Maximal Cohen-Macaulay Modules over some non-reduced Curve Singularities

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Outline of the talk

1. The Classification Problem and Theorem 1
2. Proof of Theorem 1: Reduction to a Matrix Problem
3. Examples and Classification Results

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Theorem (Drozd and Greuel, 1993)

The reduced curve singularities of type $\mathbf{P}_{pq}$ are CM-tame, where

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Idea of $\mathrm{Tri}(\check{\mathbf{P}})$: Construct a "pullback category" of top and $\_ \times \_$.

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Definition

Objects of $\mathrm{Tri}(\check{\mathbf{P}}) := \Big\{ ($

Step 2: Reconstruction of $\mathrm{CM}(\check{\mathbf{P}})$ from $\mathrm{Tri}(\check{\mathbf{P}})$

$$\begin{array}{ccc} & \mathrm{CM}(\check{\mathbf{P}}) & \dashrightarrow & \mathrm{CM}(\mathbf{S}) \\ & \vdots & & \downarrow{\scriptstyle \mathrm{top}} \\ {\color{red}V} \in & \mathrm{mod}(\mathbf{k}) & \xrightarrow[\_\times\_]{} & \mathrm{mod}(\mathbf{k} \times \mathbf{k}) \end{array}$$

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Step 2: Reconstruction of $\mathrm{CM}(\check{\mathbf{P}})$ from $\mathrm{Tri}(\check{\mathbf{P}})$

$$\begin{array}{ccc} \mathrm{CM}(\check{\mathbf{P}}) & \dashrightarrow & \mathrm{CM}(\mathbf{S}) \ni {\color{red}L} \\ \vdots & & \Big\downarrow \mathrm{top} \\ V \in \mathrm{mod}(\mathbf{k}) & \xrightarrow[\,\_\times\_\,]{} & \mathrm{mod}(\mathbf{k}\times\mathbf{k}) \end{array}$$

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\end{array}
\qquad
\begin{array}{c}
V \\ \downarrow \\ V \times V
\end{array}$$

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$$L \longmapsto \mathrm{top}(L)$$

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Objects of $\mathrm{Tri}(\check{\mathbf{P}}) := \Big\{ (V, L, \vartheta) \mid \vartheta$ surjective,

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\end{array}$$

$$\begin{array}{ccc}
V & & \\
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$$

$$
\begin{array}{ccc}
{\color{red}V} & {\color{red}\xhookrightarrow{\vartheta|_V}} & {\color{red}\mathrm{top}(L)} \\
\mathrm{diag} \downarrow & & \| \\
V \times V & \xrightarrow{\vartheta} & \mathrm{top}(L)
\end{array}
$$

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Step 2: Reconstruction of $\mathrm{CM}(\check{\mathbf{P}})$ from $\mathrm{Tri}(\check{\mathbf{P}})$

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Objects of $\mathrm{Tri}(\check{\mathbf{P}}) := \Big\{(V, L, \vartheta) \mid \vartheta$ surjective, $\vartheta|_V$ injective$\Big\}$.

Theorem (Burban and Drozd, 2012)

$$\mathrm{Tri}(\check{\mathbf{P}}) \xrightarrow{\sim} \mathrm{CM}(\check{\mathbf{P}})$$

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 & & L \\
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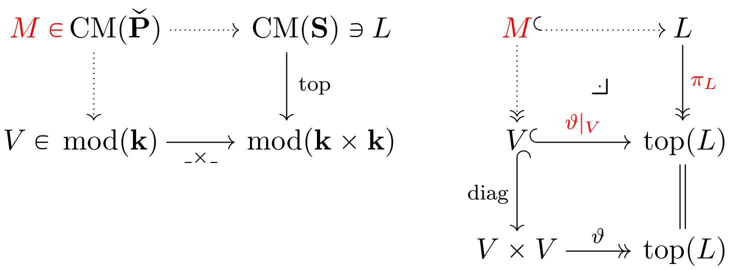

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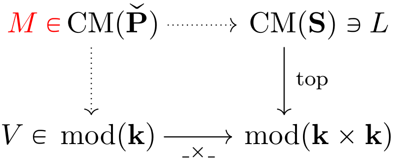

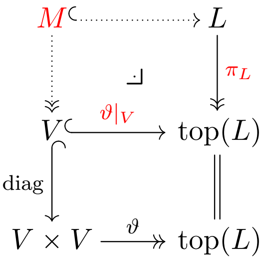

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Step 2: Conclusion on the Category of Triples

Corollary

There is a bijection of isomorphism classes

$$\left[\operatorname{ind}\operatorname{CM}(\check{\mathbf{P}})\right] \xleftrightarrow{1:1} \left[\operatorname{ind}\operatorname{Tri}(\check{\mathbf{P}})\right]$$

Step 2: Conclusion on the Category of Triples

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Proposition (Buchweitz, Greuel and Schreyer, 1987)

Let $\mathbf{A}_\infty = \mathbf{k}[\![y, w]\!]/(w^2)$. $\big[\operatorname{ind}\mathrm{CM}(\mathbf{A}_\infty)\big] = \big\{(1), (y^m, w), (w) \mid m \in \mathbb{N}\big\}$.

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Goal: Formulation of the classification problem in $\operatorname{Tri}(\check{\mathbf{P}})$ as a matrix problem.

Step 2: Conclusion on the Category of Triples

Corollary

There is a bijection of isomorphism classes

$$\begin{array}{rcl} \left[\operatorname{ind}\mathrm{CM}(\check{\mathbf{P}})\right] & \xleftrightarrow{1:1} & \left[\operatorname{ind}\mathrm{Tri}(\check{\mathbf{P}})\right] \\ M & \longleftrightarrow & V = \mathbf{k}^n, \\ & & L = L_1 \times L_2 \in \mathrm{CM}(\mathbf{A}_\infty \times \mathbf{A}_\infty), \\ & & \vartheta = \vartheta_1 \times \vartheta_2 \in \mathrm{Mat}_{m_1 \times n}(\mathbf{k}) \times \mathrm{Mat}_{m_2 \times n}(\mathbf{k}) \text{ s.t.} \\ & & \vartheta \text{ surjective and } \begin{pmatrix} \vartheta_1 \\ \vartheta_2 \end{pmatrix} \text{ is injective.} \end{array}$$

Proposition (Buchweitz, Greuel and Schreyer, 1987)

Let $\mathbf{A}_\infty = \mathbf{k}[\![y, w]\!]/(w^2)$. $\left[\operatorname{ind}\mathrm{CM}(\mathbf{A}_\infty)\right] = \left\{(1), (y^m, w), (w) \mid m \in \mathbb{N}\right\}$.

Goal: Formulation of the classification problem in $\mathrm{Tri}(\check{\mathbf{P}})$ as a matrix problem.

Step 3: Reduction to Matrix Problem

Let $\quad V \in \operatorname{mod}(\mathbf{k}), \quad L \in \mathrm{CM}(\mathbf{A}_\infty \times \mathbf{A}_\infty), \quad \vartheta : V \times V \twoheadrightarrow \operatorname{top} L\,.$

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$$\begin{array}{ccc} V \times V & \xrightarrow{\vartheta} & \operatorname{top} L \\ {\scriptstyle \phi\times\phi}\downarrow\wr & & {\scriptstyle \operatorname{top}\xi}\downarrow\wr \\ V \times V & \xrightarrow{\vartheta'} & \operatorname{top} L \end{array}$$

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Step 3: A typical part of the Matrix Problem

	ϑ_1					ϑ_2				
x^m	$*$	$*$	$*$	$*$		$*$	$*$	$*$	$*$	y^n
x^m	$*$	$*$	$*$	$*$		$*$	$*$	$*$	$*$	y^n
v	$*$	$*$	$*$	$*$		$*$	$*$	$*$	$*$	w
v	$*$	$*$	$*$	$*$		$*$	$*$	$*$	$*$	w

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	ϑ_1				ϑ_2				
x^m	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	y^n
x^m	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	y^n
v	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	w
v	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	w

Admissible transformations:

1. add a multiple of any row of x^m (resp. y^n) to any row of v (resp. w),

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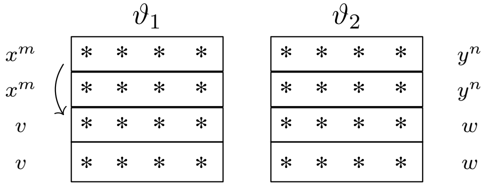

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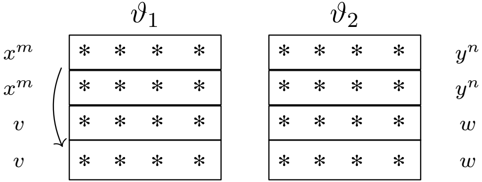

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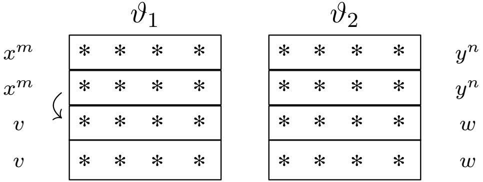

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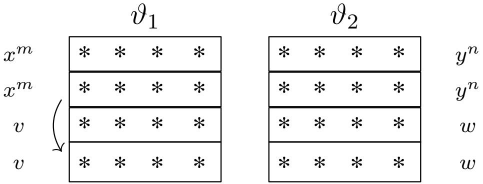

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	ϑ_1				ϑ_2				
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x^m	*	*	*	*	*	*	*	*	y^n
v	*	*	*	*	*	*	*	*	w
v	*	*	*	*	*	*	*	*	w

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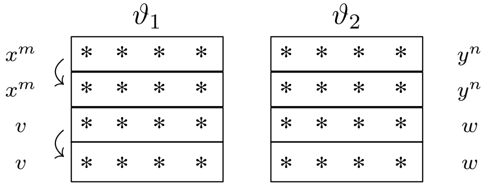

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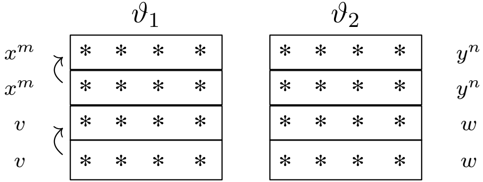

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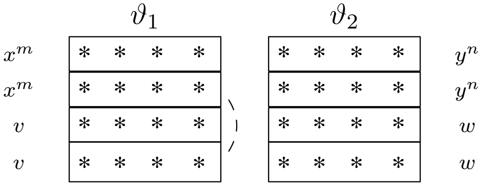

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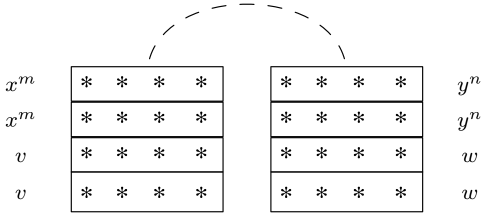

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Matrix problem : find canonical forms of ϑ_1 and ϑ_2 using only admissible transformations.

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$$
\begin{array}{c|cccc|c|cccc|c}
 & & \vartheta_1 & & & & & \vartheta_2 & & & \\
x^m & * & * & * & * & & * & * & * & * & y^n \\
x^m & * & * & * & * & & * & * & * & * & y^n \\
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Step 4: Solution of the Matrix Problem

The classification problem of $\mathrm{Tri}(\check{\mathbf{P}})$ turns out to be equivalent to a matrix problem of type "bunches of chains".

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The matrix problem over any bunch of (semi-)chains $\mathfrak{B}$ is tame.

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Summary: **Path of Reductions**

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$\mathrm{CM}(\mathbf{P})$

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Example: From a Band to a CM module

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Let $M \in \mathrm{CM}(\mathbf{P})$. M is *locally free on the punctured spectrum* (loc. free) $\iff M_{\mathfrak{q}}$ is a free module over $\mathbf{P}_{\mathfrak{q}}$ for any prime ideal $\mathfrak{q} \in \mathrm{Spec}(\mathbf{P}) \backslash \{\mathfrak{m}\}$.

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Matrix Factorizations of $a^2\,b^2$

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Thank you for listening!