Companion exercises for the lecture series
STABLE ∞-CATEGORIES: LOCALISATIONS AND RECOLLEMENTS

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Exercise 1 (Nerve and geometric realisation)**.** Let $\mathscr{C}$ be an ordinary category which admits all small colimits. Prove the following statements:

(1) Let $F\colon \mathbf{\Delta} \to \mathscr{C}$ be an arbitrary functor. The functor

$$N_F\colon \mathscr{C} \longrightarrow \mathbf{Set}_{\mathbf{\Delta}}, \qquad c \longmapsto \mathscr{C}(F(-), c)$$

admits a left adjoint.

(2) There is an adjunction

$$\tau\colon \mathbf{Set}_{\mathbf{\Delta}} \rightleftarrows \mathbf{Cat} : N$$

such that $\tau(\Delta^n) = [n]$.

(3) There is an adjunction

$$|-|\colon \mathbf{Set}_{\mathbf{\Delta}} \rightleftarrows \mathbf{Top} : \mathrm{Sing}$$

such that

$$|\Delta^n| = \left\{ \underline{t} \in [0,1]^{n+1} \,\middle|\, t_0 + t_1 + \cdots + t_n = 1 \right\}$$

endowed with the usual topology. Given a topological space X, discuss why is Sing(X) an ∞-category. □

***Exercise** 2 (Lurie's differential graded nerve)**.** For a commutative ring $\mathbb{k}$, we denote the category (small) differential graded $\mathbb{k}$-categories and differential graded functors between them by $\mathbf{dgCat}_{\mathbb{k}}$ (see [Kel06] for definitions but note that we grade chain complexes *homologically*). For $n \geq 0$, let $Q^{(n)}$ be the quiver with vertex set $[n]$ and arrows

$$I\colon i \longrightarrow j$$

for each chain $I = \{ i < k_1 < \cdots < k_\ell < j \}$ with $\ell \geq 0$. We endow $Q^{(n)}$ with a grading by setting

$$\deg(I) = \#(I \setminus \{ i, j \}) = \ell.$$

Let $\mathbb{k}Q^{(n)}$ be the free (non-negatively) graded $\mathbb{k}$-category generated by $Q^{(n)}$. Each monotone function $\sigma\colon [m] \to [n]$ induces a functor of graded $\mathbb{k}$-categories $f_\sigma\colon \mathbb{k}Q^{(m)} \to \mathbb{k}Q^{(n)}$ given by

$$f_\sigma(I) = \begin{cases} \sigma(I) & \text{if } \sigma|_I \text{ is injective,} \\ \mathrm{id}_k & \text{if } I = \{ i < j \} \text{ and } \sigma(i) = \sigma(j) = k, \\ 0 & \text{otherwise.} \end{cases}$$

We promote $\mathbb{k}Q^{(n)}$ to a differential graded $\mathbb{k}$-category $\mathbb{k}\Delta^n$ by defining the action of the differential

$$d\colon \mathbb{k}Q^{(n)}(i,j) \longrightarrow \mathbb{k}Q^{(n)}(i,j)$$

on a chain $I = \{ i < k_1 < \cdots < k_\ell < j \}$ by the formula

$$d(I) = \sum_{1 \leq m \leq \ell} (-1)^m (I \setminus \{ k_m \} - \{ k_m < \cdots < k_\ell < j \} \circ \{ i < k_1 < \cdots < k_m \}).$$

According to Exercise 1 there is an adjunction

$$\mathbb{k}\colon \mathbf{Set}_{\mathbf{\Delta}} \rightleftarrows \mathbf{dgCat}_{\mathbb{k}} : N(-)_{dg}$$

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such that $\Bbbk\Delta^n$ is the differential graded $\Bbbk$-category defined above; moreover, for each small differential graded $\Bbbk$-category $\mathscr{A}$, the simplicial set $\mathrm{N}(\mathscr{A})_{dg}$ is an ∞-category [Lur17, Prop 1.3.1.10]. The functor $\mathrm{N}(-)_{dg}$ is called the **differential graded nerve**.

Let $\mathscr{A}$ be a small differential graded $\Bbbk$-category.

(1) Describe the n-simplices of the ∞-category $\mathrm{N}(\mathscr{A})_{dg}$ for $n = 0,1,2,3$. Discuss why is $\mathrm{N}(\mathscr{A})_{dg}$ an ∞-category.

(2) Prove that

$$\mathrm{Ho}\big(\mathrm{N}(\mathscr{A})_{dg}\big) = \mathrm{H}_0(\mathscr{A}),$$

where $\mathrm{H}_0(\mathscr{A})$ is the category with the same objects as $\mathscr{A}$ and sets of morphisms

$$\mathrm{H}_0(\mathscr{A})(x,y) := \mathrm{H}_0(\mathscr{A}(x,y)).$$

□

Exercise 3. Let

$$\begin{tikzcd} \mathscr{A} \arrow[r, "i" description] & \mathscr{C} \arrow[l, bend right, "i_{\mathrm{L}}"'] \arrow[l, bend left, "i_{\mathrm{R}}"] \arrow[r, "p" description] & \mathscr{B} \arrow[l, bend right, "p_{\mathrm{L}}"'] \arrow[l, bend left, "p_{\mathrm{R}}"] \end{tikzcd}$$

be a recollement of stable ∞-categories. Prove that there is an equivalence of exact functors

$$(i_{\mathrm{R}} \circ p_{\mathrm{L}}) \simeq \Omega(i_{\mathrm{L}} \circ p_{\mathrm{R}}).$$

Hint: There is a biCartesian square of exact functors

$$\begin{tikzcd} p_{\mathrm{L}} \circ p \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{id}_{\mathscr{C}} \arrow[d] \\ 0 \arrow[r] & i \circ i_{\mathrm{L}} \end{tikzcd}$$

where $p_{\mathrm{L}} \circ p \to \mathrm{id}_{\mathscr{C}}$ and $\mathrm{id}_{\mathscr{C}} \to i \circ i_{\mathrm{L}}$ are the counit and the unit of the adjunctions $p_{\mathrm{L}} \dashv p$ and $i_{\mathrm{L}} \dashv i$, respectively. □

Exercise 4. Let $\mathrm{F}\colon \mathscr{B} \to \mathscr{A}$ be an exact functor between stable ∞-categories. Sketch the construction of recollements of stable ∞-categories

$$\begin{tikzcd} \mathscr{A} \arrow[r, "i" description] & \mathscr{L}_*(\mathrm{F}) \arrow[l, bend right, "i_{\mathrm{L}}"'] \arrow[l, bend left, "i_{\mathrm{R}}"] \arrow[r, "p" description] & \mathscr{B} \arrow[l, bend right, "p_{\mathrm{L}}"'] \arrow[l, bend left, "p_{\mathrm{R}}"] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathscr{A} \arrow[r, "j" description] & \mathscr{L}^*(\mathrm{F}) \arrow[l, bend right, "j_{\mathrm{L}}"'] \arrow[l, bend left, "j_{\mathrm{R}}"] \arrow[r, "q" description] & \mathscr{B}. \arrow[l, bend right, "q_{\mathrm{L}}"'] \arrow[l, bend left, "q_{\mathrm{R}}"] \end{tikzcd}$$

Hint: Define recollement structures on $\mathrm{Fun}(s \to t, \mathscr{A})$ *and use the defining pullback squares*

$$\begin{tikzcd} \mathscr{L}_*(\mathrm{F}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{PB}"] & \mathscr{B} \arrow[d, "\mathrm{F}"] \\ \mathrm{Fun}(s \to t, \mathscr{A}) \arrow[r, "p=t"'] & \mathrm{Fun}(t, \mathscr{A}) \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathscr{L}^*(\mathrm{F}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{PB}"] & \mathscr{B} \arrow[d, "\mathrm{F}"] \\ \mathrm{Fun}(s \to t, \mathscr{A}) \arrow[r, "q=s"'] & \mathrm{Fun}(s, \mathscr{A}) \end{tikzcd}$$

to construct morphisms of recollements

$$\begin{tikzcd} \mathscr{A} \arrow[r, "i" description] \arrow[d, "\mathrm{id}_{\mathscr{A}}"'] & \mathscr{L}_*(\mathrm{F}) \arrow[l, bend right, "i_{\mathrm{L}}"'] \arrow[l, bend left, "i_{\mathrm{R}}"] \arrow[r, "p" description] \arrow[d] & \mathscr{B} \arrow[l, bend right, "p_{\mathrm{L}}"'] \arrow[l, bend left, "p_{\mathrm{R}}"] \arrow[d, "\mathrm{F}"] \\ \mathscr{A} \arrow[r, "i" description] & \mathrm{Fun}(s \to t, \mathscr{A}) \arrow[l, bend right, "i_{\mathrm{L}}"'] \arrow[l, bend left, "i_{\mathrm{R}}"] \arrow[r, "p" description] & \mathscr{A} \arrow[l, bend right, "p_{\mathrm{L}}"'] \arrow[l, bend left, "p_{\mathrm{R}}"] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathscr{A} \arrow[r, "j" description] \arrow[d, "\mathrm{id}_{\mathscr{A}}"'] & \mathscr{L}^*(\mathrm{F}) \arrow[l, bend right, "j_{\mathrm{L}}"'] \arrow[l, bend left, "j_{\mathrm{R}}"] \arrow[r, "q" description] \arrow[d] & \mathscr{B} \arrow[l, bend right, "q_{\mathrm{L}}"'] \arrow[l, bend left, "q_{\mathrm{R}}"] \arrow[d, "\mathrm{F}"] \\ \mathscr{A} \arrow[r, "j" description] & \mathrm{Fun}(s \to t, \mathscr{A}) \arrow[l, bend right, "j_{\mathrm{L}}"'] \arrow[l, bend left, "j_{\mathrm{R}}"] \arrow[r, "q" description] & \mathscr{A} \arrow[l, bend right, "q_{\mathrm{L}}"'] \arrow[l, bend left, "q_{\mathrm{R}}"] \end{tikzcd}$$

□

Exercise 5. Let R and S be rings and M an $(\mathrm{S}^{\mathrm{op}} \otimes \mathrm{R})$-bimodule. Let $\mathscr{U}^*(- \otimes_{\mathrm{S}} \mathrm{M})$ be the ordinary category whose objects are the pairs

$$(\mathrm{X}, \varphi) = (\mathrm{X}_{\mathrm{S}}, \varphi\colon \mathrm{X} \otimes_{\mathrm{S}} \mathrm{M}_{\mathrm{R}} \to \mathrm{Y}_{\mathrm{R}})$$

where X is a right S-module and φ is a morphism between right R-modules. A morphism $(\mathrm{X}, \varphi) \to (\mathrm{X}', \varphi')$ in $\mathscr{U}^*(- \otimes_{\mathrm{S}} \mathrm{M})$ consists of morphisms

$$f : \mathrm{X} \to \mathrm{X}' \qquad \text{and} \qquad g : \mathrm{Y} \to \mathrm{Y}',$$

where f is a morphism between right S-modules and g is a morphism between right R-modules, such that the diagram

$$\begin{tikzcd} X \otimes_S M \arrow[r, "\varphi"] \arrow[d, "f \otimes_S \mathrm{id}_M"'] & Y \arrow[d, "g"] \\ X' \otimes_S M \arrow[r, "\varphi'"] & Y' \end{tikzcd}$$

commutes. Prove the following statements:

(1) There is a strict pullback diagram of categories

$$\begin{tikzcd} \mathscr{U}^*(-\otimes_M) \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{PB}"] & \mathrm{Fun}(s \to t, \mathrm{Mod}(R)) \arrow[d, "s"] \\ \mathrm{Mod}(S) \arrow[r, "-\otimes_S M"'] & \mathrm{Mod}(R) \end{tikzcd}$$

Moreover, the category $\mathscr{U}^*(- \otimes_S M)$ is abelian.

(2) There is a recollement of abelian categories

$$\begin{tikzcd} \mathrm{Mod}(R) \arrow[r, "i"] & \mathscr{U}^*(- \otimes_S M) \arrow[l, bend right, "i_L"'] \arrow[l, bend left, "i_R"] \arrow[r, "p"] & \mathrm{Mod}(S) \arrow[l, bend right, "p_L"'] \arrow[l, bend left, "p_R"] \end{tikzcd}$$

such that $- \otimes_S M = i_R \circ p_L$ (give explicit formulas for all the six functors).

(3) The category $\mathscr{U}^*(- \otimes_S M)$ is equivalent to $\mathrm{Mod}\big(\left(\begin{smallmatrix} S & M \\ 0 & R \end{smallmatrix}\right)\big)$ where

$$\left(\begin{smallmatrix} S & M \\ 0 & R \end{smallmatrix}\right) := \{ \left(\begin{smallmatrix} s & m \\ 0 & r \end{smallmatrix}\right) \mid r \in R,\ s \in S,\ m \in M \}$$

viewed as a ring with respect to the usual matrix operations.

(4) Make the appropriate modifications to define an abelian category $\mathscr{U}_*(\underline{\mathrm{Hom}}_R(M, -))$ and formulate the analogues of (1), (2) and (3) above. What do you notice? □

References

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