TWO WEEKS OF SILTING
TILTING THEORY OF PREPROJECTIVE ALGEBRAS AND COHEN-MACAULAY MODULES 2

Throughout, let Λ be a finite dimensional algebra over a field k.

1. Prove that $\mathsf{tors}\,\Lambda$ is a complete lattice.

2. Is $\mathsf{f}\text{-}\mathsf{tors}\,\Lambda$ always a lattice?

3. Draw the Hasse quiver of $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$ for the algebra $\Lambda = k[1 \xrightarrow{a} 2 \xrightarrow{b} 3]/(ab)$.

4. Prove the following statements.

(1) Let $P_1 \xrightarrow{f} P_0 \to X \to 0$ be a minimal projective presentation of $X \in \mathsf{mod}\,\Lambda$. Then X is τ-rigid if and only if the map $\mathrm{Hom}_\Lambda(P_0, X) \xrightarrow{f} \mathrm{Hom}_\Lambda(P_1, X)$ is surjective.

(2) Each τ-rigid module is rigid (i.e. $\mathrm{Ext}^1_\Lambda(X, X) = 0$). [Hint: Use (1).]

(3) Let I be a 2-sided ideal of Λ. If X is a τ-rigid Λ-module, then X/IX is a τ-rigid (Λ/I)-module. [Hint: Use (1).]

5. Prove the following statements.

(1) Each faithful τ-rigid module has projective dimension at most one. [Hint: Use **4**(1).]

(2) If (X, P) is τ-rigid, then X is a partial tilting (Λ/I)-module for the annihilator I of X. In particular, $|X| + |P| \leq |\Lambda|$. [Hint: Use (1) and **4**(2)(3).]

(3) If $(X, P) \in \mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$, then the simple Λ-modules which do *not* appear in X as composition factors are precisely the simple direct summands of $P/\operatorname{rad} P$. [Hint: Use (2).]

(4) If $(X, P), (X, Q) \in \mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$, then $\mathsf{add}\,P = \mathsf{add}\,Q$. [Hint: Use (3).]

6. Prove the following statements for $M \in \mathsf{mod}\,\Lambda$ and $\mathcal{T} \in \mathsf{tors}\,\Lambda$.

(1) $\mathsf{Fac}\,M$ is contravariantly finite subcategory of $\mathsf{mod}\,\Lambda$.

(2)* $\mathsf{Fac}\,M$ is a covariantly finite subcategory of $\mathsf{mod}\,\Lambda$. [Hint: For given $X \in \mathsf{mod}\,\Lambda$, consider a left $(\mathsf{add}\,M)$-approximation of a projective cover of X.]

(3) $\mathcal{T}$ is functorially finite if and only if there exists $M \in \mathsf{mod}\,\Lambda$ such that $\mathcal{T} = \mathsf{Fac}\,M$. [Hint: Use (1)(2).]

7. Let X be a τ-rigid Λ-module. Prove the following statements.

(1) $\mathrm{Ext}^1_\Lambda(X, \mathsf{Fac}\,X) = 0$. [Hint: Use Auslander-Reiten duality.]

(2) $\mathsf{Fac}\,X$ is a torsion class.

8. Let $\mathcal{T}_1 \subsetneq \mathcal{T}_2 \subsetneq \mathcal{T}_3 \subsetneq \cdots$ be a strictly increasing sequence of torsion classes in $\mathsf{mod}\,\Lambda$. Prove that $\bigcup_{i\geq 1} \mathcal{T}_i$ is *not* functorially finite.

For silting complexes T and U of Λ, we write $T \geq U$ if $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(T, U[i]) = 0$ for all $i > 0$. (Opposite to David's talk.)

9. Let S and T be silting complexes of Λ such that $S \geq T \geq S[1]$, and let X be an indecomposable direct summand of T. Prove the following statements.

(1) There exists a triangle $S \to T' \xrightarrow{f} T'' \to S[1]$ such that $T', T'' \in \mathsf{add}\,T$ and f is a radical morphism.
(2) X appears in precisely one of T' and T'' as a direct summand.
(3) If $X \in \mathsf{add}\,T'$ (resp. $X \in \mathsf{add}\,T''$), then the left (resp. right) mutation V of T with respect to X satisfies $T > V \geq S[1]$ (resp. $S \geq V > T$).

Recall that we have a bijection $2\text{-}\mathsf{silt}\,\Lambda \simeq \mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$ given by $T \mapsto H^0(T)$.

10. Prove that, if the Hasse quiver of $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$ has a finite connected component, then it is the whole Hasse quiver. [Hint: Use **9**(3).]

11*. Let U be a basic 2-term presilting complex of Λ such that $|U| = |\Lambda| - 1$. Prove that there are precisely two basic silting complexes of Λ having U as direct summands. [Hint: Existence: Use Bongartz completion and **9**(3). Only two: Use silting reduction.]

12*. Let $T, U \in \mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$. Prove that, there is an arrow between T and U in the Hasse quiver of $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$ if and only if they are mutation of each other.
[Hint: Use **9**(3).]

The following stronger version of **9**(3) can be used to answer **13**.

Theorem A. *Let $T \in \mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$ and $\mathcal{U} \in \mathsf{tors}\,\Lambda$. If $\mathcal{U} \supsetneq \mathsf{Fac}\,T$, then there exists a mutation S of T such that $\mathcal{T} \supseteq \mathsf{Fac}\,S \supsetneq \mathsf{Fac}\,T$.*

13*. Prove that $\mathsf{tors}\,\Lambda = \mathsf{f}\text{-}\mathsf{tors}\,\Lambda$ holds if and only if Λ is τ-tilting finite (i.e. $\#\,\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda < \infty$). [Hint: "$\Rightarrow$": Use **8**. "$\Leftarrow$": Use Theorem A.]

14* Let Π be a preprojective algebra of a Dynkin quiver Q. Prove the following statements.

(1) $I_w \in \mathsf{s}\tau\text{-}\mathsf{tilt}\,\Pi$ for any $w \in W$.
[Hint: Define an extended Dynkin quiver Q' by adding a vertex 0 to Q. The preprojective algebra Π' of Q' satisfies $\Pi = \Pi'/(e_0)$, and the 2-sided ideal I'_w of Π' satisfies $I_w = I'_w/(e_0)I'_w$. Apply a version of **4**(3) to I'_w.]
(2) We have a bijection $W \to \mathsf{s}\tau\text{-}\mathsf{tilt}\,\Pi$, $w \mapsto I_w$. [Hint: Use **10** for surjectivity.]
(3) The bijection in (2) is an anti-isomorphism of posets.
[Hint: The Hasse quiver of a finite poset determines the partial order.]