Two weeks of silting

Silting and cosilting in triangulated categories

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(1) **Generators.** Let X be an object in $\mathsf{K}^b(\mathsf{Proj}(R))$. Consider the following statements.
  (a) The smallest thick subcategory of $\mathsf{D}(R)$ containing $\mathsf{Add}(X)$ is $\mathsf{K}^b(\mathsf{Proj}(R))$.
  (b) The smallest triangulated subcategory of $\mathsf{D}(R)$ containing $\mathsf{Add}(X)$ is $\mathsf{K}^b(\mathsf{Proj}(R))$.
  (c) If $\mathrm{Hom}_{\mathcal{T}}(X, Y[k]) = 0$ for all $k \in \mathbb{Z}$, then $Y = 0$.
  (d) The smallest coproduct-closed triangulated subcategory of $\mathsf{D}(R)$ containing X is $\mathsf{D}(R)$.
  Prove that (a)$\Leftrightarrow$(b)$\Rightarrow$(c)$\Leftrightarrow$(d). What about $(c) \Rightarrow (b)$?[1]

(2) **Silting and cosilting complexes.** Let T be a silting complex in the derived category $\mathsf{D}(R)$ and let $H^0_T : \mathsf{D}(R) \longrightarrow \mathcal{H}_T$ be the cohomological functor associated to the corresponding silting t-structure and denote by P_T the object $H^0_T(T)$.
  (a) Show that the functor $\mathrm{Hom}_{\mathsf{D}(R)}(P_T, H^0_T(-))$ is naturally equivalent to $\mathrm{Hom}_{\mathsf{D}(R)}(T, -)$. Conclude that, in particular, $\mathrm{End}_{\mathsf{D}(R)}(T) \cong \mathrm{End}_{\mathsf{D}(R)}(H^0_T(T))$.
  (b) Show that P_T is a projective generator in the heart $\mathcal{H}_T$.
  (c) Prove that if T is equivalent to a compact silting complex with endomorphism ring S, then the heart $\mathcal{H}_T$ is equivalent to the module category $\mathsf{Mod}(S)$.

(3) **Purity and definability.** Let $\mathcal{T}$ be a compactly generated triangulated category.
  (a) Show that, for any family of objects $(X_i)_{i\in I}$ in $\mathcal{T}$, the canonical map

$$\coprod_{i\in I} X_i \to \prod_{i\in I} X_i$$

  is a pure monomorphism;
  (b) Let E be a pure-injective object of $\mathcal{T}$. Show that ${}^{\perp}E := \mathsf{Ker}(\mathsf{Hom}_{\mathcal{T}}(-, E))$ is definable if and only if it is product-closed;
  (c) Let P be a pure-projective object of $\mathcal{T}$. Show that $P^{\perp} := \mathsf{Ker}(\mathsf{Hom}_{\mathcal{T}}(P, -))$ is definable.
  (d) Let $\mathcal{T}$ be the derived category of a ring R and C a bounded cosilting complex in $D(R)$. Show that the following subcategory is definable

$${}^{\perp_{\mathbb{Z}}}C := \bigcap_{n\in\mathbb{Z}} \mathsf{Ker}(\mathsf{Hom}_{\mathcal{T}}(-, C[n])).$$

[1] It holds if X is compact; more generally, this is not clear to us!

(4) **Interactions with ring epimorphisms**

(a) Let R be a ring.

(i) Let $\mathcal{U}$ be a set of finitely presented modules of projective dimension at most one. Show that the perpendicular category

$$\mathcal{U}^{\perp} := \bigcap_{U \in \mathcal{U}} \mathsf{Ker}(\mathsf{Hom}_R(U, -)) \cap \mathsf{Ker}(\mathsf{Ext}^1_R(U, -))$$

is bireflective (i.e. closed under kernels, cokernels, products, coproducts) and extension-closed. Conclude that $\mathcal{U}^{\perp}$ is the essential image of the restriction of scalars functor associated to pseudoflat[2] ring epimorphism of R.

(ii) Let $f\colon R \longrightarrow S$ be a pseudoflat ring epimorphism such that S has projective dimension at most one as an R-module. Let σ_0 be a projective resolution of S as an R-module, consider the induced map $\overline{f}\colon R \longrightarrow \sigma_0$ in $D(R)$, and let σ_1 denote its cone. Show that $\sigma_0 \oplus \sigma_1$ is a silting complex. Conclude that $S \oplus \mathsf{Coker}(f)$ is a silting module.

(b) Let A denote the path algebra over a field $\mathbb{K}$ of the Kronecker quiver:

$$\bullet \rightrightarrows \bullet$$

(i) Determine the perpendicular category $U^{\perp}$ where U is an indecomposable preprojective module or an indecomposable preinjective module.

(ii) Determine all finite dimensional silting modules over the Kronecker algebra.

(iii) Show that $\mathsf{Mod}(A)$ admits infinite strictly increasing chains of subcategories which are bireflective and extension-closed.

[2] A ring epimorphism $f\colon R \longrightarrow S$ is called pseudoflat if $\mathsf{Tor}^R_1(S, S) = 0$ or, equivalently, if for any two S-modules M and N, $\mathsf{Ext}^1_S(M, N) \cong \mathsf{Ext}^1_R(M, N)$