Matrix problems, vector bundles on curves of genus one and the classical Yang–Baxter equation

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Notation:

$$r^{13} : \mathbb{C} \xrightarrow{r} \mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\rho_{13}} U \otimes U \otimes U$$

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Moreover, all elliptic and trigonometric solutions have been classified.

Weierstraß family of cubic curves

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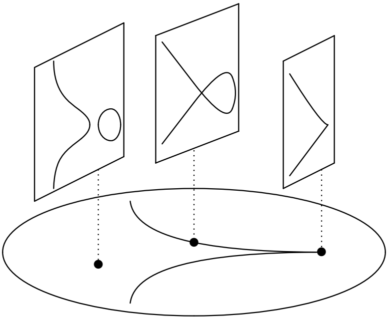
Let $E = V(y^2 - 4x^3 - g_2x - g_3)$, where $g_2, g_3 \in \mathbb{C}$.



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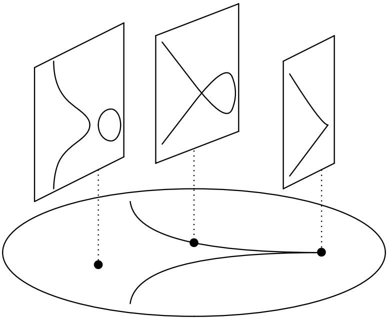

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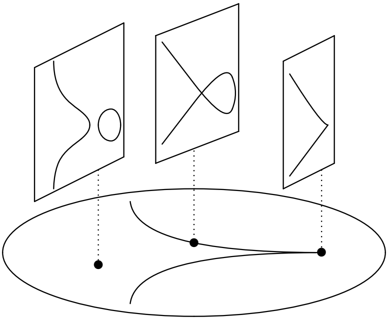

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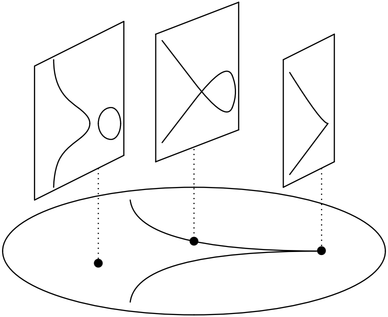

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- If $(g_2, g_3) = (0, 0)$ then E is cuspidal.

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Theorem (Bodnarchuk, Burban, Drozd, Greuel)

The last result is also true for the degenerate elliptic curves.

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Fix a Weierstraß curve E and $(n, d) \in \mathbb{N} \times \mathbb{Z} : \gcd(n, d) = 1$.

For $\mathcal{P} \in \mathsf{M}_E^{(n,d)}$ consider the sheaf of Lie algebras $\mathcal{A} = \mathit{Ad}(\mathcal{P})$:

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Let $x \neq y \in E_{reg}$. We have canonical linear maps:

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- Burban, Henrich (2012): degenerate elliptic curves, relative setting.

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Theorem (Polishchuk, Burban–Henrich)

The following diagram of vector spaces

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  which also implies unitarity: $\mathsf{m}^{12}_{x,y} = -\mathsf{m}^{21}_{y,x}$.

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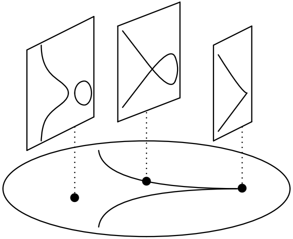

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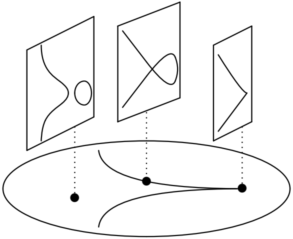

Then we get a family of solutions

$$r_{((g_2,g_3),(n,d))}(x_1, x_2) \in \mathfrak{sl}_n(\mathbb{C}) \otimes \mathfrak{sl}_n(\mathbb{C})$$

of the classical Yang–Baxter equation.

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1. E smooth: elliptic solution of Baxter

$$r(z) = \frac{\mathrm{cn}(z)}{\mathrm{sn}(z)} h \otimes h + \frac{1 + \mathrm{dn}(z)}{\mathrm{sn}(z)} (e \otimes f + f \otimes e) + (e \otimes e + f \otimes f) \frac{1 - \mathrm{dn}(z)}{\mathrm{sn}(z)}.$$

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3. E cuspidal: rational solution of Stolin

$$r(z) = \frac{1}{z}\left(\frac{1}{2} h \otimes h + e \otimes f + f \otimes e\right) + z(f \otimes h + h \otimes f) - z^3 f \otimes f.$$

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Proposition

The sheaf of Lie algebras $\mathcal{A} = \mathcal{A}_{n,d}$ on the torus $E = \mathbb{C}/\langle 1, \tau \rangle$ has the following description:

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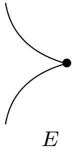

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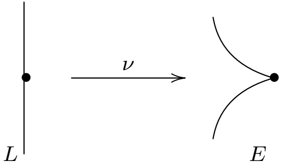

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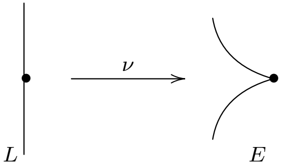

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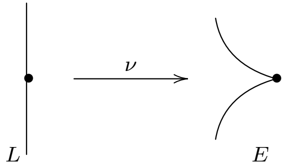

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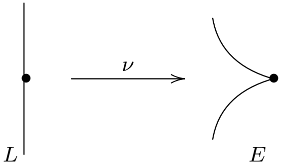

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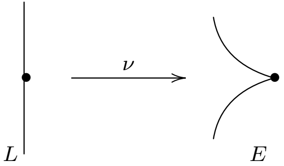

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Definition

For a cuspidal cubic curve E consider the category

$$\mathsf{S}(E) = \Big\{\mathcal{F} \in \mathsf{VB}(E) \,\Big|\, \nu^*\mathcal{F} \in \mathsf{add}\big(\mathcal{O}_L \oplus \mathcal{O}_L(1)\big)\Big\}.$$

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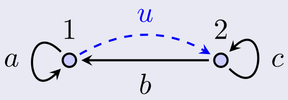

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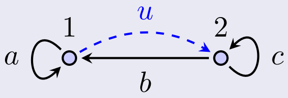

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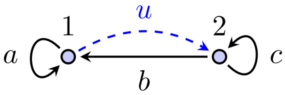
Representations of *differential biquivers* alias *bocses* (bimodule over category with coalgebra structure) resp. *bimodule problems* have been introduced in 70-th by Drozd, Kleiner, Ovsienko and Roiter.

Representations of bocses

$(\vec{Q}, \partial) =$ 

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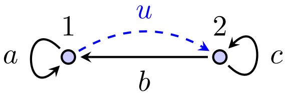
$(\vec{Q}, \partial) =$ 

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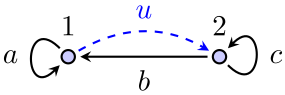
$(\vec{Q}, \partial) =$ 

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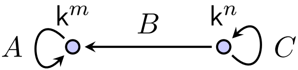

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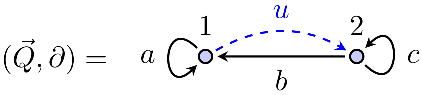

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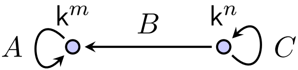

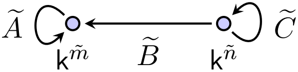

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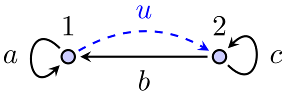

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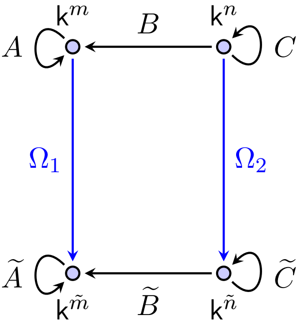

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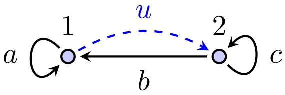

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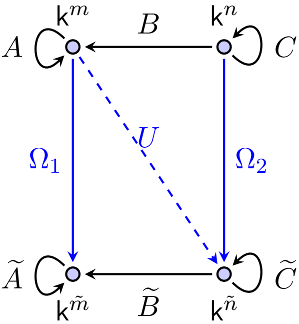

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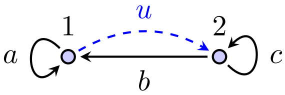
$(\vec{Q}, \partial) =$ 

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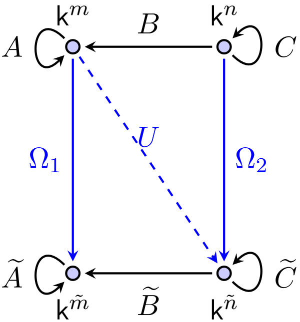
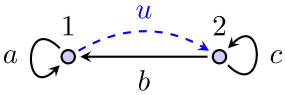


$(\vec{Q}, \partial) =$ 

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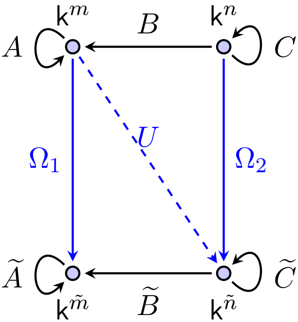

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$$J_{(e,d)} = \left(\begin{array}{c|c} J_1 & J_2 \\ \hline \times & J_3 \end{array}\right) \longrightarrow \begin{cases} J_{(e,e+d)} = \left(\begin{array}{c|cc} 0 & I & 0 \\ \hline \times & J_1 & J_2 \\ \times & 0 & J_3 \end{array}\right) \\ J_{(e+d,d)} = \left(\begin{array}{cc|c} J_1 & J_2 & 0 \\ 0 & J_3 & I \\ \hline \times & \times & 0 \end{array}\right) \end{cases}$$

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Let $(e, d) = (2, 3)$. Then $J_{(2,3)}$ is the following matrix

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$$\left[\begin{array}{c|c} 0 & 1 \\ \hline & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline & & 0 \end{array}\right] \to \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline & & 0 & 1 & 0 \\ & & 0 & 0 & 1 \\ & & 0 & 0 & 0 \end{array}\right]$$

Classification of simple objects

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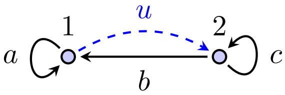

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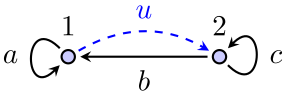
$(\vec{Q}, \partial) =$ 

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Let $\mathsf{Rep}(\vec{Q}, \partial) \ni X = \left(\begin{array}{c|c} A & B \\ \hline \times & C \end{array}\right)$ *be such that* $\mathrm{End}(X) = \mathsf{k}$.

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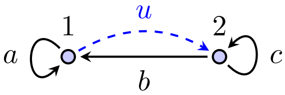
$(\vec{Q}, \partial) =$ 

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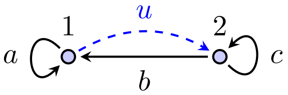
$(\vec{Q}, \partial) =$ 

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$$X \simeq \lambda I + J_{(e,d)} = \left(\begin{array}{c|c} \lambda I + J_1 & J_2 \\ \hline \times & \lambda I + J_3 \end{array}\right).$$

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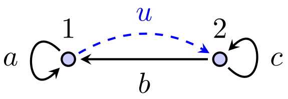
$(\vec{Q}, \partial) =$ 

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Other way around, any such object is Schurian. In terms of the equivalence of categories

$$\mathsf{Rep}(\vec{Q}, \partial) \longrightarrow \mathsf{S}(E) := \left\{ \mathcal{F} \in \mathsf{VB}(E) \,\middle|\, \nu^* \mathcal{F} \in \mathsf{add}\big(\mathcal{O}_L \oplus \mathcal{O}_L(1)\big) \right\}$$

it describes all simple vector bundles on $E = V(zy^2 - x^3)$.

Frobenius Lie algebras

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Let $\gcd(e, d) = 1$, $n = e + d$ *and*

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is non-degenerate. In other words, the Lie algebra $\mathfrak{p}$ *is Frobenius (Ooms, Elashvili).*

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  - ω *is non–degenerate on* $(\mathfrak{l} \cap \mathfrak{p}_k) \times (\mathfrak{l} \cap \mathfrak{p}_k)$.

If $H^2(\mathfrak{l}) = 0$ *then the choice of* ω *is redundant.*

Cuspidal solutions of CYBE

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Theorem (Burban, Henrich)

Let $E = V(uv^2 - w^3)$ and $0 < d < n$ be mutually prime. Then we have:

$$r_{(E,(n,d))} = r_{(\mathfrak{sl}_n(\mathbb{C}),n-d,\omega_J)}$$

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Corollary

The rational solution $r_{(\mathfrak{sl}_n(\mathbb{C}),n-d)}$ is a degeneration of Belavin's elliptic r–matrix corresponding to the root of unity $\varepsilon = \exp(\frac{2\pi i d}{n})$.

Solution $r_{(E,(n,1))}$ for E cuspidal

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$$r(x_1, x_2) = \frac{\Omega}{x_2 - x_1} +$$

$$x_1 \Big[e_{1,2} \otimes \check{h}_1 - \sum_{j=3}^{n} e_{1,j} \otimes \sum_{k=1}^{n-j+1} e_{j+k-1,k+1} \Big] - x_2 \Big[\check{h}_1 \otimes e_{1,2} - \sum_{j=3}^{n} \sum_{k=1}^{n-j+1} e_{j+k-1,k+1} \otimes e_{1,j} \Big]$$

$$+ \sum_{j=2}^{n-1} e_{1,j} \otimes \sum_{k=1}^{n-j} e_{j+k,k+1} + \sum_{i=2}^{n-1} e_{i,i+1} \otimes \check{h}_i - \sum_{j=2}^{n-1} \sum_{k=1}^{n-j} e_{j+k,k+1} \otimes e_{1,j} - \sum_{i=2}^{n-1} \check{h}_i \otimes e_{i,i+1} +$$

$$+$$

$$+ \sum_{i=2}^{n-2} \sum_{k=2}^{n-i} \sum_{l=1}^{n-i-k+1} e_{i+k+l-1,l+i} \otimes e_{i,i+k} - \sum_{i=2}^{n-2} \sum_{k=2}^{n-i} e_{i,i+k} \otimes \sum_{l=1}^{n-i-k+1} e_{i+k+l-1,l+i}.$$

Thank you for your attention!