Finding PIM's for finite groups of Lie type

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March 2013

Decomposition matrices

Representations of finite groups of Lie type $\mathrm{GL}_n(q)$, $\mathrm{Sp}_{2n}(q)$, ..., $E_8(q)$

Main goal. Extend geometric methods introduced by Deligne and Lusztig to the modular setting (representations in positive characteristic)

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- given χ an irreducible character of $G(q)$ (in char. 0), find the composition factors of any reduction of χ in positive characteristic
- given a projective indecomposable module PIM (in positive characteristic), compute the character of this module (in char. 0)

Inductive approach

M representation

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M is cuspidal → ← M is non-cuspidal

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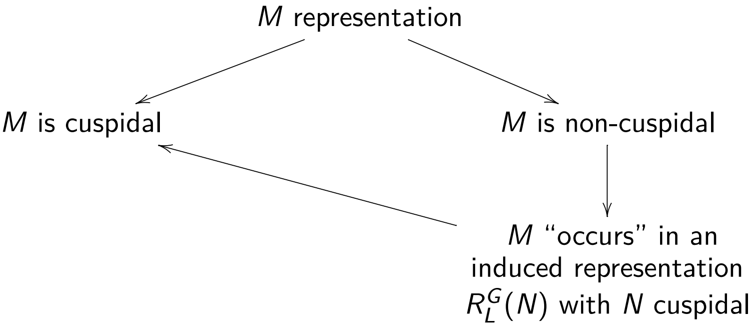
M "occurs" in an induced representation $R_L^G(N)$ with N cuspidal

Inductive approach



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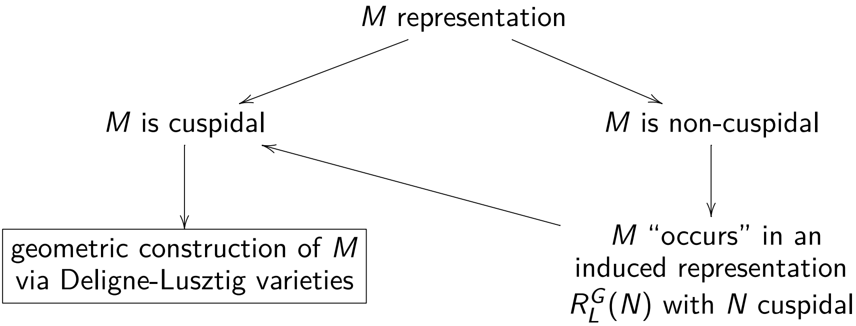

Parabolic induction

$\mathbf{G}$ reductive algebraic group over $\overline{\mathbb{F}}_p$

$F : \mathbf{G} \longrightarrow \mathbf{G}$ Frobenius endomorphism / $\mathbb{F}_q$

$\mathbf{G}^F = G(q)$ is a finite reductive group

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Parabolic induction and restriction functors, given $\mathbf{L}$ a standard F-stable Levi subgroup

$$\begin{aligned} R_L^G &: kL(q)\text{-mod} \longrightarrow kG(q)\text{-mod} \\ {}^*R_L^G &: kG(q)\text{-mod} \longrightarrow kL(q)\text{-mod} \end{aligned}$$

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Properties of induction/restriction

(i) $(R_L^G, {}^*R_L^G)$ pair of adjoint functors

(ii) They are exact if char $k \neq p$, in particular they map projective modules to projective modules

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Definition

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Consequence. it is enough to

- know the projective cover of cuspidal simple modules
- know how to decompose $R_L^G(P_N)$ (Howlett-Lehrer, Dipper-Du-James, Geck-Hiss. . .)

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$$\mathrm{H}^i_c(\mathbf{X}(w), \overline{\mathbb{Q}}_\ell) \text{ and } \mathrm{H}^i_c(\mathbf{X}(w), \overline{\mathbb{F}}_\ell)$$

give f.d. representations of $G(q)$ over $\overline{\mathbb{Q}}_\ell$ or $\overline{\mathbb{F}}_\ell$ (non-zero when $i \in \{\ell(w), \ldots, 2\ell(w)\}$ only)

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Example. Drinfeld curve $\mathbf{X} = \{(x,y) \in \overline{\mathbb{F}}_p^2 \,|\, xy^q - yx^q = 1\}$ then $\mathrm{H}_c^1(\mathbf{X})$ contains the discrete series of $\mathrm{SL}_2(q)$ (cuspidal representations)

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Problem. How to know where the representations appear?

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Let ρ be an ordinary character of $G(q)$. If w is minimal such that ρ occurs in the cohomology of $\mathbf{X}(w)$, then ρ occurs in middle degree only

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Therefore $\mathrm{H}_c^i(\mathbf{X}(w))_\rho = 0$ for $i \neq \ell(w)$

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Same result if working in the good framework

Replace individual cohomology groups by a complex $R\Gamma_c(\mathbf{X}(w), \overline{\mathbb{F}}_\ell)$

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The terms can be assumed to be projective modules and the character $[\mathrm{R}\Gamma_c(\mathbf{X}(w), \overline{\mathbb{F}}_\ell)] = \sum(-1)^i[\mathrm{H}^i_c(\mathbf{X}(w), \overline{\mathbb{F}}_\ell)]$ is a virtual projective character

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Proposition (Bonnafé-Rouquier)

Let M be a simple module and w be minimal such that

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Then there exists a representative of $\mathrm{R}\Gamma_c(\mathbf{X}(w), \overline{\mathbb{F}}_\ell)$

$$0 \longrightarrow Q_{\ell(w)} \longrightarrow Q_{\ell(w)+1} \longrightarrow \cdots \longrightarrow Q_{2\ell(w)} \longrightarrow 0$$

such that each Q_i is a finitely generated projective module and P_M is a direct summand of Q_i for $i = \ell(w)$ only

Application to decomposition matrices

$\mathbf{G} = \mathrm{Sp}_4(q)$ and $2 \neq \ell | q + 1$

$W = \langle s, t \rangle$ Weyl group of type B_2

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χ	1	1	·	·	·
χ'	1	·	1	·	·
θ_{10}	·	·	·	1	·
St	1	1	1	α	1

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and $2 \leq \alpha \leq (q-1)/2$
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$\alpha \leq 2$ therefore $\boxed{\alpha = 2}$