Representations of the partition algebra

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Introduction

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Main idea: Use the partition algebra to study the representation theory of the symmetric group.

Structure of the talk

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4. Modular representation theory of $P_r(n)$.

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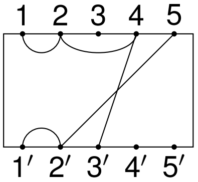
$$\{\{1, 2, 4, 3'\}, \{3\}, \{5, 1', 2'\}, \{4'\}, \{5'\}\} \quad \leftrightarrow$$



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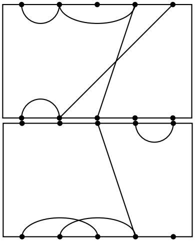
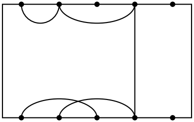


$XY = n$



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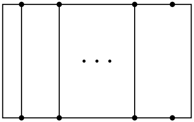
$e = \frac{1}{n}$ 

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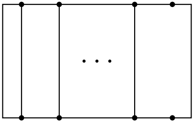
$e = \frac{1}{n}$  $\qquad e^2 = e.$

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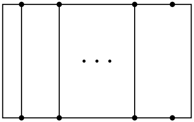
$e = \frac{1}{n}$  $\qquad e^2 = e.$

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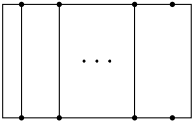

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Thus if $\operatorname{char} k = p \geq 0$ we have that the simple P_r-modules are indexed by p-regular partitions of degree $\leq r$.

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$\Lambda_{\leq r} = \{\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots),\ \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq 0,\ \sum_i \lambda_i \leq r\}$.

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A complete set of non-isomorphic simple P_r-modules is given by

$$\{L_r(\lambda) := \mathrm{hd}\, \Delta_r(\lambda), \quad \lambda \in \Lambda_{\leq r} \; p\text{-regular}\}.$$

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Example: $((2,1),(4,1))$ form a 6-pair (with $6 - |\mu| = 3$).

$$\begin{array}{|c|c|}\hline 0 & 1 \\ \hline -1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \subset \quad \begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline -1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

$P_r(n)$-blocks = maximal chains of n-pairs in $\Lambda_{\leq r}$.

$$\lambda^{(0)} \hookrightarrow_n \lambda^{(1)} \hookrightarrow_n \lambda^{(2)} \hookrightarrow_n \ldots \hookrightarrow_n \lambda^{(t)}$$

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Each cell module $\Delta_r(\lambda^{(i)})$ $(0 \leq i \leq t-1)$ has Loewy structure

$$\begin{matrix} L_r(\lambda^{(i)}) \\ L_r(\lambda^{(i+1)}) \end{matrix} \; .$$

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In the Grothendieck group we have

$$[L_r(\lambda^{(i)})] = \sum_{j=i}^{t} (-1)^{j-i} [\Delta_r(\lambda^{(j)})].$$

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Combinatorial description of $g^{\nu}_{\lambda,\mu}$?

Back to Schur-Weyl duality: As a $(\mathfrak{S}_n, P_r(n))$-bimodule we have

$$V_n^{\otimes r} = \sum S(\lambda) \otimes L_r(\lambda_{>1})$$

where the sum is over all $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ partitions of n with $\lambda_{>1} = (\lambda_2, \lambda_3, \ldots) \in \Lambda_{\leq r}$.

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Theorem: Let $\lambda, \mu, \nu \vdash n$ with $\lambda_{>1} \vdash r$ and $\mu_{>1} \vdash s$ then we have

$$g^{\nu}_{\lambda,\mu} = \begin{cases} [L_{r+s}(\nu_{>1})\downarrow_{P_r \otimes P_s} : L_r(\lambda_{>1}) \otimes L_s(\mu_{>1})] & \text{if } \nu_{>1} \in \Lambda_{\leq r+s} \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \sum_{i=0}^{t} (-1)^i [\Delta_{r+s}(\eta^{(i)})\downarrow_{P_r \otimes P_s} : L_r(\lambda_{>1}) \otimes L_s(\mu_{>1})] & \text{if } \nu_{>1} \in \Lambda_{\leq r+s} \\ 0 & \text{otherwise} \end{cases}$$

where $\nu_{>1} = \eta^{(0)} \hookrightarrow_n \eta^{(1)} \hookrightarrow_n \eta^{(2)} \hookrightarrow_n \ldots \hookrightarrow_n \eta^{(t)}$ is the $P_r(n)$-block containing $\nu_{>1}$.

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Example

$$S(1^2) \otimes S(1^2) = S(2)$$
$$S(2,1) \otimes S(2,1) = S(3) \oplus S(2,1) \oplus S(1^3)$$
$$S(3,1) \otimes S(3,1) = S(4) \oplus S(3,1) \oplus S(2,1^2) \oplus S(2^2)$$

Then for all $n \geq 4$ we have

$$S(n-1,1) \otimes S(n-1,1) = S(n) \oplus S(n-1,1) \oplus S(n-2,1^2) \oplus S(n-2,2).$$

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Note: All proofs are very elementary.

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Define a shifted action of W on $\mathbb{R}^{r+1}$ by

$$w \cdot_n x = w(x + \rho_n) - \rho_n,$$

where $\rho_n = (n, -1, -2, \ldots, -r)$.

Theorem: Assume $\mathrm{char}\, k = 0$. Let $\lambda, \mu \in \Lambda_{\leq r}$. Then λ and μ are in the same $P_r(n)$-block if and only if $\hat{\mu} \in W \cdot_n \hat{\lambda}$.

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$\hat{\mu} = (-3, 2, 1, 0, 0, 0),\ \hat{\lambda} = (-5, 4, 1, 0, 0, 0),\ \rho_6 = (6, -1, -2, -3, -4, -5)$

$$\hat{\mu} + \rho_6 = (3, 1, -1, -3, -4, -5),\ \hat{\lambda} + \rho_6 = (1, 3, -1, -3, -4, -5)$$

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$\hat{\mu} = (-3,2,1,0,0,0),\ \hat{\lambda} = (-5,4,1,0,0,0),\ \rho_6 = (6,-1,-2,-3,-4,-5)$

$$\hat{\mu} + \rho_6 = (3,1,-1,-3,-4,-5),\ \hat{\lambda} + \rho_6 = (1,3,-1,-3,-4,-5)$$

So we get

$$\hat{\mu} = s_{1,2}(\hat{\lambda} + \rho_6) - \rho_6.$$

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$$s_{i,j,mp}(\ldots x_i \ldots x_j \ldots) = (\ldots x_j + mp \ldots x_i - mp \ldots).$$

Theorem: Assume $\mathrm{char}\, k = p > 0$. Let $\lambda, \mu \in \Lambda_{\leq r}$. Then λ and μ are in the same $P_r(n)$-block if and only if $\mu \in W_p \cdot_n \lambda$.

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THANK YOU