Tilting complexes and derived equivalence

Conventions and simplifying assumptions will be as in the lectures:

- modules will be right modules unless otherwise stated,
- algebras will be over an algebraically closed field k,
- for an algebras given by quivers and admissible relations, with the vertices are labelled by a set I, then S_i, P_i and I_i denote the simple module, indecomposable projective module and indecomposable injective module associated to the vertex labelled $i \in I$ in the usual way.

1. Let A and B be derived equivalent finite dimensional algebras. Which of the following statements must be true and which may be false?

(a) $\dim_k A = \dim_k B$.
(b) A and B have the same number of isomorphism classes of simple modules.
(c) A and B have the same global dimension.
(d) If A has finite global dimension then B has finite global dimension.
(e) A and B have the same Cartan matrix.
(f) The Cartan matrices of A and B have the same determinant.

2. Let Q be the quiver

$$\begin{tikzcd} & & 3 \\ 1 \arrow[r] & 2 \arrow[ur] \arrow[dr] & \\ & & 4 \end{tikzcd}$$

and let T be the object $S_1 \oplus S_2[1] \oplus S_3[2] \oplus S_4[2]$ of the derived category $\mathcal{D}(kQ)$.

(a) Prove that T (or a projective resolution of T) is a tilting complex.
(b) Let $B = \operatorname{End}_{\mathcal{D}(kQ)}(T)$ be the endomorphism algebra of T. Calculate T in terms of a quiver with relations.
(c) Consider a derived equivalence $\mathcal{D}(kQ) \to \mathcal{D}(B)$ given by the tilting complex T (i.e., one that sends T to B). Calculate the "inverse" tilting complex (i.e., the object of $\mathcal{B}$ to which kQ is sent, which will be a tilting complex for B with endomorphism algebra isomorphic to kQ).

(d) Calculate the images of the simple kQ-modules under the derived equivalence above, and the images of the simple B-modules under its inverse.

3. Let A and B be finite dimensional algebras. For which of the following properties must a derived equivalence $\mathcal{D}(A) \simeq \mathcal{D}(B)$ restrict to an equivalence of triangulated categories between the full subcategories consisting of the objects with the given property?

(a) Objects that are isomorphic to a bounded complex of finitely generated injectives.

(b) Objects that are isomorphic to a bounded complex of finitely generated projective-injective modules.

(c) Objects that are isomorphic to a bounded above complex of finitely generated projectives with bounded dimensions.

(d) (Trick question.) Objects that are isomorphic to a bounded below complex of finitely generated projectives.

4. Let A and B be derived equivalent finite dimensional algebras. Show that the opposite algebras A^{op} and B^{op} are derived equivalent.

5. Let ${}_AX_B$ be a two-sided tilting complex inducing a derived equivalence between the derived categories $\mathcal{D}(A)$ and $\mathcal{D}(B)$ of two finite dimensional algebras, and let ${}_BY_A$ be the two-sided tilting complex inducing the inverse equivalence.

Let $A^e = A^{op} \otimes_k A$ be the enveloping algebra of A, so an A^e-module is an A-bimodule.

(a) In terms of ${}_AX_B$ and ${}_BY_A$, describe a derived equivalence

$$\mathcal{D}(A^e) \simeq \mathcal{D}(B^e).$$

(b) Calculate the image of ${}_AA_A$ under this derived equivalence.

(c) Deduce that the centres $Z(A)$ and $Z(B)$ of A and B are isomorphic as algebras.