TWO WEEKS OF SILTING
TILTING THEORY OF PREPROJECTIVE ALGEBRAS AND COHEN-MACAULAY MODULES 1

Throughout, let Q be a connected quiver without loops and Π the preprojective algebra. For $i \in Q_0$, let $I_i = (1 - e_i)$ be the 2-sided ideal of Π generated by $1 - e_i$.

1. Prove that Π is independent of the orientation of Q.

2. Prove that I_i is a tilting Π-module with $\mathrm{End}_\Pi(I_i) \simeq \Pi$ if Q is non-Dynkin.
[Hint: Follow the sketch in the lecture.]

Let $\mathcal{I}$ be the ideal monoid of Π generated by I_i for all $i \in Q_0$.

3. Prove that any $I \in \mathcal{I}$ is a tilting Π-module with $\mathrm{End}_\Pi(I_i) \simeq \Pi$ if Q is non-Dynkin. [Hint: Follow the sketch in the lecture.]

4. Prove the following.
(1) If there is no arrow between i and j in Q, then $I_i I_j = I_j I_i$.
(2) If there is precisely one arrow between i and j in Q, then $I_i I_j I_i = I_j I_i I_j$.

Let W be the Coxeter group of Q.

5. Let Q be type A_3. Calculate I_w for all $w \in W$.

6. Prove that there is a bijection $W \simeq \mathcal{I}$ sending $w \in W$ to $I_w = I_{i_1} I_{i_2} \cdots I_{i_\ell}$, where $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ is an arbitrary reduced expression. Also prove that it gives an injection $W \simeq \mathsf{tilt}\,\Pi$ if Q is non-Dynkin. [Hint: Use **7** and **8** below.]

7. Fix $w \in W$. Prove that any expression (resp. reduced expression) of w can be transformed into a fixed reduced expresision of w by a sequence of the operations (1)–(3) (resp. (1)(2)).
(1) Replace $s_i s_j$ by $s_j s_i$ if there is no arrow between i and j in Q.
(2) Replace $s_i s_j s_i$ by $s_j s_i s_j$ if there is precisely one arrow between i and j in Q.
(3) Delete $s_i s_i$.

8. Assume that Q is non-Dynkin. Let $\mathsf{fd}\,\Pi$ be the category of finite dimensional Π-modules. Then W acts on the Grothendieck group $K_0(\mathsf{fd}\,\Pi)$ by

$$s_i(x) = x - (x, S_i)[S_i],$$

where $(-,-) = \sum_{i \geq 0} (-1)^i \dim_k \mathrm{Ext}^i_\Pi(-,-)$ is the Euler form on $\mathsf{fd}\,\Pi$.

Prove that the action of $w \in W$ coincides with $[I_w \overset{\mathbf{L}}{\otimes}_\Pi -]$, where $I_w \overset{\mathbf{L}}{\otimes}_\Pi - : \mathsf{D}^{\mathrm{b}}(\mathsf{fd}\,\Pi) \to \mathsf{D}^{\mathrm{b}}(\mathsf{fd}\,\Pi)$ is an autoequivalence given by the tilting Π-module I_w.