

# Homological Methods in Representation Theory

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Tilting Preenvelopes and Cotilting Precovers

Dedicated to Professor Helmut Lenzing on his 60th birthday

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Presented by I. Reiten

**Abstract.** We relate the theory of envelopes and covers to tilting and cotilting theory, for (infinitely generated) modules over arbitrary rings. Our main result characterizes tilting torsion classes as the pretorsion classes providing special preenvelopes for all modules. A dual characterization is proved for cotilting torsion-free classes using the new notion of a cofinendo module. We also construct unique representing modules for these classes.

**Mathematics Subject Classifications (2000):** 16D90, 16E30, 16G10.

**Key words:** preenvelopes, precovers, tilting modules, cotilting modules, finendo, cofinendo.

## Introduction

Tilting and cotilting modules have first been considered in the context of finitely generated modules over finite-dimensional algebras. The category equivalences and dualities induced by them have provided tools for exploring the structure of module categories over their endomorphism algebras, as well as the structure of tilting torsion, and cotilting torsion-free, classes of modules. The pioneering works of Brenner and Butler [4], Happel and Ringel [16], Assem [1] and Smalø [19] have later been extended to the setting of (infinitely generated) modules over arbitrary

(minimal) left and right approximations of modules developed by Auslander and Reiten [5] for finitely generated modules over arbitrary rings. In fact, the

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## A SOLUTION TO THE BAER SPLITTING PROBLEM

LIDIA ANGELERI HÜGEL, SILVANA BAZZONI, AND DOLORS HERBERA

**ABSTRACT.** Let  $R$  be a commutative domain. We prove that an  $R$ -module  $B$  is projective if and only if  $\text{Ext}_R^1(B, T) = 0$  for any torsion module  $T$ . This answers in the affirmative a question raised by Kaplansky in 1962.

A module  $B$  over a commutative domain  $R$  is called a Baer module when  $\text{Ext}_R^1(B, T) = 0$  for every torsion  $R$ -module  $T$ . This definition goes back to 1936 when R. Baer [3] posed the question of characterizing the class of all abelian (torsion-free) groups  $G$  such that any extension of  $G$  with a torsion group splits. In the language of homological algebra, the problem asks which groups  $G$  satisfy  $\text{Ext}_R^1(G, T) = 0$  for all torsion groups  $T$ . Baer proved that every countably generated group  $G$  with this property must be free [3, Theorem 8.6 and Footnote 11 p. 781].

In 1961 Rotman [19] introduced the terminology of Baer groups or B-groups and put this problem, together with the Whitehead problem, in the more general setting of describing, for a given class of abelian groups  $S$ , the groups  $B$  satisfying that  $\text{Ext}_R^1(B, S) = 0$  for any abelian group  $S \in S$ .

In 1962 Kaplansky [16] considered the case of modules over commutative domains. He raised the question of whether Baer modules over commutative domains are well known tools of homological algebra, he proved that Baer modules are flat, hence torsion-free, modules of projective dimension at most one.

The answer to the original problem raised by Baer was only given in 1969, when Griffith [10] proved that the only Baer groups are the free groups. Grimaldi [12] later generalized the result to modules over Dedekind domains proving that the

A real breakthrough came in 1988 when Eklöf and Fuchs and Shelton [11] showed that set-theoretically, the only Baer modules are projective. In 1988 Eklöf and Fuchs and Shelton [11] showed that set-theoretically, the only Baer modules are projective.

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Handbook of Tilting Theory

Edited by Lidia Angeleri Hügel, Dieter Happel, and Henning Krause

## Tilting modules and Gorenstein rings

Lidia Angeleri Hügel, Dolores Herbera and Jan Trlifaj  
(Communicated by Rüdiger Göbel)

Dedicated to Claus Michael Ringel on the occasion of his 60th birthday

**Abstract.** We apply tilting theory to the study of finite and cofinite classes of modules. We then focus on the case of Gorenstein rings. We prove that for a Gorenstein ring  $R$ , the following conditions are equivalent: (1)  $R$  is a tilting module over itself; (2)  $R$  is a Gorenstein ring; (3)  $R$  is a tilting module over itself. We also prove that for a Gorenstein ring  $R$ , the following conditions are equivalent: (1)  $R$  is a tilting module over itself; (2)  $R$  is a Gorenstein ring; (3)  $R$  is a tilting module over itself.

## Infinitely generated tilting modules of finite projective dimension

Lidia Angeleri Hügel and Flávio Ulhoa Coelho  
(Communicated by Rüdiger Göbel)

**Abstract.** We extend Miyashita's notion of a tilting module of finite projective dimension to the case of infinitely generated modules over an arbitrary ring  $R$  and characterize the classes induced by such tilting modules in terms of the existence of  $\mathcal{X}$ -preenvelopes. We also prove that for a ring  $R$ , the following conditions are equivalent: (1)  $R$  is a tilting module of finite projective dimension; (2)  $R$  is a Gorenstein ring; (3)  $R$  is a tilting module of finite projective dimension.

**Mathematics Subject Classification:** 16D90, 16E30, 16G10.

Tilting theory was introduced in the early eighties in the context of finitely generated modules over artin algebras by Brenner and Butler [5] and by Happel and Ringel [16] and since then it has played a central role in the development of the representation theory of artin algebras. While the first papers were dealing with tilting modules of finite projective dimension at most one, the theory was later extended to tilting modules of arbitrary projective dimension by Auslander and Reiten [3]. This also shed light on the connection between tilting theory and the notion of covariantly finite subcategory introduced by Auslander and Smalø in [4]. This also shed light on the Homological Conjectures, which still are a challenging topic in tilting theory.

This paper is to extend the results of Auslander and Reiten at most one projective dimension jointly with Tonolo and Trlifaj [17] for infinitely generated modules over arbitrary rings. Hereby we generalize previous results of projective dimension at most one.

It is well known that these dimensions may be infinite. Moreover, they do not depend on the choice of the tilting module. In fact, the first author jointly with Tonolo and Trlifaj [17] for infinitely generated modules over arbitrary rings. Hereby we generalize previous results of projective dimension at most one.

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## Silting Modules

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Dedicated to the memory of Dieter Happel

**Abstract.** We introduce the new concept of silting modules over a finite dimensional algebra recently introduced by Adachi, Iyama, and Reiten. We show that silting modules generalize tilting modules and that every partial tilting module is a silting module. Furthermore, we prove that silting modules generate torsion classes that provide left approximations, and that every partial tilting module is a silting module. We also see how some of these results are related to the derived module category. We also see how some of these results are related to the derived module category.

**Mathematics Subject Classification:** 16D90, 16E30, 16G10.

**Key words:** silting modules, tilting modules, torsion classes, derived module category.

**Introduction.** Tilting and cotilting modules have first been considered in the context of finitely generated modules over finite-dimensional algebras. The category equivalences and dualities induced by them have provided tools for exploring the structure of module categories over their endomorphism algebras, as well as the structure of tilting torsion, and cotilting torsion-free, classes of modules. The pioneering works of Brenner and Butler [4], Happel and Ringel [16], Assem [1] and Smalø [19] have later been extended to the setting of (infinitely generated) modules over arbitrary

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## TILTING THEORY AND THE FINITISTIC DIMENSION CONJECTURES

LIDIA ANGELERI HÜGEL AND JAN TRLIFAJ

**Abstract.** We study the relationship between the finitistic dimension conjecture and the tilting theory. We prove that for a ring  $R$ , the following conditions are equivalent: (1)  $R$  is a tilting module of finite projective dimension; (2)  $R$  is a Gorenstein ring; (3)  $R$  is a tilting module of finite projective dimension.

**Mathematics Subject Classification:** 16D90, 16E30, 16G10.

**Key words:** tilting modules, Gorenstein rings, finitistic dimension conjecture.

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## Ladders and simplicity of derived module categories

Lidia Angeleri Hügel<sup>a,1</sup>, Steffen Koenig<sup>b</sup>, Qunhua Liu<sup>b,c,\*,2</sup>, Dong Yang<sup>b,d,3</sup>

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**Abstract.** Recollections of derived module categories are introduced using a new technique, ladders of recollements, maximal mutation sequences. The position in is shown to control whether a recollement is unbounded to another level of derived category also turn out to control derived simplicity on a algebra is derived simple if its derived category is unbounded to another level of derived category.

**Mathematics Subject Classification:** 16D90, 16E30, 16G10.

**Key words:** derived module categories, recollements, ladders, mutation sequences, derived simplicity.

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## COVERS AND ENVELOPES VIA ENDOPROPERTIES OF MODULES

LIDIA ANGELERI HÜGEL

**Abstract.** Precovers and preenvelopes were introduced in the early eighties by Enochs and, independently, by Auslander and Smalø [12]. Enochs gave a general definition in terms of commutative diagrams for modules over arbitrary rings, whereas Auslander and Smalø, mainly concerned with the case of finitely generated modules over finite-dimensional algebras, stressed the functorial viewpoint and coined the terminology of contravariant and covariant finiteness. Both approaches turned out to be extremely fruitful for general module theory as well as for representation theory.

**Mathematics Subject Classification:** 16D90, 16E30, 16G10.

**Key words:** covers, envelopes, endoproperties, module theory.

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