Two weeks of silting
Silting theory in commutative algebra

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1. **First examples of silting modules.** Let R be a ring.
   (a) Prove that the zero module is a silting module.
   (b) Prove that if $R = S \times T$ as rings, then S is a silting R-module.
   (c) Prove that $\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$ is a silting $\mathbb{Z}$-module.

2. **Basic geometric examples and properties of commutative rings**[1]**.** Let $I \subseteq \mathbb{C}[x_1, \dots, x_n]$ be an ideal and $X = V(I) \subseteq \mathbb{C}^n$ the set of zeros of all the polynomials $f \in I$.
   (a) (Hilbert's Nullstellensatz, [3, §§1.7–1.10]) Show that the maximal ideals of $\mathbb{C}[x_1, \dots, x_n]/I$, the *coordinate ring* of X, bijectively correspond to the elements of X via
   $$\underline{a} = (a_1, \dots, a_n) \in X \mapsto \{f \mid f(\underline{a}) = 0\}.$$
   (b) If R is a commutative ring, the *Krull dimension* of R is defined as
   $$\dim R = \sup\{d \geq 0 \mid \exists \text{ a chain } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d \text{ in } \operatorname{Spec} R\}.$$
   Show that if $0 \neq f \in \mathbb{C}[x, y]$, the Krull dimension of $R = \mathbb{C}[x, y]/(f)$ equals one and describe the poset of prime ideals of R with respect to the inclusion (*Hint*: [3, §1.6]).
   (c) A commutative local noetherian ring $(R, \mathfrak{m})$ is called *regular* [2, §10.3] if the vector space dimension of $\mathfrak{m}/\mathfrak{m}^2$ over the residue field $R/\mathfrak{m}$ equals the Krull dimension of R.
   Show that if R is the localization of $\mathbb{C}[x_1, \dots, x_n]/I$ as above at the maximal ideal $\mathfrak{m}_{\underline{a}}$ corresponding to a point $\underline{a} \in X$, then R is regular if and only if the set X is smooth at $\underline{a}$ [2, §16.6]. Here, *smooth* means that if we fix a set of generators $f_1, \dots, f_m \in \mathbb{C}[x_1, \dots, x_n]$ for I, then the rank of the Jacobi matrix
   $$\left(\frac{\partial f_i}{\partial x_j}(\underline{a})\right)_{1 \leq i \leq m, 1 \leq j \leq n}$$
   (which is a matrix of complex numbers) equals $n - \dim R$. This in fact equivalently says that one can use the implicit function theorem to analytically express X as a function of $\dim R$ parameters around the point $\underline{a}$).
   (d) A general commutative noetherian ring is *regular* if the localization $R_\mathfrak{m}$ at all maximal ideals is regular. Show that $\mathbb{C}[x, y]/(y^2 - x(x^2 - 1))$ is regular and one-dimensional (such rings are called *Dedekind domains*), while the ring $\mathbb{C}[x, y]/(y^2 - x^2(x + 1))$ is one-dimensional, but not regular.

[1] This part should mainly serve as a source of concrete examples for the following exercises.

3. **One-dimensional commutative noetherian domains and silting classes and modules over them.**

(a) If R is a one-dimensional commutative noetherian domain and $\mathfrak{m} \subseteq R$ is a maximal ideal and $0 \neq x \in \mathfrak{m}$, then we have the following pullback diagram:

$$\begin{tikzcd} R \arrow[r, "\subseteq"] & R_{\mathfrak{m}} \\ I \arrow[u, "\subseteq"] \arrow[r, "\subseteq"'] & x \cdot R_{\mathfrak{m}} \arrow[u, "\subseteq"'] \end{tikzcd}$$

Prove that the ideal I is stalk-wise projective, and therefore projective over R. Moreover, show that the support of R/I is precisely $\mathfrak{m}$.

(b) If Q is the quotient field of R, we obtain a subring S of Q by the following construction.

$$\begin{aligned} f\colon R \overset{\subseteq}{\to} S &= \{q \in Q \mid (\exists n)(q \cdot I^n = 0)\} \\ &= \varinjlim(R \subseteq I^{-1} \subseteq I^{-2} \subseteq \cdots). \end{aligned}$$

Show that f is the flat ring epimorphism corresponding to the specialization closed subset $V = \{\mathfrak{m}\}$ of $\mathrm{Spec}(R)$.

(c) Show that the silting class corresponding to $V = \{\mathfrak{m}\}$ is

$$\mathrm{Gen}(S) = \{X \in \text{Mod-}R \mid X\mathfrak{m} = X\}$$

and the corresponding silting module is $S \oplus S/R$. The cosilting class and module are dual.

(d) Show that any silting class $\mathcal{D}$ is induced by a silting module of the form $S \oplus \mathrm{Coker}(\lambda)$, where $\lambda\colon R \to S$ is a flat ring epimorphism.

4. **Regular commutative noetherian domains and silting classes and modules over them.** Show that a construction similar to the previous exercise works in case that R is a regular domain, and $V = V(\mathfrak{p})$, where $\mathfrak{p}$ is a prime ideal of height one. Then R is locally a factorial domain (= the localizations at all prime ideals are unique factorization domains [2, Theorem 19.19]), so $\mathfrak{p}$ is locally a principal ideal, so locally projective, so a projective ideal. Then again we have that the flat ring epimorphism corresponding to $V(\mathfrak{p})$ can be constructed as

$$\begin{aligned} f\colon R \overset{\subseteq}{\to} S &= \{q \in Q \mid (\exists n)(q \cdot \mathfrak{p}^n = 0)\} \\ &= \varinjlim(R \subseteq \mathfrak{p}^{-1} \subseteq \mathfrak{p}^{-2} \subseteq \cdots), \end{aligned}$$

and $S \oplus S/R$ is a silting module corresponding to $V(\mathfrak{p})$.

5. **The Picard group of Dedekind domains**. Let R be a commutative ring.

(a) An R-module M is *invertible* if there is an R-module N such that $M \otimes_R N \simeq R$.

(b) Show that any invertible R-module is projective and finitely generated.

(c) Show that a finitely generated R-module is projective if and only if it is locally free of rank 1.

(d) Let R be a Dedekind domain. Show that invertible R-modules are, up to isomorphism, precisely the fractional ideals of R.

(e) The *Picard group* is a commutative group consisting of all invertible R-modules, together with the operation $\otimes_R$ and unit R. Show that if R is a Dedekind domain such that the Picard group contains a non-torsion element[2], then there is a universal localization of R which is not a classical localization.

6. **Tilting modules.** Let $\sigma : P_1 \to P_0$ be a two-term silting complex and let $T = \operatorname{Coker}(\sigma)$ be the associated silting module.

   (a) Let $\sigma' : P_1 \to \operatorname{Im}(\sigma)$ be the corestriction of σ onto its image. Show that

   $$\mathcal{D}_\sigma = \mathcal{D}_{\sigma'} \cap \operatorname{Ker} \operatorname{Ext}^1_R(T, -).$$

   Reminder: If $\lambda : Q_1 \to Q_0$ is a map between modules, we define the class of modules $\mathcal{D}_\lambda = \{M \mid \operatorname{Hom}_R(\lambda, M) \text{ is surjective}\}$.

   (b) Assume that T is *faithful*, that is, its annihilator $\operatorname{Ann} T = \{r \in R \mid T \cdot r = 0\}$ is zero. Then T is a silting module with respect to a monomorphic map between projectives. In particular, $\operatorname{pd}_R T \leq 1$. (*Hint*: P_1 is a submodule of some module from $\operatorname{gen}(T)$.)

   (c) Conclude that a silting module T is faithful if and only if it is a *tilting* module, that is, if and only if the condition $\operatorname{Gen}(T) = \operatorname{Ker} \operatorname{Ext}^1_R(T, -)$ holds.

   (d) Suppose that T is a tilting module. Show that any monomorphic projective presentation of T is a two-term silting complex.

7. **A warning example.** For a general $V \subseteq \operatorname{Spec} R$, which is specialization closed and corresponding to a flat epimorphisms $f : R \to S$, we *cannot* always construct the silting module as $S \oplus S/R$. The simplest such example seems to be $R = \mathbb{C}[x, y]$, $S = \mathbb{C}(x, y)$, and the natural flat epimorphism $f : R \overset{\subseteq}{\to} S$, which corresponds to the specialization set $V = \operatorname{Spec} \mathbb{C}[x, y] \setminus \{0\}$.

   (a) The projective dimension of S over R equals two [5][3].

   (b) Conclude that $S \oplus S/R$ is not a silting module.

   (c) Construct the corresponding silting module (one can use the construction of Fuchs from [4, Example 13.4]).

References

[1] L. Claborn, *Every abelian group is a class group*, Pacific J. Math. **18** (1966), 219–222.

[2] D. Eisenbud, *Commutative Algebra, with a view toward algebraic geometry*, Graduate Texts in Mathematics, 150, Springer-Verlag, New York, 1995.

[3] W. Fulton, *Algebraic curves, An introduction to algebraic geometry*, Notes written with the collaboration of Richard Weiss, Reprint of 1969 original, Advanced Book Classics, Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989. Available on-line: `http://www.math.lsa.umich.edu/~wfulton/CurveBook.pdf`

[4] R. Göbel, J. Trlifaj, *Approximations and endomorphism algebras of modules, Volume 1. Approximations*, Second revised and extended edition, De Gruyter, 2012.

[5] I. Kaplansky, *The homological dimension of a quotient field*, Nagoya Math. J., Volume 27, Part 1 (1966), 139–142.

[2] Such Dedekind domains do exist! Indeed, Claborn [1] showed that *any* abelian group can be realized as the Picard group of a Dedekind domain.

[3] This is a very non-trivial theorem!