Workshop on Postmodern Ringel duality, bocses and contramodules

Titles and abstracts

Raymundo Bautista

Four talks on homological systems and bocses

Homological systems. (See [5])

Let Λ be a finite-dimensional algebra over an algebraically closed field k.

A preordered set $(\mathcal{P}, \leq)$ is a non-empty set $\mathcal{P}$ equipped with a reflexive and transitive relation $\leq$. Two elements $i, j \in \mathcal{P}$ are equivalent if $i \leq j$ and $j \leq i$.

A (finite) homological system $(P, \leq, \Delta_i)_{i\in\mathcal{P}}$ for Λ consists of a finite preordered set $(P, \leq)$ and a family of pair-wise non-isomorphic indecomposable finite-dimensional Λ-modules $\{\Delta_i\}_{i\in\mathcal{P}}$ satisfying the following two conditions:

1. $\mathrm{Hom}_\Lambda(\Delta_i, \Delta_j) \neq 0$ implies $i \leq j$.
2. $\mathrm{Ext}^1_\Lambda(\Delta_i, \Delta_j) \neq 0$ implies $i \leq j$ and i non equivalent to j.

We take $\Delta = \oplus_{i\in\mathcal{P}}\Delta_i$ and $\mathcal{F}(\Delta)$ consists of the full subcategory of Λ-mod of the trivial module 0 and of those modules M for which there is Δ a filtration

$$0 = M_t \subset M_{t-1} \subset \cdots \subset M_1 \subset M_0 = M$$

with each M_i/M_{i+1} isomorphic to some Δ_j.

Bocses

We recall that a bocs $\mathcal{B} = (B, W, \mu, \epsilon)$ consists of a k-algebra B a B-B bimodule W, $\mu : W \to W \otimes_B W$ a co-associative co-multiplication and a co-unit $\epsilon : W \to B$.

Take Γ a finite oriented quiver with two kinds of arrows, full arrows and dashed arrows. We denote by Q the quiver consisting of all vertices and of the full arrows of Γ.

We say that the bocs $\mathcal{B} = (B, W, \mu, \epsilon)$ is associated to Γ if $B = kQ/J$ with J an admissible ideal of kQ and and there is an exact sequence of B-B-bimodules:

$$0 \to \overline{W} \to W \xrightarrow{\epsilon} B \to 0$$

with $\overline{W} = \oplus_{\alpha:x\to y;\alpha\in D} Be_y \otimes_k e_x B$, where D is the set of dashed arrows of Γ.

The category of left representations of a bocs

Let $\mathcal{B} = (B, W, \mu, \epsilon)$ be a bocs the category of representations of $\mathcal{B}$ denoted by $\mathcal{B}-\mathrm{mod}$ is the category whose objects are the finitely generated left B-modules and if M and N are finitely generated left B-modules then $\mathrm{Hom}_\mathcal{B}(M, N) = \mathrm{Hom}_B(W \otimes_B M, N)$ if f is a morphism from M to N and g is a morphism from N to L its composition is given by the composition of B-morphisms

$$V \otimes M \overset{\mu\otimes 1_M}{\to} V \otimes_B V \otimes_B M \overset{1_V\otimes f}{\to} V \otimes N \overset{g}{\to} L$$

The category of right representations mod-$\mathcal{B}$ is defined in a similar way.

The algebra $R(\mathcal{B}) = \mathrm{End}_\mathcal{B}({}_BB)^{op}$ is called the right algebra of $\mathcal{B}$ and the algebra $L(\mathcal{B}) = \mathrm{Hom}_\mathcal{B}(B_B)$ is called the left algebra of $\mathcal{B}$.

During the first three talks the following result will be proved.

Main Theorem (See [1])

Let $(P, \leq, \Delta_i)_{i\in\mathcal{P}}$ be a homological system for Λ such that $\Lambda \in \mathcal{F}(\Delta)$ then there is a bocs $\mathcal{B} = (B, W, \mu, \epsilon)$ associated to a quiver Γ such that

(i) There is an exact equivance $\mathcal{B}$-mod $\to \mathcal{F}(\Delta)$

(ii) If there is an oriented path in Γ from i to j, and i is equivalent to j then the path consists of dashed arrows only. In case $\mathcal{P}$ is a partially ordered set then the only oriented cycles in Γ consists of dashed loops.

(iii) The algebra B is an exact Borel-subalgebra of $R(\mathcal{B})$

(iv) There is a Morita equivalence $L : \Lambda$-mod $\to R(\mathcal{B})$-mod such that for each $i \in \mathcal{P}$,

$$L(\Delta_i) \cong R(\mathcal{B}) \otimes_B S_i$$

with S_i a simple B-module.

Conversely if $\mathcal{B}$ is a bocs with associated quiver Γ and the oriented paths between equivalent vertices consists of dashed arrows only, then $(\mathcal{P}, \leq, \{L(\mathcal{B}) \otimes_B S_i\}_{i\in\mathcal{P}})$ is an homological sistem for $L(\mathcal{B})$ with $\Lambda \in \mathcal{F}(\Delta')$, $\Delta' = \oplus_{i\in\mathcal{P}} R(\mathcal{B}) \otimes_B S_i$.

This theorem is a generalization of the main result in [3].

The fourth tallk will be devoted to see some properties of generic modules for $\mathcal{F}(\Delta)$, in the case Λ is cuasi-hereditary.

Plan

Talk 1

1. Homological Systems and their properties.
2. Graded duals
3. Strictly unital A_∞-structure for the Yoneda algebra of a Homological System.

Talk 2

1. The bar construction
2. A_∞ categories and b-categories.
3. The category of twisted objects and $\mathcal{F}(\Delta)$

Talk 3

1. Weak differential tensor algebras and bocses.
2. Interlaced ideals
3. Description of $H^0(\mathrm{tw}(\mathcal{A}))$ in terms of a weak differential tensor algebra.
4. On the Keller and Lefevre-Hasegawa Theorem (see [2]and [4]).

Talk 4

1. BT II in $\mathcal{F}(\Delta)$ and Generic modules.
2. BT II and a representation embedding

$$k]x]\text{-mod} \to \mathcal{F}(\Delta)$$

3. Quasi-hereditary algebras where $\mathcal{F}(\Delta)$ is tame and generic modules.

References

1. R. Bautista, E. Pérez, L. Salmerón. Homological systems and bocses. Journal of Algebra 617 (2023) 192-274.

2. Keller B., Introduction to A_∞-algebras and modules. Homology, Homotopyand Applications, vol. 3, 1 (2001) 1-35.

3. Koenig S., Külshammer J., Ovsienko S., Quasi-hereditary algebras, exact Borel subalgebras, A1-categories and boxes. Advances in Mathematics 262 (2014) 546-592.

4. Lefevre-Hasegawa K., Sur les A_∞-categories, These de doctorat, Universite Denis Diderot (Paris 7), 2003. arXiv: math/0310337v1.

5. Mendoza O., Sáenz C., Xi Ch., Homological systems in module categories over preordered sets. Quarterly J. Math. 60, 1, (2009) 75 -103.

Teresa Conde

First talk
Quasi-hereditary algebras with regular exact Borel subalgebras

Exact Borel subalgebras of quasi-hereditary algebras emulate the role of classic Borel subalgebras of complex semi-simple Lie algebras. Not every quasi-hereditary algebra A has an exact Borel subalgebra. However, a theorem by Koenig, Külshammer and Ovsienko establishes that there always exists a quasi-hereditary algebra Morita equivalent to A that has a (regular) exact Borel subalgebra. Despite that, an explicit characterisation of such 'special' Morita representatives is not directly obtainable from their work. In this talk, I shall present a criterion to decide whether a quasi-hereditary algebra contains a regular exact Borel subalgebra, and a method to compute all Morita representatives of A that have a regular exact Borel subalgebra. We shall also see that the Cartan matrix of a regular exact Borel subalgebra of a quasi-hereditary algebra A only depends on the composition factors of the standard and costandard A-modules, and on the dimension of the Hom-spaces between standard A-modules. I will conclude the talk with a characterisation of the basic quasi-hereditary algebras that admit a regular exact Borel subalgebra.

Second talk
Exact Borel subalgebras, good quotients and good subalgebras

The class of standardly stratified algebras includes all quasi-hereditary algebras and does not restrict to algebras of finite global dimension. Koenig, Külshammer and Ovsienko's existence result concerning exact Borel subalgebras of quasi-hereditary algebras extends to this context too. Work by Bautista, Pérez and Salmerón and by Goto guarantees the existence of exact Borel subalgebras of standardly stratified algebras up to Morita equivalence. By their recursive nature, standardly stratified algebras come equipped with a chain of standardly stratified quotient algebras (good quotients) and another chain of standardly stratified centraliser subalgebras (good subalgebras). In this talk, we shall see that exact Borel subalgebras are compatible, in more than one way, with good quotients and good subalgebras of standardly stratified algebras.

This is based on joint work in progress with Julian Külshammer.

Tiago Cruz

Ringel self-duality via relative dominant dimension

It is well known that the blocks of the Bernstein-Gelfand-Gelfand category O are Ringel self-dual. In this talk, we discuss an integral version of Soergel's Struktursatz and deformations of the blocks of the BGG category O based on Gabber and Joseph. By studying relative dominant dimension over these deformations together with cover theory in the sense of Rouquier, we reprove Ringel-self duality of the blocks of the BGG category O.

Julian Külshammer

(i) Regular exact Borel subalgebras of hereditary algebras

In this talk, I will present Markus Thuresson's work on Ext-algebras of standard modules over a hereditary algebra with a quasi-hereditary structure. In particular, building on work by Flores, Kimura and Rognerud, I will present a combinatorial description of the quiver and relations of this Ext-algebra in the case of a linearly oriented type A quiver. Furthermore, applying Conde's criterion, it turns out that each such algebra has a regular exact Borel subalgebra. The results can be extended to more general hereditary algebras using a gluing procedure.

(ii) Uniqueness of regular exact Borel subalgebras

Teresa Conde showed that the quasi-hereditary algebra admitting a basic regular exact Borel subalgebra is unique up to isomorphism. In this talk, I will present a different proof of her result using the machinery of A_∞-Koszul duality. This alternative proof shows in particular that also the basic regular exact Borel subalgebra is unique up to isomorphism and can be obtained by applying A_∞-Koszul duality to the positive part of the Ext-algebra of the standard modules. This is joint work with Vanessa Miemietz.

(iii) Uniqueness of basic bocses

Given the result of the previous talk, the question remains, whether the inclusion of the basic regular exact Borel subalgebra into the quasi-hereditary algebra is unique up to isomorphism. The joint work with Vanessa Miemietz gives an affirmative answer to this question. We will sketch the proof of this theorem, which constructs an A_∞-structure on the Ext-algebra of the standard modules from a presentation of the dual coring of the inclusion of the basic regular exact Borel subalgebra into a quasi-hereditary algebra.

Leonid Positselski

Lecture 1: Contramodules over coalgebras and topological rings

I will start with defining a contramodule over a coalgebra over a field and discussing the case of the coalgebra dual to the algebra of formal power series k[[x]]. Nonseparated contramodules over k[[x]] and contramodule Nakayama lemma for coalgebras over a field will be mentioned. Then I will pass to the more general setting of topological rings, define discrete modules, and introduce contramodules over a topological ring as modules over the corresponding monad on the category of sets. Time permitting, I may also define the contratensor product of a discrete module and a contramodule.

Lecture 2: Differential graded Koszul duality

I will start with defining the bar- and cobar-constructions for algebras, modules, coalgebras, and comodules (over a field). Then I will pass to the differential graded case, and say a few words about curved DG-algebras and curved DG-coalgebras. Finally, I will introduce the coderived and contraderived categories, and state the conilpotent and nonconilpotent versions of the derived nonhomogeneous Koszul duality theorem.

Lecture 3: Infinity-tilting-cotilting correspondence and generalized tilting theory

I will start with explaining how to produce an abelian category out of a given object in an idempotent-complete additive category with coproducts. Then I will define infinity-tilting and infinity-cotilting objects in abelian categories, infinity-tilting and infinity-cotilting pairs, and state the infinity-tilting-cotilting correspondence theorem. Finally, I will restrict to the special cases of module and comodule categories and explain the role of contramodules over topological rings in the generalized tilting theory.

Lecture 4: Underived and derived full-and-faithfulness theorems

I will start with explaining how to recover a contramodule over k[[x]] from its underlying k[x]-module structure, then proceed to formulate the full-and-faithfulness theorems for contramodule forgetful functors over coalgebras and topological rings. Passing to derived full-and-faithfulness, I will say a few words about Ext spaces over graded algebras and coalgebras before stating the general "if and only if" results describing the situation with the comodule inclusion and contramodule forgetful functors preserve Ext spaces.

Anna Rodriguez Rasmussen

Quasi-hereditary skew group algebras

Let A be a finite-dimensional algebra A and let G be a group acting on A. In this setting, a natural object of study is the skew group algebra A*G. In this talk I will show how the quasi-hereditary structures on A relate to the quasi-hereditary structures on A*G. More precisely, I will show that, under a natural compatibility condition, a partial order on the set of isomorphism classes of simple A-modules gives a similar order for simple A*G-simple modules. Further, the order on A-modules gives a quasi-hereditary structure on A if and only if the corresponding order gives a quasi-hereditary structure on A*G. Finally, if A is quasi-hereditary and B is an exact Borel subalgebra of A which, as a set, is invariant under the G-action, then B*G becomes an exact Borel subalgebra of A*G

Catharina Stroppel

Ringel duality for (semiinfinite) stratified categories

In this talk I will explain a semiinfinite version of stratified categories generalizing work of Dlab, Ringel, Donkin, ... In these contexts tilting modules exist and one can define a version of Ringel duality. I will explain the construction and give some examples. If time allows I will mention how these constructions fit into algebras with a triangular decomposition. This is joint work with Jon Brundan.