# Silting: mutation, completion and reduction

*Warm up.* True or false?

(a) Any two basic silting objects have the same number of indecomposable summands.

(b) A t-structure can also be a co-t-structure.

(c) $\mathsf{D}^b(k[x]/(x^2))$ has a bounded co-t-structure.

(d) The 2-cluster category $\mathsf{C}_2(\mathbf{k}A_n)$ has no nontrivial co-t-structures.

(e) The 2-cluster category $\mathsf{C}_2(\mathbf{k}A_n)$ has nontrivial t-structures.

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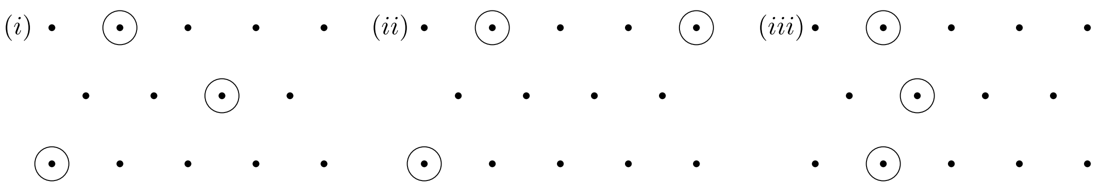
**Exercise 1.** (a) Decide whether the following objects in $\mathsf{D}^b(\mathbf{k}A_3)$ are silting objects. Note that the arrows in the AR quiver have been omitted. (Where the corresponding candidate silting object is the direct sum of the circled objects. Feel free to label these objects as objects in $\mathsf{D}^b(\mathbf{k}A_3)$, choosing some orientation on the $A_3$ quiver.)




(b) Decide whether the following objects in $\mathsf{D}^b(\mathbf{k}\widetilde{A_2})$ are silting objects, where $\widetilde{A_2}$ is the quiver

$$\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[rr] & & 3 \end{tikzcd} .$$

$$(i) \quad m = P_2 \oplus S_2 \oplus P_1 \quad \text{and} \quad (ii) \quad n = P_2 \oplus S_2 \oplus \tau P_1,$$

where $\tau$ denotes the Auslander–Reiten translate.

(c) For the silting objects in part (a), calculate the corresponding algebraic t-structures and their hearts. Identify the simple objects in the hearts. What do you notice about the simple objects and the summands of the silting object?

(d) Right mutate some of the silting subcategories $\mathsf{M}$ with respect to some $\mathsf{M}' \subset \mathsf{M}$ from (a).

(e) (Extra exercise) Calculate the corresponding algebraic t-structures $(\mathsf{X}_{\mathsf{R}_{\mathsf{M}'}(\mathsf{M})}, \mathsf{Y}_{\mathsf{R}_{\mathsf{M}'}(\mathsf{M})})$. Find an (abelian) torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathsf{H}_\mathsf{M}$ such that $\mathsf{X}_{\mathsf{R}_{\mathsf{M}'}(\mathsf{M})} = \mathsf{X}_\mathsf{M} * \Sigma^{-1}\mathcal{T}$ and $\mathsf{Y}_{\mathsf{R}_{\mathsf{M}'}(\mathsf{M})} = \Sigma^{-1}(\mathcal{F} * \mathsf{Y}_\mathsf{M})$. (Hint: think about the extension closure of the simples and the relationship observed in (c).)

**Exercise 2.** Let $(\mathsf{X}, \mathsf{Y})$ be a co-t-structure in $\mathsf{D}$ with coheart $\mathsf{M} = \mathsf{X} \cap \Sigma^{-1}\mathsf{Y}$. Show that the following conditions are equivalent.

(1) $(\mathsf{X}, \mathsf{Y})$ is bounded.

(2) For each object $d$ in $\mathsf{D}$ there exist integers $i < j$ such that $d \in \Sigma^j\mathsf{X} \cap \Sigma^i\mathsf{Y}$.

(3) For each object $0 \neq d \in \mathsf{D}$, there exists a Postnikov tower

$$\begin{tikzcd}
0 = d_0 \arrow[r] & d_1 \arrow[r] \arrow[d] & d_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & d_{n-1} \arrow[r] & d_n = d \arrow[d] \,, \\
& \Sigma^{i_1} m_1 \arrow[ul, squiggly] & \Sigma^{i_2} m_2 \arrow[ul, squiggly] & & & \Sigma^{i_n} m_n \arrow[ul, squiggly]
\end{tikzcd}$$

with $m_j \in \mathsf{M}$ and $i_1 < i_2 < \cdots < i_n$.

**Exercise 3.** Let $(\mathsf{M}, \mathsf{M}')$ be a silting pair in $\mathsf{D}$.

(a) Prove that $\mathsf{R}_{\mathsf{M}'}(\mathsf{M}) = \mathsf{add}(\mathsf{M}' \cup \{x_m \mid m \in \mathsf{M}\})$, where for $m \in \mathsf{M}$ is given by taking the triangle $\Sigma^{-1} m \longrightarrow x_m \longrightarrow m'_m \xrightarrow{f} m$ in which $f$ is a right $\mathsf{M}'$-approximation of $m$.

(b) Verify that $\mathsf{R}_{\mathsf{M}'}(\mathsf{M})$ is also a silting subcategory of $\mathsf{D}$.

(c) Is the (right) silting mutation of a tilting subcategory again a tilting subcategory? If not, think of a homological condition which is both necessary and sufficient for the mutation to preserve the tilting property.

**Exercise 4.** Consider the $A_5$ quiver with the following orientation $1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \leftarrow 5$ and the presilting object $P = P_3 \oplus \Sigma S_3$. Write $\mathsf{P} = \mathsf{add}(P)$.

(a) Calculate the co-t-structures $({}^{\perp}(\Sigma^{>0}\mathsf{P}), \mathsf{susp}(\Sigma\mathsf{P}))$ and $(\mathsf{cosusp}(\Sigma^{-1}\mathsf{P}), (\Sigma^{<0}\mathsf{P})^{\perp})$ and their cohearts in $\mathsf{D}^b(\mathbf{k}A_5)$.

(b) Considering $P$ as an object in the 3-cluster category $\mathsf{C}_3(\mathbf{k}A_5)$, write down the AR quiver of the Iyama-Yoshino subfactor category $\mathsf{Z}/[\mathsf{P}]$, where $\mathsf{Z} = \bigcap_{i=1}^{2} (\Sigma^{-i}\mathsf{P})^{\perp}$

**Exercise 5.** Determine the $w \in \mathbb{Z}$ such that $\mathsf{C}_2(\mathbf{k}A_3)$ is $w$-Calabi-Yau (on objects). (Note: to do this properly, we should also check this on morphisms.)

**Exercise 6.** Suppose $\mathsf{C} \subset \mathsf{D}$ is a $w$-cluster-tilting subcategory. Show that

$$\mathsf{D} = \mathsf{C} * \Sigma\mathsf{C} * \cdots * \Sigma^{w-1}\mathsf{C}.$$

(Hint: Let $\mathsf{X}_n = \mathsf{C} * \Sigma\mathsf{C} * \cdots * \Sigma^{n}\mathsf{C}$ for $0 \leq n < w-1$, and $\mathsf{Y}_n = \bigcap_{i=0}^{n} (\Sigma^{i}\mathsf{C})^{\perp} = \mathsf{X}_n^{\perp}$. Show by induction on $n$ that $(\mathsf{X}_n, \mathsf{Y}_n)$ is a torsion pair in $\mathsf{D}$.)

**Exercise 7.** Let $\mathsf{R}$ be a rigid subcategory of $\mathsf{D}$ and suppose $(\mathsf{X}, \mathsf{Y})$ is an $\mathsf{R}$-mutation pair.

(a) Show that for each $y \in (\mathsf{R} * \Sigma\mathsf{X}) \cap {}^{\perp}(\Sigma\mathsf{R})$ there exists a triangle $x \xrightarrow{f} r \xrightarrow{g} y \longrightarrow \Sigma x$ such that $f$ is a left $\mathsf{R}$-approximation.

(b) Using part (a) and $\mathsf{X} \subset (\Sigma^{-1}\mathsf{Y} * \mathsf{R}) \cap (\Sigma^{-1}\mathsf{R})^{\perp}$, show that $(\mathsf{R} * \Sigma\mathsf{X}) \cap {}^{\perp}(\Sigma\mathsf{R}) \subset \mathsf{Y}$.

(Hint: find another left $\mathsf{R}$-approximation of $x$, and use the basic properties of approximations to say something about $y$ in part (a).)

**Exercise 8.** Let $\mathsf{R}$ be a rigid $\mathbb{S}_w$-subcategory of $\mathsf{D}$ and set $\mathsf{Z} = \bigcap_{i=1}^{w-1} {}^{\perp}(\Sigma^{i}\mathsf{R})$.

(a) Show that $\mathsf{Z} = \bigcap_{i=1}^{w-1} (\Sigma^{-i}\mathsf{R})^{\perp}$.

(b) Show that $(\mathsf{Z}, \mathsf{Z})$ is an $\mathsf{R}$-mutation pair.

**Exercise 9.** Consider the $A_3$ quiver with the orientation $1 \to 2 \to 3$. Using Calabi-Yau reduction classify all 2-cluster-tilting objects in $\mathsf{C}_2(\mathbf{k}A_3)$ containing the simple module $S_2$ as a summand.

**Exercise 10.** This exercise will take you through the proof of the following theorem in the lectures.

**Theorem** (Aihara-Iyama)**.** *Let* $\mathsf{P} \subset \mathsf{D}$ *be presilting. Suppose* $\mathsf{S} = \mathsf{thick}(\mathsf{P})$ *is a functorially finite subcategory of* $\mathsf{D}$*. Let* $\rho\colon \mathsf{D} \to \mathsf{U} = \mathsf{D}/\mathsf{S}$ *be the canonical functor. Then there is a bijection*

$$\{\textit{silting subcategories } \mathsf{M} \textit{ of } \mathsf{D} \textit{ with } \mathsf{P} \subset \mathsf{M}\} \overset{1-1}{\longleftrightarrow} \{\textit{silting subcategories of } \mathsf{U}\}.$$

Note, we can identify $\mathsf{U} \simeq \mathsf{S}^{\perp}$ and observe that the functorial finiteness of $\mathsf{S}$ in $\mathsf{D}$ means we have a stable t-structure $(\mathsf{S}, \mathsf{S}^{\perp})$ in $\mathsf{D}$.

For parts (a) - (c) suppose $\mathsf{M}$ is a silting subcategory of $\mathsf{D}$ with $\mathsf{P} \subset \mathsf{M}$.

(a) For $m \in \mathsf{M}$ show that in the decomposition triangle with respect to $(\mathsf{S}, \mathsf{S}^{\perp})$,

$$s \longrightarrow m \longrightarrow \rho(m) \longrightarrow \Sigma s$$

we have $s \in \mathsf{susp}(\mathsf{P})$. (Recall, $(\mathsf{cosusp}(\Sigma^{-1}\mathsf{P}), \mathsf{susp}(\mathsf{P}))$ forms a bounded co-t-structure in $\mathsf{S}$.)

(b) Show that $\mathsf{thick}_{\mathsf{U}}(\rho(\mathsf{M})) = \mathsf{U}$. (Hint: think about Exercise 2.)

(c) Show that $\rho(\mathsf{M})$ is presilting in $\mathsf{U}$ by using the triangle from part (a) to show that

   (i) $\mathrm{Hom}(\mathsf{M}, \Sigma^{>0}\rho(\mathsf{M})) = 0$; and,

   (ii) $\mathrm{Hom}(\rho(\mathsf{M}), \Sigma^{>0}\rho(\mathsf{M})) = 0$.

(d) Show that for silting subcategories $\mathsf{P} \subset \mathsf{M}$ and $\mathsf{P} \subset \mathsf{N}$, if $\rho(\mathsf{M}) \leq \rho(\mathsf{N})$ then $\mathsf{M} \leq \mathsf{N}$. Hence conclude that the assignment is injective. (Hint: use the triangle from part (a) for $\mathsf{M}$ and $\mathsf{N}$, the approximation property and the definition of the partial order.)

(e) Show that $({}^{\perp}(\Sigma^{>0}\mathsf{P}), \mathsf{susp}(\Sigma\mathsf{P}))$ is a co-t-structure in $\mathsf{D}$ with coheart $\mathsf{P}$.

(f) Let $\mathsf{N} \subset \mathsf{U} \simeq \mathsf{S}^{\perp}$ be a silting subcategory. For $n \in \mathsf{N}$, take the decomposition triangle with respect to the co-t-structure in part (e):

$$s_n \longrightarrow m_n \longrightarrow n \longrightarrow \Sigma s_n \text{ with } \Sigma s_n \in \mathsf{susp}(\Sigma\mathsf{P}) \text{ and } m_n \in {}^{\perp}(\Sigma^{>0}\mathsf{P}).$$

Set $\mathsf{M} = \mathsf{add}(\mathsf{P} \cup \{m_n \mid n \in \mathsf{N}\})$. Show that $\mathsf{M}$ is presilting in $\mathsf{D}$. (Hint: you will need two steps like in part (c).)

(g) Show that $\mathsf{D} = \mathsf{thick}(\mathsf{M})$, where $\mathsf{M}$ is defined in part (f).