Irreducible Components of Quiver Grassmannians

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Quiver Grassmannians

A quiver Grassmannian is a projective variety of the form

$$\mathrm{Gr}_Q\binom{M}{\underline{d}} := \{U \subset M \text{ subrepresentation}, \ \underline{\dim} U = \underline{d}\}$$

where

- Q is a quiver
- M is a representation of Q
- $\underline{d}$ is a dimension vector for Q.

The aim of this talk is to give a decomposition theorem for irreducible components of quiver Grassmannians, analogous to the canonical decomposition of quiver representations.

Representation Varieties

Let Q be a quiver and $\underline{d}$ a dimension vector. Consider the scheme

$$\mathsf{rep}_Q^{\underline{d}} := \prod_{i \to j} \mathbb{M}_{d_j \times d_i}$$

This is isomorphic to affine space, so is smooth and irreducible.

The group scheme

$$\mathsf{GL}_{\underline{d}} := \prod_i \mathsf{GL}_{d_i}$$

acts by conjugation, and for any field L we have a bijection

$$\mathsf{GL}_{\underline{d}}(L)\text{-orbits} \leftrightarrow \text{isoclasses of } Q\text{-reps over } L \text{ of dimension } \underline{d}$$

Schur Roots

One can ask about general properties of quiver representations, so those properties shared by all representations in an open dense subset of some $\mathrm{rep}_Q^{\underline{d}}$.

For example, we call $\underline{d}$ a **Schur root** if a general representation is indecomposable.

Real Schur Roots

A Schur root $\underline{d}$ is **real** if $q(\underline{d}) = 1$, where

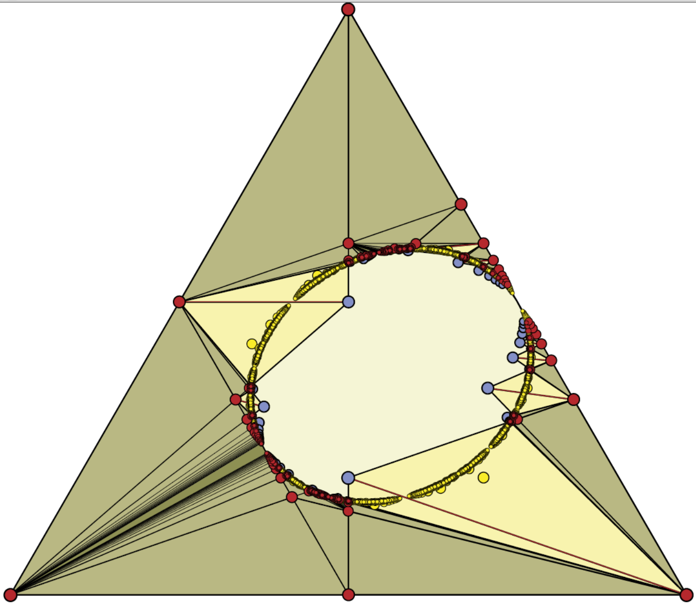
$$q(\underline{d}) := \sum_i d_i^2 - \sum_{i \to j} d_i d_j.$$

Equivalently, there is a unique indecomposable up to isomorphism.

It follows that the real Schur roots are the **rigid** ones: a small perturbation of an indecomposable yields an isomorphic indecomposable.

Real Schur roots are in bijection with the (non-trivial) cluster variables in the cluster algebra.

Schur Roots



Direct Sum Map

The direct sum of representations induces a closed immersion

$$\mathrm{rep}_Q^{\underline{d}} \times \mathrm{rep}_Q^{\underline{e}} \to \mathrm{rep}_Q^{\underline{d}+\underline{e}}.$$

Its image consists of those tuples of matrices which are in block-diagonal form.

Combining with the group action we obtain the morphism

$$\Theta = \Theta_{\underline{d},\underline{e}} \colon \mathrm{GL}_{\underline{d}+\underline{e}} \times \mathrm{rep}_Q^{\underline{d}} \times \mathrm{rep}_Q^{\underline{e}} \to \mathrm{rep}_Q^{\underline{d}+\underline{e}}.$$

Canonical Decomposition

Recall that $\mathrm{rep}_Q^{\underline{d}}$ is irreducible.

So, if $\underline{d}$ is not a Schur root, then

$$\mathrm{rep}_Q^{\underline{d}} = \bigcup_{0<\underline{e}<\underline{d}} \overline{\operatorname{Im}\Theta_{\underline{e},\underline{d}-\underline{e}}}.$$

We can thus find some $\underline{e}$ with

$$\mathrm{rep}_Q^{\underline{d}} = \overline{\operatorname{Im}\Theta_{\underline{e},\underline{d}-\underline{e}}}.$$

It follows that a general representation M of dimension $\underline{d}$ satisfies $M \cong M' \oplus M''$ with $\underline{\dim} M' = \underline{e}$.

Canonical Decomposition

Repeating we obtain a decomposition

$$\underline{d} = \underline{d}_1 + \cdots + \underline{d}_n$$

called the **canonical decomposition**, such that

- each $\underline{d}_i$ is a Schur root
- a general representation M of dimension $\underline{d}$ satisfies $M \cong \bigoplus_i M_i$ with M_i indecomposable of dimension $\underline{d}_i$.

Canonical Decomposition

A decomposition into Schur roots

$$\underline{d} = \underline{d}_1 + \cdots + \underline{d}_n$$

is the canonical decomposition if the general representations M_i of dimension $\underline{d}_i$ have no extensions with each other

$$\mathrm{Ext}^1_\Lambda(M_i, M_j) = 0 \quad \text{for } i \neq j.$$

We usually write this as

$$\mathrm{ext}(\underline{d}_i, \underline{d}_j) = 0 \quad \text{for } i \neq j.$$

Extension to Other Algebras

Crawley-Boevey and Schröer extended these results to all finitely-generated algebras.

Let K be algebraically-closed and Λ a finitely-generated K-algebra.

For an integer d we have the functor on commutative K-algebras

$$R \mapsto \mathrm{rep}^d_\Lambda(R) := \mathrm{Hom}_{K\text{-alg}}(\Lambda, \mathbb{M}_d(R)).$$

This is an affine scheme on which GL_d acts by conjugation, and

$$\mathrm{GL}_d(L)\text{-orbits} \leftrightarrow \text{isoclasses of } L \otimes_K \Lambda\text{-modules of dimension } d$$

Direct Sum Map

We have a closed immersion

$$\mathrm{rep}_\Lambda^d \times \mathrm{rep}_\Lambda^e \to \mathrm{rep}_\Lambda^{d+e},$$

which we combine with the group action to get

$$\Theta\colon \mathrm{GL}_{d+e} \times \mathrm{rep}_\Lambda^d \times \mathrm{rep}_\Lambda^e \to \mathrm{rep}_\Lambda^{d+e}.$$

If $X \subset \mathrm{rep}_\Lambda^d$ and $Y \subset \mathrm{rep}_\Lambda^e$ are irreducible components, write

$$\overline{X \oplus Y} := \overline{\Theta(\mathrm{GL}_{d+e} \times X \times Y)},$$

an irreducible subset of $\mathrm{rep}_\Lambda^{d+e}$.

Decomposition of Irreducible Components

Theorem [Crawley-Boevey, Schröer]

1. Every irreducible component of rep_Λ^d can be written as a finite direct sum of generally indecomposable irreducible components.

2. Let $X \subset \mathrm{rep}_\Lambda^d$ and $Y \subset \mathrm{rep}_\Lambda^e$ be irreducible components. Then $\overline{X \oplus Y} \subset \mathrm{rep}_\Lambda^{d+e}$ is an irreducible component if and only if the general representations of X and Y have no extensions with each other

$$\mathrm{ext}(X, Y) = 0 = \mathrm{ext}(Y, X).$$

Grassmannians

Given a Λ-module M we can consider the Grassmannian $\mathrm{Gr}_\Lambda\binom{M}{d}$, a subfunctor of the usual Grassmannian.

For each commutative K-algebra R we take those R-submodules $U \subset R \otimes_K M$ which are

- direct summands of rank d
- Λ-submodules

This time it is the group scheme $\mathrm{Aut}_\Lambda(M)$ that acts on $\mathrm{Gr}_\Lambda\binom{M}{d}$.

Direct Sum Map

We again have a closed immersion

$$\mathrm{Gr}_\Lambda\binom{M}{d} \times \mathrm{Gr}_\Lambda\binom{N}{e} \to \mathrm{Gr}_\Lambda\binom{M \oplus N}{d+e},$$

and so we obtain the morphism

$$\Theta\colon\ \mathrm{Aut}_\Lambda(M \oplus N) \times \mathrm{Gr}_\Lambda\binom{M}{d} \times \mathrm{Gr}_\Lambda\binom{N}{e} \to \mathrm{Gr}_\Lambda\binom{M \oplus N}{d+e}.$$

A Principal Bundle

We want to understand when an irreducible component of the Grassmannian is generally indecomposable. For this we recall the construction of the Grassmannian as a principal GL_d-bundle.

Let $\mathsf{inj}_{m\times d} = \mathsf{rank}_d \subset \mathbb{M}_{m\times d}$ be the open subscheme of matrices of rank d, so those matrices whose d minors generate the unit ideal.

The group GL_d acts and the morphism

$$\pi\colon\ \mathsf{inj}_{m\times d} \to \mathsf{Gr}_K \binom{m}{d}$$

is a principal GL_d-bundle.

A Principal Bundle

Fix a Λ-module M of dimension m.

Consider the scheme $\operatorname{rep}\operatorname{inj}_\Lambda^{(d,M)}$ consisting of

- a Λ-module M' of dimension d
- an injective homomorphism $f\colon M' \to M$.

The map

$$\operatorname{rep}\operatorname{inj}_\Lambda^{(d,M)} \to \operatorname{inj}_{m\times d}, \quad (M', f) \mapsto f,$$

is a closed immersion, and commutes with the natural GL_d-actions.

A Principal Bundle

We have a commutative diagram

$$\begin{array}{ccc} \operatorname{rep}\operatorname{inj}_\Lambda^{(d,M)} & \longrightarrow & \operatorname{inj}_{m\times d} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \operatorname{Gr}_\Lambda\binom{M}{d} & \longrightarrow & \operatorname{Gr}_K\binom{m}{d} \end{array}$$

where the horizontal maps are closed immersions and the vertical maps are principal GL_d-bundles.

A Tensor Algebra

Let $\Lambda(2) \subset \mathbb{M}_2(\Lambda)$ be the algebra of upper-triangular matrices.

We can also write this as the tensor algebra

$$K\Delta \otimes_K \Lambda$$

where Δ is the quiver $1 \to 2$.

A $\Lambda(2)$-module is a triple (M', M, f) where

- M', M are Λ-modules
- $f\colon M' \to M$ is a homomorphism.

We then have the scheme $\mathrm{rep}_{\Lambda(2)}^{(d,m)}$ parameterising those modules with $\dim M' = d$ and $\dim M = m$.

Generally Indecomposable Subsets

We can therefore regard $\operatorname{rep} \operatorname{inj}_{\Lambda}^{(d,M)}$ as a subscheme of $\operatorname{rep}_{\Lambda(2)}^{(d,m)}$.

Thus if $U \in \operatorname{Gr}_{\Lambda} \binom{M}{d}$, then we regard $(U \subset M)$ as a $\Lambda(2)$-representation.

If $X \subset \operatorname{Gr}_{\Lambda} \binom{M}{d}$ is an irreducible component, then $\pi^{-1}(X) \subset \operatorname{rep}_{\Lambda(2)}^{(d,m)}$ is also irreducible.

We say that X is generally indecomposable provided $\pi^{-1}(X)$ is generally indecomposable. This means that there is an open dense subset of X consisting of submodules $U \in \operatorname{Gr}_{\Lambda} \binom{M}{d}$ such that $(U \subset M)$ is an indecomposable $\Lambda(2)$-representation.

A Vanishing Condition

We now want to understand when the direct sum of two irreducible components will again be an irreducible component.

Recall that this held for representation varieties if generally there were no extensions.

This arises through Voigt's Lemma, which says that for each representation M of dimension d there is an exact sequence

$$0 \to \mathrm{End}_\Lambda(M) \to \underset{\displaystyle \mathrm{Lie}(\mathrm{GL}_d)}{\underset{\|}{\mathbb{M}_d(K)}} \to \underset{\displaystyle T_M\,\mathrm{rep}_\Lambda^d}{\underset{\|}{\mathrm{Der}_K(\Lambda, \mathrm{End}_K(M))}} \to \mathrm{Ext}^1_\Lambda(M, M) \to 0$$

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In fact, this can be naturally identified with the beginning of the Hochschild cohomology complex for Λ with values in $\mathrm{End}_K(M) = \mathbb{M}_d(K)$.

A Vanishing Condition

We want to extend this to Grassmannians.

Let $U \in \mathrm{Gr}_\Lambda \binom{M}{d}$ and consider the following $\Lambda(2)$-modules.

- $\widetilde{U} := (U \subset M)$
- $\widetilde{M} := (M = M)$
- $\widetilde{M}/\widetilde{U} := (M/U \to 0)$.

We have the identification

$$T_U \, \mathrm{Gr}_\Lambda \binom{M}{d} \cong \mathrm{Hom}_\Lambda(U, M/U) \cong \mathrm{Hom}_{\Lambda(2)}(\widetilde{U}, \widetilde{M}/\widetilde{U}).$$

Also,

$$\mathrm{Lie}\left(\mathrm{Aut}_\Lambda(M)\right) = \mathrm{End}_\Lambda(M) \cong \mathrm{Hom}_{\Lambda(2)}(\widetilde{U}, \widetilde{M}).$$

A Vanishing Condition

The analogue of Voigt's Lemma for Grassmannians is the following.

$$\begin{array}{ccccccccc} 0 \to & \mathrm{End}_{\Lambda(2)}(\widetilde{U}) & \to & \mathrm{End}_{\Lambda(2)}(\widetilde{M}) & \to & \mathrm{Hom}_{\Lambda(2)}(\widetilde{U}, \widetilde{M}/\widetilde{U}) & \to & \overline{\mathrm{Ext}}(\widetilde{U}, \widetilde{U}) & \to 0 \\ & & & \| & & \| & & & \\ & & & \mathrm{Lie}(\mathrm{Aut}_\Lambda(M)) & & T_U \,\mathrm{Gr}_\Lambda \binom{M}{d} & & & \end{array}$$

A Vanishing Condition

The analogue of Voigt's Lemma for Grassmannians is the following.

$$0 \to \operatorname{End}_{\Lambda(2)}(\widetilde{U}) \to \operatorname{End}_{\Lambda(2)}(\widetilde{M}) \to \operatorname{Hom}_{\Lambda(2)}(\widetilde{U}, \widetilde{M}/\widetilde{U}) \to \overline{\operatorname{Ext}}(\widetilde{U}, \widetilde{U}) \to 0$$

This can be obtained from applying $\operatorname{Hom}_{\Lambda(2)}(\widetilde{U}, -)$ to the short-exact sequence

$$0 \to \widetilde{U} \to \widetilde{M} \to \widetilde{M}/\widetilde{U} \to 0.$$

Thus

$$\overline{\operatorname{Ext}}(\widetilde{U}, \widetilde{U}) \subset \operatorname{Ext}^1_{\Lambda(2)}(\widetilde{U}, \widetilde{U})$$

consists of those extension classes which are pull-backs along some homomorphism $\widetilde{U} \to \widetilde{M}/\widetilde{U}$.

Decomposition of Irreducible Components

Theorem

1. Every irreducible component of $\mathrm{Gr}_\Lambda \binom{M}{d}$ can be written as a finite direct sum of generally indecomposable irreducible components.

2. If $X \subset \mathrm{Gr}_\Lambda \binom{M}{d}$ and $Y \subset \mathrm{Gr}_\Lambda \binom{N}{e}$ are irreducible components, then $\overline{X \oplus Y} \subset \mathrm{Gr}_\Lambda \binom{M \oplus N}{d+e}$ is an irreducible component if and only if $\overline{\mathrm{Ext}}(\widetilde{U}, \widetilde{V}) = 0 = \overline{\mathrm{Ext}}(\widetilde{V}, \widetilde{U})$ for all (U, V) in an open dense subset of $X \times Y$.

Examples

Let $\Lambda = K[X]/(X^4)$, and let $S_i = K[X]/(X^i)$ for $i = 1, 2, 3, 4$ be representatives for the indecomposable modules.
Then

$$\mathrm{Gr}_\Lambda \binom{S_1 \oplus S_3}{2} \cong \mathrm{Proj}\left(K[x, y, s, t]/(xt - ys, s^3, st, t^3)\right)$$

so this looks like $\mathbb{P}^1$ but is generically non-reduced.
The general representation is of the form

$$\begin{pmatrix} 0 & -x \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} : S_2 \hookrightarrow S_1 \oplus S_3$$

which is indecomposable as a $\Lambda(2)$-module.

Examples

Let $\Lambda = K\Delta \subset \mathbb{M}_2(K)$ be the subalgebra of upper-triangular matrices. This has indecomposables

- the simple projective S_2
- the simple injective S_1
- the indecomposable projective-injective T

Let $M = S_1 \oplus S_2 \oplus T$. Then

$$\mathrm{Gr}_\Lambda \binom{M}{(1,1)} \cong \mathrm{Proj}\left(K[x,y,z]/(xz)\right)$$

so is a union of two projective lines intersecting in a point.

Example

The two irreducible components of $\mathrm{Gr}_\Lambda \binom{M}{(1,1)}$ can be decomposed as

$$\overline{X_1 \oplus X_2 \oplus X_3} \quad \text{and} \quad \overline{Y_1 \oplus Y_2 \oplus Y_3}$$

where

$$X_1 = \mathrm{Gr}_\Lambda \binom{S_1}{(1,0)} \quad X_2 = \mathrm{Gr}_\Lambda \binom{S_2}{(0,1)} \quad X_3 = \mathrm{Gr}_\Lambda \binom{T}{(0,0)}$$

and

$$Y_1 = \mathrm{Gr}_\Lambda \binom{S_1}{(0,0)} \quad Y_2 = \mathrm{Gr}_\Lambda \binom{S_2}{(0,0)} \quad Y_3 = \mathrm{Gr}_\Lambda \binom{T}{(1,1)}.$$

A General Setting

Many schemes arising from representation theory come equipped with a group action and a direct sum map, so we consider the following general setting:

- G is a smooth, connected group scheme
- G acts on a scheme Y
- $X \subset Y$ is an irreducible subscheme
- $\Theta\colon G \times X \to Y$ is the restriction of the group action.

The question then becomes: when is $\overline{\operatorname{Im}\Theta}$ an irreducible component of Y?

A General Setting

This question has a nice answer in terms of infinitesimal deformations of the schemes, provided the morphism Θ is separable.

In general it seems difficult to determine this. In the situations we were interested in, though, we had a subgroup $H \leq G$, also smooth and connected, and fixing X.

Sufficient conditions for Θ to be separable are then

- the stabilisers are smooth
- a G-orbit intersected with X decomposes into only finitely many H-orbits
- the map on the conormals to the orbits is injective

A General Setting

For us these always hold:

- the stabilisers are smooth
  they are open in the endomorphism algebra
- a G-orbit intersected with X decomposes into only finitely many H-orbits
  follows from the Krull-Remak-Schmidt Theorem
- the map on the conormals to the orbits is injective
  follows from the cohomological interpretation of Voigt's Lemma

So this method applies, and hence the decomposition theorem holds, much more generally, to many types of schemes arising from representation theory.