Twisted Category Algebras and Quasi-Heredity

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Overview

1. Quasi-hereditary algebras
2. Twisted category algebras
3. Simple $k_\alpha\mathcal{C}$-modules
4. Partial orders
5. Results
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Quasi-hereditary algebras

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A is called **quasi-hereditary** w.r.t $(\Lambda, \leqslant)$ if every P_λ has a filtration

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- prominent example of a quasi-hereditary algebra: Schur algebra of the symmetric group, which relates representations of $\mathfrak{S}_n$ to representations of GL_n

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- algebras related to biset functors and double Burnside rings

Aims

- Define a partial order on an indexing set of the isoclasses of simple $k_\alpha\mathcal{C}$-modules such that $k_\alpha\mathcal{C}$ is quasi-hereditary w.r.t. this partial order, whenever $\mathrm{char}(k) = 0$.
- Characterize the corresponding standard modules, and get information about their composition factors.
- Apply this to the previous examples.

This is joint work with Robert Boltje.

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Theorem (Ganyushkin–Mazorchuk–Steinberg 2009, Linckelmann–Stolorz 2011)

The A-modules
$$D_{(i,r)} := \mathrm{top}(Ae_i' \otimes_{e_i'Ae_i'} T_{(i,r)}) \quad (i = 1, \ldots, n,\ r = 1, \ldots, l_i)$$
are representatives of the isoclasses of simple A-modules.

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   • $(i,r) \sqsubset (j,s) :\Leftrightarrow (i,r) < (j,s)$ and the $k_\alpha\Gamma_{e_i}$-module $T_{(i,r)}$ is related to the $k_\alpha\Gamma_{e_j}$-module $T_{(j,s)}$ via a particular bimodule.

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Remark

The relation $\sqsubseteq$ is in general not transitive!

Results

Theorem (Boltje–D. 2012/13)

Let $\mathrm{char}(k) = 0$.

(a) *The* k*-algebra* $A = k_\alpha \mathcal{C}$ *is quasi-hereditary w.r.t.* $(\Lambda, \leqslant)$*, and the standard module corresponding to* (i, r) *is* $\Delta_{(i,r)} := Ae'_i \otimes_{e'_i A e'_i} T_{(i,r)}$.

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- The partial order $\leqslant$ is a proper refinement of $\trianglelefteq$. $\rightsquigarrow$ (b) gives new information about composition factors of the standard modules $\Delta_{(i,r)}$.

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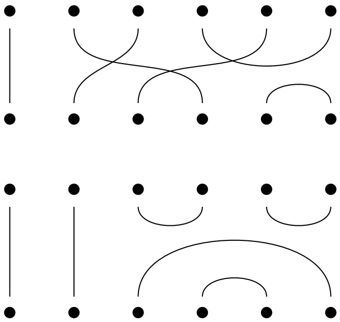

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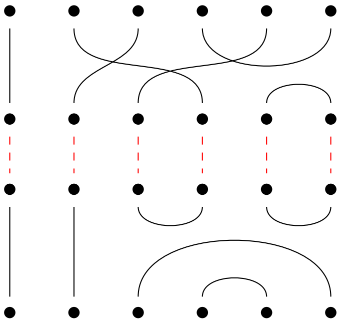

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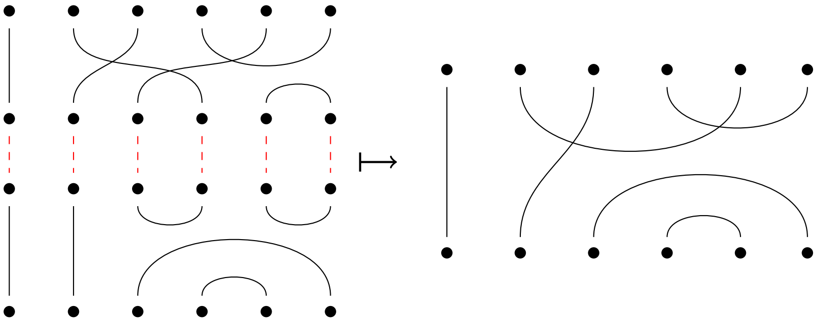

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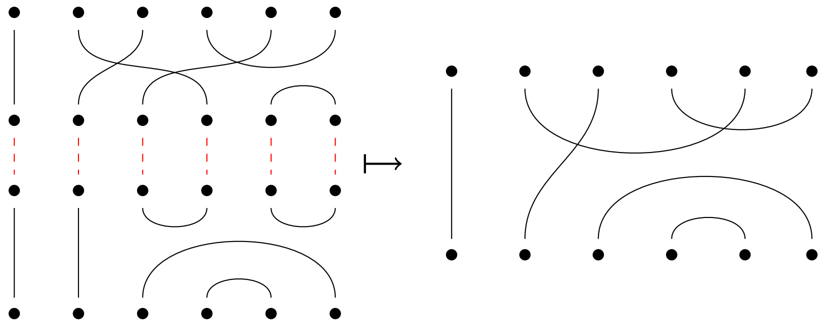

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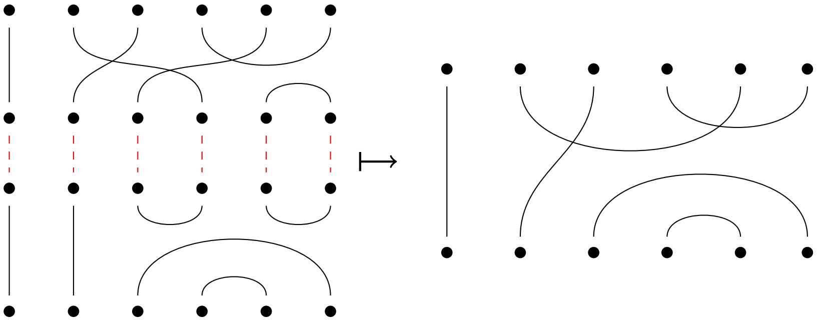

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- $\alpha : S \times S \to k^{\times}$, $(s, t) \mapsto \delta^{\text{number of cycles in concat. of } s \text{ and } t}$
- $-^{\circ} : S \to S$ corresponds to 'flipping diagrams'

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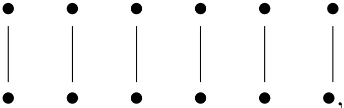,

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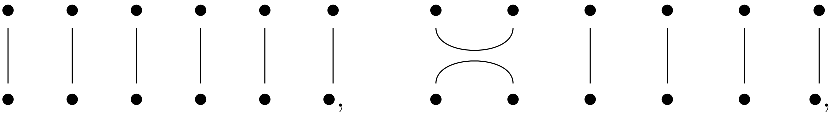

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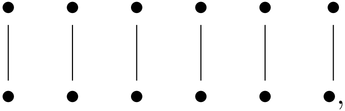

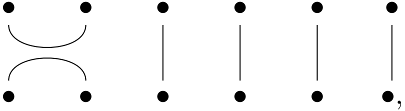

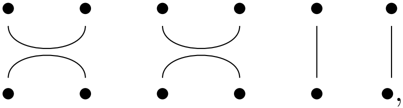

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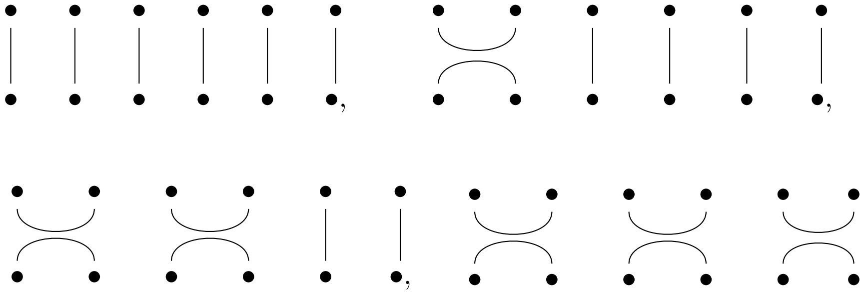

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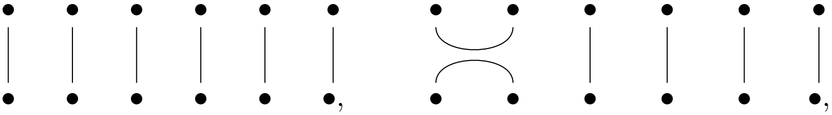

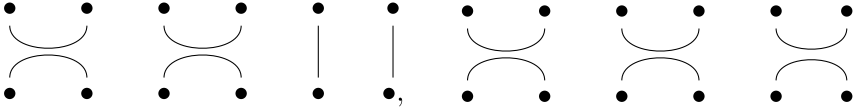

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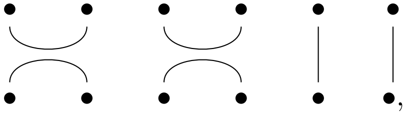

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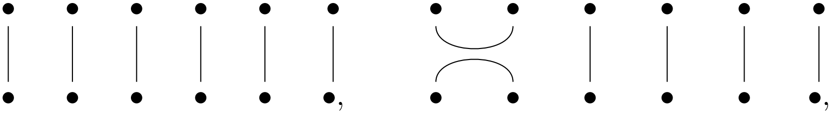

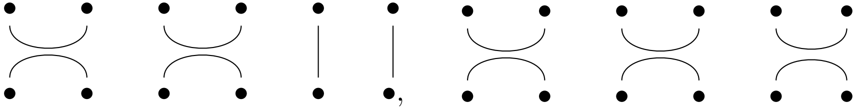

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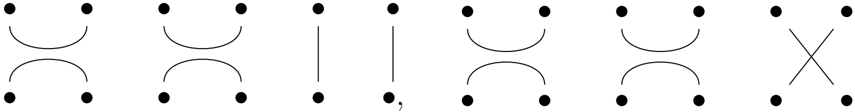

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$$(i,r) \vartriangleleft (j,s) \Leftrightarrow k_i < k_j \text{ and } [\mathrm{Ind}_{\mathfrak{S}_{n-2k_j} \times \mathfrak{S}_2 \times \cdots \times \mathfrak{S}_2}^{\mathfrak{S}_{n-2k_i}} (T_{(j,s)} \otimes k \otimes \cdots \otimes k) : T_{(i,r)}] \neq 0$$

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  - $(G, S) \sqsubset (H, T) \Leftrightarrow (G, S) < (H, T)$ and $T \otimes S^*$ is a composition factor of a certain permutation $\mathbb{Q}[\mathrm{Aut}(H) \times \mathrm{Aut}(G)]$-module.