\} \in \mathcal{D}^b(A)$$

4. Relative singularity categories

Example

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & P_1 \arrow[r, "ba\cdot"] & P_1 \arrow[r, "ba\cdot"] & \cdots \arrow[r, "ba\cdot"] & P_1 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots & \in K^b(\mathrm{proj}-R) \arrow[dd, hook] \\
\cdots \arrow[r] & 0 \arrow[r] & P_2 \arrow[r, "b\cdot"] & P_1 \arrow[r, "ba\cdot"] & P_1 \arrow[r, "ba\cdot"] & \cdots \arrow[r, "ba\cdot"] & P_1 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots & \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & P_1 \arrow[r, "ba\cdot"] & P_1 \arrow[r, "ba\cdot"] & \cdots \arrow[r, "ba\cdot"] & P_1 \arrow[r, "a\cdot"] & P_2 \arrow[r] & 0 \arrow[r] & \cdots & \Bigg\} \in \mathcal{D}^b(A) \\
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\end{tikzcd}$$

Now, $K^b(\mathrm{proj}-R)$ vanishes in the relative singularity category $\Delta_R(A)$

4. Relative singularity categories

Example

$$\begin{array}{ll}
\cdots\longrightarrow 0\longrightarrow 0\longrightarrow P_1\xrightarrow{ba\cdot}P_1\xrightarrow{ba\cdot}\cdots\xrightarrow{ba\cdot}P_1\longrightarrow 0\longrightarrow 0\longrightarrow\cdots & \in K^b(\mathrm{proj}\,{-}R) \\
 & \qquad\rotatebox[origin=c]{-90}{$\hookrightarrow$} \\
\left.\begin{array}{l}
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\end{array}\right\} & \in \mathcal{D}^b(A)
\end{array}$$

Now, $K^b(\mathrm{proj}\,{-}R)$ vanishes in the relative singularity category $\Delta_R(A)$ and as in the computations for singularity categories, the morphism

$$\begin{tikzcd}
X \arrow[r, phantom, "="] \arrow[d, "f"] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & P_1 \arrow[r, "ba\cdot"] \arrow[d, "\mathrm{id}"'] \arrow[rrr, phantom, bend left=40, "\overbrace{\hspace{5cm}}^{\mathbb{N}\ni l \text{ terms}}"] & P_1 \arrow[r, "ba\cdot"] \arrow[d, "\mathrm{id}"'] & \cdots \arrow[r, "ba\cdot"] & P_1 \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \arrow[r] \arrow[d] & \cdots & \in K^b(\mathrm{proj}\,{-}R) \\
Y \arrow[r, phantom, "="] & \cdots \arrow[r] & 0 \arrow[r] & P_2 \arrow[r, "b\cdot"] & P_1 \arrow[r, "ba\cdot"] & P_1 \arrow[r, "ba\cdot"] & \cdots \arrow[r, "ba\cdot"] & P_1 \arrow[r] & 0 \arrow[r] & \cdots &
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4. Relative singularity categories

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$\mathbb{N} \ni l$ terms

$$\begin{tikzcd}
X = \cdots \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & P_1 \arrow[r, "ba\cdot"] \arrow[d, "\mathrm{id}"'] & P_1 \arrow[r, "ba\cdot"] \arrow[d, "\mathrm{id}"'] & \cdots \arrow[r, "ba\cdot"] & P_1 \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \arrow[r] \arrow[d] & \cdots & \in K^b(\mathrm{proj}\,-R) \\
Y = \cdots \arrow[r] & 0 \arrow[r] & P_2 \arrow[r, "b\cdot"] & P_1 \arrow[r, "ba\cdot"] & P_1 \arrow[r, "ba\cdot"] & \cdots \arrow[r, "ba\cdot"] & P_1 \arrow[r] & 0 \arrow[r] & \cdots &
\end{tikzcd}$$

shows that there exists $n \in \mathbb{Z}$ such that

$$Y \cong \Sigma^n(P_2) \qquad \text{in } \Delta_R(A).$$

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$$\begin{tikzcd}
 & & & & & \overbrace{\qquad\qquad\qquad\qquad}^{\mathbb{N} \ni l \text{ terms}} & & & & & \\
X \arrow[d, "f"] & = \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & P_1 \arrow[r, "ba\cdot"] \arrow[d, "\mathrm{id}"'] & P_1 \arrow[r, "ba\cdot"] \arrow[d, "\mathrm{id}"'] & \cdots \arrow[r, "ba\cdot"] & P_1 \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \arrow[r] \arrow[d] & \cdots & \in K^b(\mathrm{proj} -R) \\
Y & = \cdots \arrow[r] & 0 \arrow[r] & P_2 \arrow[r, "b\cdot"] & P_1 \arrow[r, "ba\cdot"] & P_1 \arrow[r, "ba\cdot"] & \cdots \arrow[r, "ba\cdot"] & P_1 \arrow[r] & 0 \arrow[r] & \cdots &
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shows that there exists $n \in \mathbb{Z}$ such that

$$Y \cong \Sigma^n(P_2) \qquad \text{in } \Delta_R(A).$$

Similarly, one obtains isomorphisms

$$\cdots \longrightarrow 0 \longrightarrow P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1 \xrightarrow{a\cdot} P_2 \longrightarrow 0 \longrightarrow \cdots \cong \Sigma^n(P_2).$$

4. Relative singularity categories

Example

One can show that the remaining objects, i.e. shifts of

$$P_2 \quad \text{and} \quad \cdots \longrightarrow 0 \longrightarrow P_2 \xrightarrow{b\cdot} P_1 \xrightarrow{ba\cdot} P_1 \xrightarrow{ba\cdot} \cdots \xrightarrow{ba\cdot} P_1 \xrightarrow{a\cdot} P_2 \longrightarrow 0 \longrightarrow \cdots$$

are indecomposable and pairwise non-isomorphic in $\Delta_R(A)$.

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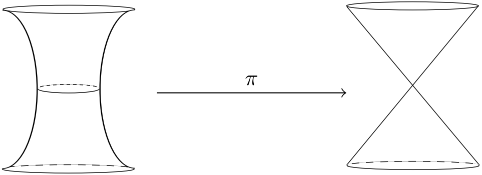
are indecomposable and pairwise non-isomorphic in $\Delta_R(A)$. Moreover, the **quiver of irreducible morphisms** of $\Delta_R(A)$ consists of one $\mathbb{Z}A_\infty$-component and one equioriented A_∞^∞-component.

Overview

③ Resolutions ① Singularities



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Aim

Use representation theory to describe the triangulated quotients

⑤ *Relations*

$$\Delta_X(Y) := \frac{\mathcal{D}^b(\mathsf{Coh}\, Y)}{\mathsf{Perf}(X)} \longleftrightarrow \mathcal{D}_{sg}(X) := \frac{\mathcal{D}^b(\mathsf{Coh}\, X)}{\mathsf{Perf}(X)}$$

④ *Relative singularity categories* ② *Singularity categories*

5. Relations: Knörrer's Periodicity

Knörrer's Periodicity Theorem yields a relation between singularity categories for different Krull dimensions:

$$\mathcal{D}_{sg}\big(S/(f)\big) \xrightarrow{\sim} \mathcal{D}_{sg}\big(S[\![x,y]\!]/(f+x^2+y^2)\big),$$

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The implication $(ii) \Rightarrow (i)$ holds more generally for arbitrary NCRs A and A' of arbitrary isolated Gorenstein singularities R and R'. In fact, there **always exists a quotient functor** *$\Delta_R(A) \to \mathcal{D}_{sg}(R)$.*

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$\wr\Big\downarrow$ Derived McKay

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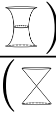

$$\begin{tikzcd}
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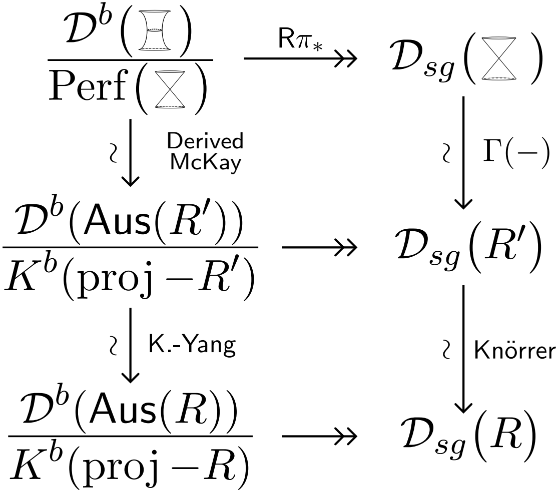

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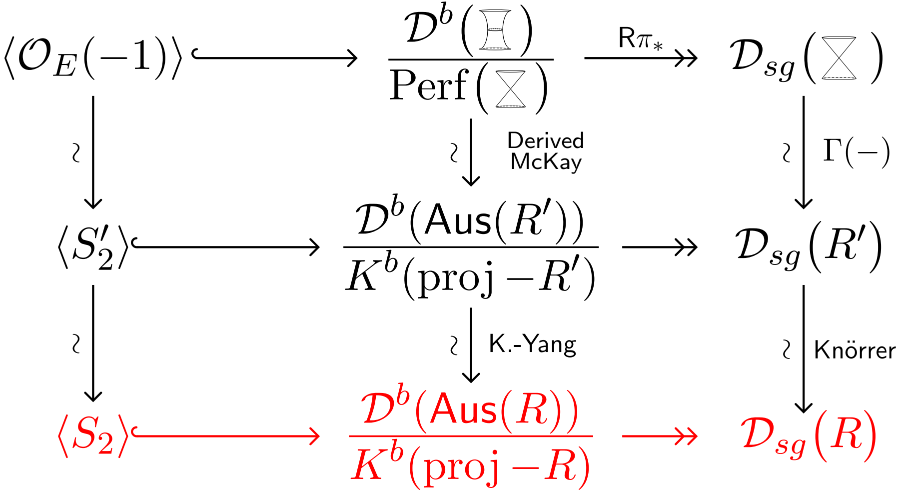

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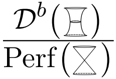

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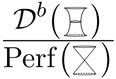

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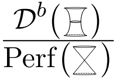

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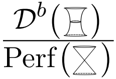

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- More generally for ADE-singularities R, the **relative Auslander singularity categories** $\Delta_R(\mathsf{Aus}(R))$ admit **explicit dg descriptions**.
- Finally, in dimension 3, the '**conifold**'

$$X = V(W^2 + X^2 + Y^2 + Z^2) \subseteq \mathbb{A}^4$$

  has a crepant resolution Y and the **relative singularity category** $\Delta_X(Y)$ admits a very similar representation theoretic description.

Thank you!