On Graded Division Rings

Javier Sánchez

joint with Daniel Eiti Nishida Kawai

Departamento de Matemática - IME
Universidade de São Paulo, Brazil

Supported by FAPESP and CNPq

Daniel E. N. Kawai, Javier Sánchez, *On graded division rings*.

https://arxiv.org/abs/2010.09146

Notation

- Rings are associative with 1.
- Homomorphisms preserve 1.
- Modules are right modules, unless otherwise stated.

Notation

- Rings are associative with 1.
- Homomorphisms preserve 1.
- Modules are right modules, unless otherwise stated.
- If $\varphi\colon R \to S$ is a ring homomorphism and $(a_{ij}) = A \in M_{m\times n}(R)$,

$$A^{\varphi} := (\varphi(a_{ij})) \in M_{m\times n}(S).$$

Notation

- Rings are associative with 1.
- Homomorphisms preserve 1.
- Modules are right modules, unless otherwise stated.
- If $\varphi\colon R \to S$ is a ring homomorphism and $(a_{ij}) = A \in M_{m\times n}(R)$,

$$A^{\varphi} := (\varphi(a_{ij})) \in M_{m\times n}(S).$$

- Γ will be a multiplicative group with identity element e.
  Usually, elements of Γ will be denoted by γ, γ'
  Usually, elements of Γ^m will be denoted by $\overline{\alpha} = (\alpha_1, \dots, \alpha_m)$.

Classical problem and commutative case

Let R be a ring.

- An epic R-division ring is a ring homomorphism

$$\varphi\colon R \to D$$

where D is a division ring generated by $\operatorname{im}\varphi$, or equivalently, φ is a ring epimorphism.

Classical problem and commutative case

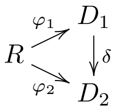
Let R be a ring.

- An epic R-division ring is a ring homomorphism

$$\varphi\colon R \to D$$

  where D is a division ring generated by $\operatorname{im}\varphi$, or equivalently, φ is a ring epimorphism.

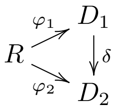
- Two epic R-division rings $\varphi_1\colon R \to D_1$, $\varphi_2\colon R \to D_2$ are isomorphic if there exists a ring isomorphism $\delta\colon D_1 \to D_2$ such that $\delta\varphi_1 = \varphi_2$.



Classical problem and commutative case

Let R be a ring.

- An epic R-division ring is a ring homomorphism
$$\varphi\colon R \to D$$
where D is a division ring generated by $\operatorname{im}\varphi$, or equivalently, φ is a ring epimorphism.
- Two epic R-division rings $\varphi_1\colon R \to D_1$, $\varphi_2\colon R \to D_2$ are isomorphic if there exists a ring isomorphism $\delta\colon D_1 \to D_2$ such that $\delta\varphi_1 = \varphi_2$.



Can we determine the different epic R-division rings?

Classical problem and commutative case

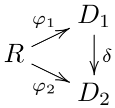
Let R be a ring.

- An epic R-division ring is a ring homomorphism

$$\varphi\colon R \to D$$

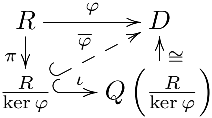
  where D is a division ring generated by $\operatorname{im}\varphi$, or equivalently, φ is a ring epimorphism.

- Two epic R-division rings $\varphi_1\colon R \to D_1$, $\varphi_2\colon R \to D_2$ are isomorphic if there exists a ring isomorphism $\delta\colon D_1 \to D_2$ such that $\delta\varphi_1 = \varphi_2$.



Can we determine the different epic R-division rings?

- If R is commutative, up to isomorphism, the epic R-division rings are parametrized by the prime ideals of R.



General situation

- In general,
  - Prime ideals are not enough to differenciate epic R-division rings.
  - There are domains not embeddable in division rings.

General situation

- In general,
  - Prime ideals are not enough to differenciate epic R-division rings.
  - There are domains not embeddable in division rings.

- Let $\varphi\colon R \to D$ be an epic R-division ring. It induces functions:

$$\begin{array}{rcl} \{\text{ Finitely presented } R\text{-modules}\} & \longrightarrow & \mathbb{N} \\ M & \longmapsto & \dim_D(M \otimes_R D) \end{array}$$

General situation

- In general,
  - Prime ideals are not enough to differenciate epic R-division rings.
  - There are domains not embeddable in division rings.
- Let $\varphi\colon R \to D$ be an epic R-division ring. It induces functions:

$$\begin{array}{ccc} \{\text{ Finitely presented } R\text{-modules}\} & \longrightarrow & \mathbb{N} \\ M & \longmapsto & \dim_D(M \otimes_R D) \end{array}$$

$$\begin{array}{ccc} \{\text{ Matrices over } R\ \} & \longrightarrow & \mathbb{N} \\ A & \longmapsto & \operatorname{rank} A^{\varphi} \end{array}$$

General situation

- In general,
  - Prime ideals are not enough to differenciate epic R-division rings.
  - There are domains not embeddable in division rings.

- Let $\varphi\colon R \to D$ be an epic R-division ring. It induces functions:

$$\begin{array}{ccc} \{ \text{Finitely presented } R\text{-modules}\} & \longrightarrow & \mathbb{N} \\ M & \longmapsto & \dim_D(M \otimes_R D) \end{array}$$

$$\begin{array}{ccc} \{ \text{Matrices over } R \} & \longrightarrow & \mathbb{N} \\ A & \longmapsto & \operatorname{rank} A^{\varphi} \end{array}$$

P. Malcolmson showed that two epic R-division rings are isomorphic if and only if they induce the same dimension function if and only if they induce the same rank function.

General situation

- In general,
  - Prime ideals are not enough to differenciate epic R-division rings.
  - There are domains not embeddable in division rings.

- Let $\varphi\colon R \to D$ be an epic R-division ring. It induces functions:

$$\begin{array}{ccc} \{\text{ Finitely presented } R\text{-modules}\} & \longrightarrow & \mathbb{N} \\ M & \longmapsto & \dim_D(M \otimes_R D) \end{array}$$

$$\begin{array}{ccc} \{\text{ Matrices over } R\ \} & \longrightarrow & \mathbb{N} \\ A & \longmapsto & \operatorname{rank} A^{\varphi} \end{array}$$

  P. Malcolmson showed that two epic R-division rings are isomorphic if and only if they induce the same dimension function if and only if they induce the same rank function.

- We develop a similar theory for group graded rings.
- Most of our proofs are natural extensions of the ones by Cohn, Malcolmson et al.

Basics on Graded Rings

Let R be a ring. R is Γ-graded if

- $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ where each R_γ is an additive group, and
- $R_\gamma \cdot R_{\gamma'} \subseteq R_{\gamma\gamma'}$ for all $\gamma, \gamma' \in \Gamma$.

Basics on Graded Rings

Let R be a ring. R is Γ-graded if

- $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ where each R_γ is an additive group, and
- $R_\gamma \cdot R_{\gamma'} \subseteq R_{\gamma\gamma'}$ for all $\gamma, \gamma' \in \Gamma$.

$\mathrm{h}(R) = \bigcup_{\gamma \in \Gamma} R_\gamma =$ homogeneous elements

Basics on Graded Rings

Let R be a ring. R is Γ-graded if

- $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ where each R_γ is an additive group, and
- $R_\gamma \cdot R_{\gamma'} \subseteq R_{\gamma\gamma'}$ for all $\gamma, \gamma' \in \Gamma$.

$\mathrm{h}(R) = \bigcup_{\gamma\in\Gamma} R_\gamma =$ homogeneous elements

- Let $X = \{x_i\}_{i\in I}$ be a set with a map $X \to \Gamma$, $x_i \mapsto \gamma_i$.

  The free ring $\mathbb{Z}\langle X\rangle$ is a Γ-graded ring with

$$\mathbb{Z}\langle X\rangle_\gamma = \left\{ \begin{array}{c} \mathbb{Z}\text{-linear span of monomials } x_{i_1}x_{i_2}\cdots x_{i_r} \\ \text{such that } \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_r} = \gamma \end{array} \right.$$

Basics on Graded Rings

Let R be a ring. R is Γ-graded if

- $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ where each R_γ is an additive group, and
- $R_\gamma \cdot R_{\gamma'} \subseteq R_{\gamma\gamma'}$ for all $\gamma, \gamma' \in \Gamma$.

$\mathrm{h}(R) = \bigcup_{\gamma\in\Gamma} R_\gamma =$ homogeneous elements

- Let $X = \{x_i\}_{i\in I}$ be a set with a map $X \to \Gamma$, $x_i \mapsto \gamma_i$.

  The free ring $\mathbb{Z}\langle X\rangle$ is a Γ-graded ring with

$$\mathbb{Z}\langle X\rangle_\gamma = \left\{ \begin{array}{c} \mathbb{Z}\text{-linear span of monomials } x_{i_1}x_{i_2}\cdots x_{i_r} \\ \text{such that } \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_r} = \gamma \end{array} \right.$$

- A homomorphism of Γ-graded rings

$$\varphi\colon R = \bigoplus_{\gamma\in\Gamma} R_\gamma \longrightarrow D = \bigoplus_{\gamma\in\Gamma} D_\gamma$$

  is a ring homomorphism such that $\varphi(R_\gamma) \subseteq D_\gamma$ for all $\gamma \in \Gamma$.

Basics on Graded Rings

Let R be a Γ-graded ring.

- R is a Γ-graded division ring if $R \neq \{0\}$ and every nonzero homogeneous element is invertible.

Basics on Graded Rings

Let R be a Γ-graded ring.

- R is a Γ-graded division ring if $R \neq \{0\}$ and every nonzero homogeneous element is invertible.
- R is a Γ-graded local ring if $R \neq \{0\}$ and the ideal $\mathfrak{m}$ generated by the noninvertible homogeneous elements is a proper ideal.

  Then $R/\mathfrak{m}$ is a Γ-graded division ring.

Basics on Graded Modules

Let R be a Γ-graded ring and M be a (right) R-module.

- M is a Γ-graded module if
  - $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ where each M_γ is an additive group, and
  - $M_\gamma \cdot R_{\gamma'} \subseteq M_{\gamma\gamma'}$.

Basics on Graded Modules

Let R be a Γ-graded ring and M be a (right) R-module.

- M is a Γ-graded module if
  - $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ where each M_γ is an additive group, and
  - $M_\gamma \cdot R_{\gamma'} \subseteq M_{\gamma\gamma'}$.
- A homomorphism of Γ-graded R-modules $f\colon M \to N$ is a homomorphism of modules such that $f(M_\gamma) \subseteq N_\gamma$ for all $\gamma \in \Gamma$.

Basics on Graded Modules

Let R be a Γ-graded ring and M be a (right) R-module.

- M is a Γ-graded module if
  - $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ where each M_γ is an additive group, and
  - $M_\gamma \cdot R_{\gamma'} \subseteq M_{\gamma\gamma'}$.
- A homomorphism of Γ-graded R-modules $f\colon M \to N$ is a homomorphism of modules such that $f(M_\gamma) \subseteq N_\gamma$ for all $\gamma \in \Gamma$.
- Let $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ be a Γ-graded R-module.
  If $\theta \in \Gamma$, the θ-shift $M(\theta)$ of M is the Γ-graded R-module

$$M(\theta) = \bigoplus_{\gamma \in \Gamma} M(\theta)_\gamma, \text{ where } M(\theta)_\gamma = M_{\theta\gamma}.$$

Basics on Graded Modules

Let R be a Γ-graded ring and M be a (right) R-module.

- M is a Γ-graded module if
  - $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ where each M_γ is an additive group, and
  - $M_\gamma \cdot R_{\gamma'} \subseteq M_{\gamma\gamma'}$.
- A homomorphism of Γ-graded R-modules $f\colon M \to N$ is a homomorphism of modules such that $f(M_\gamma) \subseteq N_\gamma$ for all $\gamma \in \Gamma$.
- Let $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ be a Γ-graded R-module.
  If $\theta \in \Gamma$, the θ-shift $M(\theta)$ of M is the Γ-graded R-module

$$M(\theta) = \bigoplus_{\gamma \in \Gamma} M(\theta)_\gamma, \text{ where } M(\theta)_\gamma = M_{\theta\gamma}.$$

$$R[x] \quad = \quad \cdots \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus R \oplus Rx \oplus Rx^2 \oplus Rx^3 \oplus Rx^4 \oplus \cdots$$

Basics on Graded Modules

Let R be a Γ-graded ring and M be a (right) R-module.

- M is a Γ-graded module if
  - $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ where each M_γ is an additive group, and
  - $M_\gamma \cdot R_{\gamma'} \subseteq M_{\gamma\gamma'}$.
- A homomorphism of Γ-graded R-modules $f\colon M \to N$ is a homomorphism of modules such that $f(M_\gamma) \subseteq N_\gamma$ for all $\gamma \in \Gamma$.
- Let $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ be a Γ-graded R-module.
  If $\theta \in \Gamma$, the θ-shift $M(\theta)$ of M is the Γ-graded R-module

$$M(\theta) = \bigoplus_{\gamma \in \Gamma} M(\theta)_\gamma, \text{ where } M(\theta)_\gamma = M_{\theta\gamma}.$$

$$R[x] = \cdots \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus R \oplus Rx \oplus Rx^2 \oplus Rx^3 \oplus Rx^4 \oplus \cdots$$

$$R[x](3) = \cdots \oplus 0 \oplus R \oplus Rx \oplus Rx^2 \oplus Rx^3 \oplus Rx^4 \oplus Rx^5 \oplus Rx^6 \oplus Rx^7 \oplus \cdots$$

Graded free modules

Let $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ be a Γ-graded ring.

- If $M_1, \ldots, M_r$ are Γ-graded R-modules, then $M_1 \oplus \cdots \oplus M_r$ is a Γ-graded R-module with

$$(M_1 \oplus \cdots \oplus M_r)_\gamma = \{(x_1, \ldots, x_r) \mid x_i \in (M_i)_\gamma\}$$

Graded free modules

Let $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ be a Γ-graded ring.

- If $M_1, \ldots, M_r$ are Γ-graded R-modules, then $M_1 \oplus \cdots \oplus M_r$ is a Γ-graded R-module with

$$(M_1 \oplus \cdots \oplus M_r)_\gamma = \{(x_1, \ldots, x_r) \mid x_i \in (M_i)_\gamma\}$$

- M is a graded free R-module if M is a free R-module with a homogeneous basis.

Graded free modules

Let $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ be a Γ-graded ring.

- If $M_1, \ldots, M_r$ are Γ-graded R-modules, then $M_1 \oplus \cdots \oplus M_r$ is a Γ-graded R-module with

$$(M_1 \oplus \cdots \oplus M_r)_\gamma = \{(x_1, \ldots, x_r) \mid x_i \in (M_i)_\gamma\}$$

- M is a graded free R-module if M is a free R-module with a homogeneous basis.

- Finitely generated graded free R-modules are of the form:

$$R^n(\overline{\beta}) = R(\beta_1) \oplus R(\beta_2) \oplus \cdots \oplus R(\beta_n),$$

for $\overline{\beta} = (\beta_1, \beta_2, \ldots, \beta_n) \in \Gamma^n$.

Morphisms between f. g. graded free modules

- A matrix $(a_{ij}) = A \in M_{m\times n}(R)[\overline{\alpha}][\overline{\beta}]$ if there are $\overline{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \Gamma^m$, $\overline{\beta} = (\beta_1, \ldots, \beta_n) \in \Gamma^n$ such that

$$a_{ij} \in R_{\alpha_i \beta_j^{-1}}$$

Morphisms between f. g. graded free modules

- A matrix $(a_{ij}) = A \in M_{m\times n}(R)[\overline{\alpha}][\overline{\beta}]$ if there are $\overline{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \Gamma^m$, $\overline{\beta} = (\beta_1, \ldots, \beta_n) \in \Gamma^n$ such that

$$a_{ij} \in R_{\alpha_i \beta_j^{-1}}$$

In other words, A defines a morphism of Γ-graded R-modules

$$R(\beta_1) \oplus \cdots \oplus R(\beta_n) \xrightarrow{A} R(\alpha_1) \oplus \cdots \oplus R(\alpha_m).$$

Morphisms between f. g. graded free modules

- A matrix $(a_{ij}) = A \in M_{m\times n}(R)[\overline{\alpha}][\overline{\beta}]$ if there are $\overline{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \Gamma^m$, $\overline{\beta} = (\beta_1, \ldots, \beta_n) \in \Gamma^n$ such that

$$a_{ij} \in R_{\alpha_i \beta_j^{-1}}$$

In other words, A defines a morphism of Γ-graded R-modules

$$R(\beta_1) \oplus \cdots \oplus R(\beta_n) \xrightarrow{A} R(\alpha_1) \oplus \cdots \oplus R(\alpha_m).$$

- Given $B \in M_{r\times n}(R)[\overline{\lambda}][\overline{\beta}]$ and $C \in M_{m\times r}(R)[\overline{\alpha}][\overline{\lambda}]$, we say that C, B are compatible.

Morphisms between f. g. graded free modules

- A matrix $(a_{ij}) = A \in M_{m\times n}(R)[\overline{\alpha}][\overline{\beta}]$ if there are $\overline{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \Gamma^m$, $\overline{\beta} = (\beta_1, \ldots, \beta_n) \in \Gamma^n$ such that

$$a_{ij} \in R_{\alpha_i \beta_j^{-1}}$$

  In other words, A defines a morphism of Γ-graded R-modules

$$R(\beta_1) \oplus \cdots \oplus R(\beta_n) \xrightarrow{A} R(\alpha_1) \oplus \cdots \oplus R(\alpha_m).$$

- Given $B \in M_{r\times n}(R)[\overline{\lambda}][\overline{\beta}]$ and $C \in M_{m\times r}(R)[\overline{\alpha}][\overline{\lambda}]$, we say that C, B are compatible.
  Then $CB \in M_{m\times n}[\overline{\alpha}][\overline{\beta}]$ defines the composition of homomorphisms of Γ-graded free R-modules

$$R(\beta_1)\oplus\cdots\oplus R(\beta_n) \xrightarrow{B} R(\lambda_1)\oplus\cdots\oplus R(\lambda_r) \xrightarrow{C} R(\alpha_1)\oplus\cdots\oplus R(\alpha_m)$$

Morphisms between f. g. graded free modules

- A matrix $(a_{ij}) = A \in M_{m\times n}(R)[\overline{\alpha}][\overline{\beta}]$ if there are $\overline{\alpha} = (\alpha_1, \dots, \alpha_m) \in \Gamma^m$, $\overline{\beta} = (\beta_1, \dots, \beta_n) \in \Gamma^n$ such that

$$a_{ij} \in R_{\alpha_i \beta_j^{-1}}$$

In other words, A defines a morphism of Γ-graded R-modules

$$R(\beta_1) \oplus \cdots \oplus R(\beta_n) \xrightarrow{A} R(\alpha_1) \oplus \cdots \oplus R(\alpha_m).$$

- Given $B \in M_{r\times n}(R)[\overline{\lambda}][\overline{\beta}]$ and $C \in M_{m\times r}(R)[\overline{\alpha}][\overline{\lambda}]$, we say that C, B are compatible.
Then $CB \in M_{m\times n}[\overline{\alpha}][\overline{\beta}]$ defines the composition of homomorphisms of Γ-graded free R-modules

$$R(\beta_1)\oplus\cdots\oplus R(\beta_n) \xrightarrow{B} R(\lambda_1)\oplus\cdots\oplus R(\lambda_r) \xrightarrow{C} R(\alpha_1)\oplus\cdots\oplus R(\alpha_m)$$

- $\mathfrak{M}(R) = \bigcup_{n,\overline{\alpha},\overline{\beta}} M_n(R)[\overline{\alpha}][\overline{\beta}], \qquad \mathfrak{M}_\bullet(R) = \bigcup_{m,n,\overline{\alpha},\overline{\beta}} M_{m\times n}(R)[\overline{\alpha}][\overline{\beta}].$

Universal localization

Let R be a Γ-graded ring and $\Sigma \subseteq \mathfrak{M}(R)$.

The universal localization of R at Σ is a pair (R_Σ, λ) where

- $\lambda\colon R \to R_\Sigma$ is a ring homomorphism.
- For every $A \in \Sigma$, A^λ is invertible. (λ is Σ-inverting)
- For any other ring homomorphism $f\colon R \to S$ such that A^f is invertible for all $A \in \Sigma$, there exists a unique ring homomorphism $F\colon R_\Sigma \to S$ with $f = F\lambda$.

Universal localization

Let R be a Γ-graded ring and $\Sigma \subseteq \mathfrak{M}(R)$.

The universal localization of R at Σ is a pair (R_Σ, λ) where

- $\lambda\colon R \to R_\Sigma$ is a ring homomorphism.
- For every $A \in \Sigma$, A^λ is invertible. (λ is Σ-inverting)
- For any other ring homomorphism $f\colon R \to S$ such that A^f is invertible for all $A \in \Sigma$, there exists a unique ring homomorphism $F\colon R_\Sigma \to S$ with $f = F\lambda$.

The universal localization of R at Σ always exists.

Proposition (Kawai, S.)

Let R be a Γ-graded ring and $\Sigma \subseteq \mathfrak{M}(R)$. The universal localization $\lambda\colon R \to R_\Sigma$ is a homomorphism of Γ-graded rings.

Graded epic R-division rings

Theorem (Kawai, S.)

Let R be a Γ-graded ring.

- *If $\Sigma \subseteq \mathfrak{M}(R)$ is such that R_Σ is a Γ-graded local ring with maximal graded ideal $\mathfrak{m}$, then the natural homomorphism $R \to R_\Sigma/\mathfrak{m}$ is a Γ-graded epic R-division ring.*

Graded epic R-division rings

Theorem (Kawai, S.)

Let R be a Γ-graded ring.

- *If $\Sigma \subseteq \mathfrak{M}(R)$ is such that R_Σ is a Γ-graded local ring with maximal graded ideal $\mathfrak{m}$, then the natural homomorphism $R \to R_\Sigma/\mathfrak{m}$ is a Γ-graded epic R-division ring.*
- *Let $\varphi \colon R \to K$ be a Γ-graded epic R-division ring and set*

$$\Sigma = \{A \in \mathfrak{M}(R) \mid A^\varphi \textit{ is invert. over } D\}.$$

Then R_Σ is a Γ-graded local ring with maximal ideal $\mathfrak{m}$ and there exists commutative diagram

$$\begin{array}{ccccc} R & \xrightarrow{\lambda} & R_\Sigma & \xrightarrow{\pi} & R_\Sigma/\mathfrak{m} \\ & {\scriptstyle\varphi}\searrow & \downarrow{\scriptstyle\Phi} & \swarrow{\scriptstyle\widetilde{\Phi}} & \\ & & K & & \end{array}$$

with $\widetilde{\Phi}$ and isomorphism of Γ-graded rings.

Graded modules over graded division rings

Let $D = \bigoplus_{\gamma \in \Gamma} D_\gamma$ be a Γ-graded division ring.
Let $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ be a Γ-graded D-module.

- Every Γ-graded D-module M is a Γ-graded free module.

Graded modules over graded division rings

Let $D = \bigoplus_{\gamma \in \Gamma} D_\gamma$ be a Γ-graded division ring.
Let $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ be a Γ-graded D-module.

- Every Γ-graded D-module M is a Γ-graded free module.
- Any two homogeneous bases of M have the same cardinality $= \dim_D M$.

Graded modules over graded division rings

Let $D = \bigoplus_{\gamma\in\Gamma} D_\gamma$ be a Γ-graded division ring.
Let $M = \bigoplus_{\gamma\in\Gamma} M_\gamma$ be a Γ-graded D-module.

- Every Γ-graded D-module M is a Γ-graded free module.
- Any two homogeneous bases of M have the same cardinality $= \dim_D M$.
- If $A \in M_{m\times n}(D)[\overline{\alpha}][\overline{\beta}]$, we can define the rank of A:

$$A = (a_{ij})\colon D(\beta_1) \oplus \cdots \oplus D(\beta_n) \to D(\alpha_1) \oplus \cdots \oplus D(\alpha_m).$$

$$\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} \in \Big(D(\alpha_1) \oplus \cdots \oplus D(\alpha_m)\Big)_{\beta_j^{-1}}$$

rank A $= \dim_D$ of graded submodule gen. by columns of A.

gr-Sylvester matrix rank functions

Let R be a Γ-graded ring.

A gr-Sylvester matrix rank function for R is a map $\mathrm{r}\colon \mathfrak{M}_\bullet(R) \to \mathbb{N}$ with

gr-Sylvester matrix rank functions

Let R be a Γ-graded ring.

A gr-Sylvester matrix rank function for R is a map $\mathrm{r}\colon \mathfrak{M}_\bullet(R) \to \mathbb{N}$ with

(MatRF1) $\mathrm{r}((1)) = 1$, where (1) is the identity matrix of size 1×1.

gr-Sylvester matrix rank functions

Let R be a Γ-graded ring.

A gr-Sylvester matrix rank function for R is a map $\mathrm{r}\colon \mathfrak{M}_\bullet(R) \to \mathbb{N}$ with

(MatRF1) $\mathrm{r}((1)) = 1$, where (1) is the identity matrix of size 1×1.

(MatRF2) $\mathrm{r}(AB) \leq \min\{\mathrm{r}(A), \mathrm{r}(B)\}$ for compatible matrices $A, B \in \mathfrak{M}_\bullet(R)$.

gr-Sylvester matrix rank functions

Let R be a Γ-graded ring.

A gr-Sylvester matrix rank function for R is a map $\mathrm{r}\colon \mathfrak{M}_\bullet(R) \to \mathbb{N}$ with

(MatRF1) $\mathrm{r}((1)) = 1$, where (1) is the identity matrix of size 1×1.

(MatRF2) $\mathrm{r}(AB) \leq \min\{\mathrm{r}(A), \mathrm{r}(B)\}$ for compatible matrices $A, B \in \mathfrak{M}_\bullet(R)$.

(MatRF3) $\mathrm{r}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \mathrm{r}(A) + \mathrm{r}(B)$ for all $A, B \in \mathfrak{M}_\bullet(R)$.

gr-Sylvester matrix rank functions

Let R be a Γ-graded ring.

A gr-Sylvester matrix rank function for R is a map $\mathrm{r}\colon \mathfrak{M}_\bullet(R) \to \mathbb{N}$ with

(MatRF1) $\mathrm{r}((1)) = 1$, where (1) is the identity matrix of size 1×1.

(MatRF2) $\mathrm{r}(AB) \leq \min\{\mathrm{r}(A), \mathrm{r}(B)\}$ for compatible matrices $A, B \in \mathfrak{M}_\bullet(R)$.

(MatRF3) $\mathrm{r}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \mathrm{r}(A) + \mathrm{r}(B)$ for all $A, B \in \mathfrak{M}_\bullet(R)$.

(MatRF4) $\mathrm{r}\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) \geq \mathrm{r}(A) + \mathrm{r}(B)$ for all $A, B, C \in \mathfrak{M}_\bullet(R)$ s.t.$\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) \in \mathfrak{M}_\bullet(R)$.

gr-Sylvester matrix rank functions

Let R be a Γ-graded ring.

A gr-Sylvester matrix rank function for R is a map $\mathrm{r}\colon \mathfrak{M}_\bullet(R) \to \mathbb{N}$ with

(MatRF1) $\mathrm{r}((1)) = 1$, where (1) is the identity matrix of size 1×1.

(MatRF2) $\mathrm{r}(AB) \leq \min\{\mathrm{r}(A), \mathrm{r}(B)\}$ for compatible matrices $A, B \in \mathfrak{M}_\bullet(R)$.

(MatRF3) $\mathrm{r}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \mathrm{r}(A) + \mathrm{r}(B)$ for all $A, B \in \mathfrak{M}_\bullet(R)$.

(MatRF4) $\mathrm{r}\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \geq \mathrm{r}(A) + \mathrm{r}(B)$ for all $A, B, C \in \mathfrak{M}_\bullet(R)$ s.t. $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \mathfrak{M}_\bullet(R)$.

If $\varphi\colon R \to D$ is a homomorphism of Γ-graded rings with D a Γ-graded division ring then

$$\mathfrak{M}_\bullet(R) \to \mathbb{N}, \quad A \mapsto \operatorname{rank}(A^\varphi)$$

is a gr-Sylvester matrix rank function.

gr-Sylvester matrix rank functions

Theorem (Kawai, S.)

Let R be a Γ-graded ring. There is an anti-isomorphism of partially ordered sets

$$\left\{ \begin{array}{c} \Gamma\text{-graded epic } R\text{-division rings} \\ \varphi\colon R \to K \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{gr-Sylvester matrix rank} \\ \text{functions for } R \end{array} \right\}$$

$$\begin{array}{rcl} \varphi\colon R \to K & \longmapsto & \mathrm{r}_{\varphi} \\ \varphi_{\mathrm{r}}\colon R \to R_{\Sigma_{\mathrm{r}}}/\mathfrak{m} & \longleftarrow & \mathrm{r} \end{array}$$

defined as follows.

gr-Sylvester matrix rank functions

Theorem (Kawai, S.)

Let R be a Γ-graded ring. There is an anti-isomorphism of partially ordered sets

$$\left\{ \begin{array}{c} \Gamma\textit{-graded epic } R\textit{-division rings} \\ \varphi\colon R \to K \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \textit{gr-Sylvester matrix rank} \\ \textit{functions for } R \end{array} \right\}$$

$$\varphi\colon R \to K \longmapsto \mathrm{r}_\varphi$$

$$\varphi_{\mathrm{r}}\colon R \to R_{\Sigma_{\mathrm{r}}}/\mathfrak{m} \longleftarrow \mathrm{r}$$

defined as follows.

- *If (K, φ) is a Γ-graded epic R-division ring, then*

$$\mathrm{r}_\varphi(A) = \mathrm{rank}(A^\varphi).$$

gr-Sylvester matrix rank functions

Theorem (Kawai, S.)

Let R be a Γ-graded ring. There is an anti-isomorphism of partially ordered sets

$$\left\{ \begin{array}{c} \Gamma\textit{-graded epic } R\textit{-division rings} \\ \varphi\colon R \to K \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \textit{gr-Sylvester matrix rank} \\ \textit{functions for } R \end{array} \right\}$$

$$\begin{array}{rcl} \varphi\colon R \to K & \longmapsto & \mathrm{r}_\varphi \\ \varphi_{\mathrm{r}}\colon R \to R_{\Sigma_{\mathrm{r}}}/\mathfrak{m} & \longleftarrow & \mathrm{r} \end{array}$$

defined as follows.

- *If (K, φ) is a Γ-graded epic R-division ring, then*

$$\mathrm{r}_\varphi(A) = \mathrm{rank}(A^\varphi).$$

- *Conversely, given a gr-Sylvester matrix rank function $\mathrm{r}\colon \mathfrak{M}_\bullet(R) \to \mathbb{N}$, then set*

$$\Sigma_{\mathrm{r}} = \{A \in \mathfrak{M}(R)\colon\ \mathrm{r}(A) = \textit{size of } A\}.$$

  Then $R_{\Sigma_{\mathrm{r}}}$ is a Γ-graded local ring with maximal graded ideal $\mathfrak{m}$.

Tensor product of graded modules

Let R be a Γ-graded ring.

Let M be a Γ-graded R-module and N be a Γ-graded left R-module.

- $M \otimes_R N = \bigoplus_{\gamma \in \Gamma} (M \otimes_R N)_\gamma$ is a Γ-graded abelian group with

$$(M \otimes_R N)_\gamma = \left\{ \sum_i m_i \otimes n_i \mid m_i \in M_{\gamma_i},\ n_i \in N_{\gamma_i'},\ \gamma_i \gamma_i' = \gamma \right\}$$

Tensor product of graded modules

Let R be a Γ-graded ring.

Let M be a Γ-graded R-module and N be a Γ-graded left R-module.

- $M \otimes_R N = \bigoplus_{\gamma \in \Gamma} (M \otimes_R N)_\gamma$ is a Γ-graded abelian group with

$$(M \otimes_R N)_\gamma = \left\{ \sum_i m_i \otimes n_i \mid m_i \in M_{\gamma_i},\ n_i \in N_{\gamma_i'},\ \gamma_i \gamma_i' = \gamma \right\}$$

- If $R \to D$ is a homomorphism of Γ-graded rings, then $M \otimes_R D$ is a Γ-graded D-module.

gr-Sylvester module rank functions

Let R be a Γ-graded ring.

A gr-Sylvester module rank function for R is a function d on the class of finitely presented Γ-graded R-modules with values on $\mathbb{N}$ such that

(ModRF1) $\mathrm{d}(R) = 1$, where R is considered as a Γ-graded R-module

gr-Sylvester module rank functions

Let R be a Γ-graded ring.

A gr-Sylvester module rank function for R is a function d on the class of finitely presented Γ-graded R-modules with values on $\mathbb{N}$ such that

(ModRF1) $\mathrm{d}(R) = 1$, where R is considered as a Γ-graded R-module

(ModRF2) $\mathrm{d}(M_1 \oplus M_2) = \mathrm{d}(M_1) + \mathrm{d}(M_2)$.

gr-Sylvester module rank functions

Let R be a Γ-graded ring.

A gr-Sylvester module rank function for R is a function d on the class of finitely presented Γ-graded R-modules with values on $\mathbb{N}$ such that

(ModRF1) $\mathrm{d}(R) = 1$, where R is considered as a Γ-graded R-module

(ModRF2) $\mathrm{d}(M_1 \oplus M_2) = \mathrm{d}(M_1) + \mathrm{d}(M_2)$.

(ModRF3) For any exact sequence $M_1 \to M_2 \to M_3 \to 0$ of graded homomorphisms $\mathrm{d}(M_3) \leq \mathrm{d}(M_2) \leq \mathrm{d}(M_1) + \mathrm{d}(M_3)$.

gr-Sylvester module rank functions

Let R be a Γ-graded ring.

A gr-Sylvester module rank function for R is a function d on the class of finitely presented Γ-graded R-modules with values on $\mathbb{N}$ such that

(ModRF1) $\mathrm{d}(R) = 1$, where R is considered as a Γ-graded R-module

(ModRF2) $\mathrm{d}(M_1 \oplus M_2) = \mathrm{d}(M_1) + \mathrm{d}(M_2)$.

(ModRF3) For any exact sequence $M_1 \to M_2 \to M_3 \to 0$ of graded homomorphisms $\mathrm{d}(M_3) \leq \mathrm{d}(M_2) \leq \mathrm{d}(M_1) + \mathrm{d}(M_3)$.

(ModRF4) Let $f\colon R^n(\overline{\beta}) \to R^m(\overline{\alpha})$ and $f'\colon R^n(\overline{\beta'}) \to R^m(\overline{\alpha'})$ be homomorphisms of Γ-graded R-modules. If $\mathcal{F}(f) = \mathcal{F}(f')$, then $\mathrm{d}(\mathrm{coker}\, f) = \mathrm{d}(\mathrm{coker}\, f')$, where $\mathcal{F}$ denotes the forgetful functor from the category of finitely presented Γ-graded R-modules to the category of finitely presented R-modules.

gr-Sylvester module rank functions

Let R be a Γ-graded ring.

A gr-Sylvester module rank function for R is a function d on the class of finitely presented Γ-graded R-modules with values on $\mathbb{N}$ such that

(ModRF1) $\mathrm{d}(R) = 1$, where R is considered as a Γ-graded R-module

(ModRF2) $\mathrm{d}(M_1 \oplus M_2) = \mathrm{d}(M_1) + \mathrm{d}(M_2)$.

(ModRF3) For any exact sequence $M_1 \to M_2 \to M_3 \to 0$ of graded homomorphisms $\mathrm{d}(M_3) \leq \mathrm{d}(M_2) \leq \mathrm{d}(M_1) + \mathrm{d}(M_3)$.

(ModRF4) Let $f\colon R^n(\overline{\beta}) \to R^m(\overline{\alpha})$ and $f'\colon R^n(\overline{\beta'}) \to R^m(\overline{\alpha'})$ be homomorphisms of Γ-graded R-modules. If $\mathcal{F}(f) = \mathcal{F}(f')$, then $\mathrm{d}(\operatorname{coker} f) = \mathrm{d}(\operatorname{coker} f')$, where $\mathcal{F}$ denotes the forgetful functor from the category of finitely presented Γ-graded R-modules to the category of finitely presented R-modules.

If $\varphi\colon R \to D$ is a homomorphism of Γ-graded rings with D a Γ-graded division ring then
$$M \mapsto \dim_D(M \otimes_R D)$$
is a gr-Sylvester module rank function.

gr-Sylvester module rank functions

Theorem (Kawai, S.)

Let R be a Γ-graded ring. There is an anti-isomorphism of partially ordered sets

$$\left\{ \begin{array}{c} \textit{gr-Sylvester matrix rank} \\ \textit{functions for } R \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \textit{gr-Sylvester module rank} \\ \textit{functions for } R \end{array} \right\}$$

$$\mathrm{r} \longmapsto \mathrm{d}_{\mathrm{r}}$$

$$\mathrm{r}_{\mathrm{d}} \longleftarrow \mathrm{d}$$

defined as follows.

gr-Sylvester module rank functions

Theorem (Kawai, S.)

Let R be a Γ-graded ring. There is an anti-isomorphism of partially ordered sets

$$\left\{ \begin{array}{c} \textit{gr-Sylvester matrix rank} \\ \textit{functions for } R \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \textit{gr-Sylvester module rank} \\ \textit{functions for } R \end{array} \right\}$$

$$\mathrm{r} \longmapsto \mathrm{d}_{\mathrm{r}}$$

$$\mathrm{r}_{\mathrm{d}} \longleftarrow \mathrm{d}$$

defined as follows.

- *If r is a gr-Sylvester matrix rank function for R and M is a finitely presented Γ-graded R-module with presentation $R^n(\overline{\beta}) \xrightarrow{A} R^m(\overline{\alpha}) \to M \to 0$, where $A \in M_{m\times n}(R)[\overline{\alpha}][\overline{\beta}]$, then*

$$\mathrm{d}_{\mathrm{r}}(M) = m - \mathrm{r}(A).$$

gr-Sylvester module rank functions

Theorem (Kawai, S.)

Let R be a Γ-graded ring. There is an anti-isomorphism of partially ordered sets

$$\left\{ \begin{array}{c} \textit{gr-Sylvester matrix rank} \\ \textit{functions for } R \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \textit{gr-Sylvester module rank} \\ \textit{functions for } R \end{array} \right\}$$

$$\begin{array}{ccc} \mathrm{r} & \longmapsto & \mathrm{d}_{\mathrm{r}} \\ \mathrm{r}_{\mathrm{d}} & \longleftarrow & \mathrm{d} \end{array}$$

defined as follows.

- *If r is a gr-Sylvester matrix rank function for R and M is a finitely presented Γ-graded R-module with presentation $R^n(\overline{\beta}) \xrightarrow{A} R^m(\overline{\alpha}) \to M \to 0$, where $A \in M_{m\times n}(R)[\overline{\alpha}][\overline{\beta}]$, then*
$$\mathrm{d}_{\mathrm{r}}(M) = m - \mathrm{r}(A).$$
- *Conversely, let d be a gr-Sylvester module rank function for R. If $A \in M_{m\times n}(R)[\overline{\alpha}][\overline{\beta}]$, we consider $A\colon R^n(\overline{\beta}) \to R^m(\overline{\alpha})$ and define*
$$\mathrm{r}(A) = m - \mathrm{d}\left(\frac{R^m(\overline{\alpha})}{A(R^n(\overline{\beta}))}\right).$$

Lidia's question

Let k be a field and $k\langle x, y\rangle$ the free k-algebra on two letters

Lidia's question

Let k be a field and $k\langle x, y\rangle$ the free k-algebra on two letters

Is there a classification of the universal localizations of $k\langle x, y\rangle$?

The free field

The universal localization of $k\langle x, y\rangle$ at the set Φ of full (square) matrices is a division ring, called the free field **(P. M. Cohn)**.

The free field

The universal localization of $k\langle x, y\rangle$ at the set Φ of full (square) matrices is a division ring, called the free field **(P. M. Cohn)**.

- Let F be the free group on two letters x, y

The free field

The universal localization of $k\langle x, y\rangle$ at the set Φ of full (square) matrices is a division ring, called the free field **(P. M. Cohn)**.

- Let F be the free group on two letters x, y
- F is an orderable group:

  There exists a total (linear) order in F such that, for all $u, v, z \in F$,

$$u < v \implies uz < vz \quad \text{and} \quad zu < zv.$$

The free field

The universal localization of $k\langle x,y\rangle$ at the set Φ of full (square) matrices is a division ring, called the free field **(P. M. Cohn)**.

- Let F be the free group on two letters x, y
- F is an orderable group:

  There exists a total (linear) order in F such that, for all $u, v, z \in F$,

$$u < v \implies uz < vz \quad \text{and} \quad zu < zv.$$

- Group k-algebra $k[F]$

$$k[F] = \left\{ f = \sum_{g \in F} a_g g \;\middle|\; a_g \in k,\ |\operatorname{supp} f| < \infty \right\}$$

The free field

The universal localization of $k\langle x, y\rangle$ at the set Φ of full (square) matrices is a division ring, called the free field **(P. M. Cohn)**.

- Let F be the free group on two letters x, y
- F is an orderable group:

  There exists a total (linear) order in F such that, for all $u, v, z \in F$,

  $$u < v \implies uz < vz \quad \text{and} \quad zu < zv.$$

- Group k-algebra $k[F]$

  $$k[F] = \left\{ f = \sum_{g \in F} a_g g \;\middle|\; a_g \in k,\ |\operatorname{supp} f| < \infty \right\}$$

- Malcev-Neumann series ring: Fix $<$ in F as before. Then

  $$k((F;<)) = \left\{ f = \sum_{g \in F} a_g g \;\middle|\; a_g \in k,\ \operatorname{supp} f \text{ is well-ordered} \right\}$$

The free field

The universal localization of $k\langle x, y\rangle$ at the set Φ of full (square) matrices is a division ring, called the free field **(P. M. Cohn)**.

- Let F be the free group on two letters x, y
- F is an orderable group:

  There exists a total (linear) order in F such that, for all $u, v, z \in F$,

$$u < v \implies uz < vz \quad \text{and} \quad zu < zv.$$

- Group k-algebra $k[F]$

$$k[F] = \left\{ f = \sum_{g \in F} a_g g \;\middle|\; a_g \in k,\ |\operatorname{supp} f| < \infty \right\}$$

- Malcev-Neumann series ring: Fix $<$ in F as before. Then

$$k((F;<)) = \left\{ f = \sum_{g \in F} a_g g \;\middle|\; a_g \in k,\ \operatorname{supp} f \text{ is well-ordered} \right\}$$

  $k((F;<))$ is a division ring **(Malcev, Neumann)**

The free field

The universal localization of $k\langle x, y\rangle$ at the set Φ of full (square) matrices is a division ring, called the free field **(P. M. Cohn)**.

- Let F be the free group on two letters x, y
- F is an orderable group:

  There exists a total (linear) order in F such that, for all $u, v, z \in F$,

  $$u < v \implies uz < vz \quad \text{and} \quad zu < zv.$$

- Group k-algebra $k[F]$

  $$k[F] = \left\{ f = \sum_{g \in F} a_g g \;\middle|\; a_g \in k,\ |\operatorname{supp} f| < \infty \right\}$$

- Malcev-Neumann series ring: Fix $<$ in F as before. Then

  $$k((F;<)) = \left\{ f = \sum_{g \in F} a_g g \;\middle|\; a_g \in k,\ \operatorname{supp} f \text{ is well-ordered} \right\}$$

  $k((F;<))$ is a division ring **(Malcev, Neumann)**

- $k(F)$, the division subring of $k((F;<))$ generated by $k[F]$ is the free field **(J. Lewin)**

Graded division rings inside $k(F)$

- Let H be a *torsion-free* group generated by two elements a, b.
  Let $\varphi\colon F \to H$, $x \mapsto a$, $y \mapsto b$ be a group homomorphism.

Graded division rings inside $k(F)$

- Let H be a *torsion-free* group generated by two elements a, b.
  Let $\varphi\colon F \to H$, $x \mapsto a$, $y \mapsto b$ be a group homomorphism.
- φ induces an H-gradation of $k[F]$:

$$k[F] = \bigoplus_{h \in H} k[F]_h, \quad k[F]_h = \left\{ \sum_{w \in F} a_w w \in k[F] \;\middle|\; \varphi(w) = h \right\}$$

Graded division rings inside $k(F)$

- Let H be a *torsion-free* group generated by two elements a, b. Let $\varphi\colon F \to H$, $x \mapsto a$, $y \mapsto b$ be a group homomorphism.
- φ induces an H-gradation of $k[F]$:

$$k[F] = \bigoplus_{h \in H} k[F]_h, \quad k[F]_h = \left\{ \sum_{w \in F} a_w w \in k[F] \;\middle|\; \varphi(w) = h \right\}$$

- Also H-graded division rings inside $k(F)$:

$$D_\varphi(H) = \bigoplus_{h \in H} D_\varphi(H)_h, \quad D_\varphi(H)_h = \left\{ \sum_{w \in F} a_w w \in k(F) \;\middle|\; \varphi(w) = h \right\}$$

Graded division rings inside $k(F)$

- Let H be a *torsion-free* group generated by two elements a, b.
  Let $\varphi\colon F \to H$, $x \mapsto a$, $y \mapsto b$ be a group homomorphism.
- φ induces an H-gradation of $k[F]$:

$$k[F] = \bigoplus_{h \in H} k[F]_h, \quad k[F]_h = \left\{ \sum_{w \in F} a_w w \in k[F] \;\middle|\; \varphi(w) = h \right\}$$

- Also H-graded division rings inside $k(F)$:

$$D_\varphi(H) = \bigoplus_{h \in H} D_\varphi(H)_h, \quad D_\varphi(H)_h = \left\{ \sum_{w \in F} a_w w \in k(F) \;\middle|\; \varphi(w) = h \right\}$$

- If $H_1 \not\cong H_2$, then $\ker \varphi_1 \neq \ker \varphi_2$. Let $1 < w \in \ker \varphi_1 \setminus \ker \varphi_2$.

Graded division rings inside $k(F)$

- Let H be a *torsion-free* group generated by two elements a, b. Let $\varphi\colon F \to H$, $x \mapsto a$, $y \mapsto b$ be a group homomorphism.
- φ induces an H-gradation of $k[F]$:

$$k[F] = \bigoplus_{h \in H} k[F]_h, \quad k[F]_h = \left\{ \sum_{w \in F} a_w w \in k[F] \;\middle|\; \varphi(w) = h \right\}$$

- Also H-graded division rings inside $k(F)$:

$$D_\varphi(H) = \bigoplus_{h \in H} D_\varphi(H)_h, \quad D_\varphi(H)_h = \left\{ \sum_{w \in F} a_w w \in k(F) \;\middle|\; \varphi(w) = h \right\}$$

- If $H_1 \not\cong H_2$, then $\ker \varphi_1 \neq \ker \varphi_2$. Let $1 < w \in \ker \varphi_1 \setminus \ker \varphi_2$. $1 - w$ is homogeneous of degree e and invertible in $D_{\varphi_1}(H_1)$,

Graded division rings inside $k(F)$

- Let H be a *torsion-free* group generated by two elements a, b. Let $\varphi\colon F \to H$, $x \mapsto a$, $y \mapsto b$ be a group homomorphism.
- φ induces an H-gradation of $k[F]$:

$$k[F] = \bigoplus_{h \in H} k[F]_h, \quad k[F]_h = \left\{ \sum_{w \in F} a_w w \in k[F] \;\middle|\; \varphi(w) = h \right\}$$

- Also H-graded division rings inside $k(F)$:

$$D_\varphi(H) = \bigoplus_{h \in H} D_\varphi(H)_h, \quad D_\varphi(H)_h = \left\{ \sum_{w \in F} a_w w \in k(F) \;\middle|\; \varphi(w) = h \right\}$$

- If $H_1 \not\cong H_2$, then $\ker \varphi_1 \neq \ker \varphi_2$. Let $1 < w \in \ker \varphi_1 \setminus \ker \varphi_2$. $1 - w$ is homogeneous of degree e and invertible in $D_{\varphi_1}(H_1)$, but $1 - w$ is not homogeneous and not invertible in $D_{\varphi_2}(H_2)$.

Graded division rings inside $k(F)$

- Let H be a *torsion-free* group generated by two elements a, b. Let $\varphi\colon F \to H$, $x \mapsto a$, $y \mapsto b$ be a group homomorphism.
- φ induces an H-gradation of $k[F]$:

$$k[F] = \bigoplus_{h\in H} k[F]_h, \quad k[F]_h = \left\{ \sum_{w\in F} a_w w \in k[F] \;\middle|\; \varphi(w) = h \right\}$$

- Also H-graded division rings inside $k(F)$:

$$D_\varphi(H) = \bigoplus_{h\in H} D_\varphi(H)_h, \quad D_\varphi(H)_h = \left\{ \sum_{w\in F} a_w w \in k(F) \;\middle|\; \varphi(w) = h \right\}$$

- If $H_1 \ncong H_2$, then $\ker\varphi_1 \neq \ker\varphi_2$. Let $1 < w \in \ker\varphi_1 \setminus \ker\varphi_2$. $1 - w$ is homogeneous of degree e and invertible in $D_{\varphi_1}(H_1)$, but $1 - w$ is not homogeneous and not invertible in $D_{\varphi_2}(H_2)$.
- There are uncountably many nonisomorphic torsion-free groups H Thus uncountably many $D_\varphi(H)$ inside $K(H)$ which are not R-isomorphic. If $\ker\varphi_1 \subseteq \ker\varphi_2$, then $D_{\varphi_1}(H_1) \subseteq D_{\varphi_2}(H_2)$.

Each $D_\varphi(H)$ is a universal localization

$A \in \mathfrak{M}(R)$ is nonfull if there exist compatible homogeneous matrices

$$\begin{array}{ccc} R(\beta_1)\oplus\cdots\oplus R(\beta_n) & \xrightarrow{\quad A \quad} & R(\alpha_1)\oplus\cdots\oplus R(\alpha_n) \\ & {\searrow}^{B} \qquad {\nearrow}^{C} & \\ & R(\lambda_1)\oplus\cdots\oplus R(\lambda_{n-1}) & \end{array}$$

Each $D_\varphi(H)$ is a universal localization

$A \in \mathfrak{M}(R)$ is nonfull if there exist compatible homogeneous matrices

$$\begin{array}{ccc} R(\beta_1) \oplus \cdots \oplus R(\beta_n) & \xrightarrow{A} & R(\alpha_1) \oplus \cdots \oplus R(\alpha_n) \\ & {\scriptstyle B} \searrow \quad \nearrow {\scriptstyle C} & \\ & R(\lambda_1) \oplus \cdots \oplus R(\lambda_{n-1}) & \end{array}$$

Let H and $\varphi\colon F \to H$ be as before and consider the induced gradation on $k\langle x, y\rangle$.

Consider $\vartheta\colon k\langle x, y\rangle \hookrightarrow D_\varphi(H)$

Each $D_\varphi(H)$ is a universal localization

$A \in \mathfrak{M}(R)$ is nonfull if there exist compatible homogeneous matrices

$$\begin{array}{ccc} R(\beta_1)\oplus\cdots\oplus R(\beta_n) & \xrightarrow{\quad A \quad} & R(\alpha_1)\oplus\cdots\oplus R(\alpha_n) \\ {\scriptstyle B}\searrow & & \nearrow{\scriptstyle C} \\ & R(\lambda_1)\oplus\cdots\oplus R(\lambda_{n-1}) & \end{array}$$

Let H and $\varphi\colon F \to H$ be as before and consider the induced gradation on $k\langle x, y\rangle$.

Consider $\vartheta\colon k\langle x, y\rangle \hookrightarrow D_\varphi(H)$

- **Reutenauer's method:**
  Let $A \in \mathfrak{M}^H(k\langle x, y\rangle)$ such that A^ϑ is not invertible.

Each $D_\varphi(H)$ is a universal localization

$A \in \mathfrak{M}(R)$ is nonfull if there exist compatible homogeneous matrices

$$\begin{array}{ccc} R(\beta_1) \oplus \cdots \oplus R(\beta_n) & \xrightarrow{\quad A \quad} & R(\alpha_1) \oplus \cdots \oplus R(\alpha_n) \\ & {\scriptstyle B} \searrow \qquad \nearrow {\scriptstyle C} & \\ & R(\lambda_1) \oplus \cdots \oplus R(\lambda_{n-1}) & \end{array}$$

Let H and $\varphi \colon F \to H$ be as before and consider the induced gradation on $k\langle x, y\rangle$.

Consider $\vartheta \colon k\langle x, y\rangle \hookrightarrow D_\varphi(H)$

- **Reutenauer's method:**
  Let $A \in \mathfrak{M}^H(k\langle x, y\rangle)$ such that A^ϑ is not invertible.

$$A^\vartheta \colon D_\varphi(H)(\overline{\sigma}) \to D_\varphi(H)(\overline{\tau}), \qquad \text{is not injective}$$

Each $D_\varphi(H)$ is a universal localization

$A \in \mathfrak{M}(R)$ is nonfull if there exist compatible homogeneous matrices

$$\begin{array}{ccc} R(\beta_1) \oplus \cdots \oplus R(\beta_n) & \xrightarrow{A} & R(\alpha_1) \oplus \cdots \oplus R(\alpha_n) \\ \searrow^{B} & & \nearrow^{C} \\ & R(\lambda_1) \oplus \cdots \oplus R(\lambda_{n-1}) & \end{array}$$

Let H and $\varphi\colon F \to H$ be as before and consider the induced gradation on $k\langle x, y\rangle$.

Consider $\vartheta\colon k\langle x,y\rangle \hookrightarrow D_\varphi(H)$

- **Reutenauer's method:**
  Let $A \in \mathfrak{M}^H(k\langle x,y\rangle)$ such that A^ϑ is not invertible.

  $$A^\vartheta\colon D_\varphi(H)(\overline{\sigma}) \to D_\varphi(H)(\overline{\tau}), \qquad \text{is not injective}$$

  Then there exist an identity matrix and $P, Q \in GL(k\langle x,y\rangle)$ such that $P \begin{pmatrix} A & \\ & I \end{pmatrix} Q$ has one row of zeros. Thus $\begin{pmatrix} A & \\ & I \end{pmatrix}$ is not gr-full.

Each $D_\varphi(H)$ is a universal localization

$A \in \mathfrak{M}(R)$ is nonfull if there exist compatible homogeneous matrices

$$\begin{array}{ccc} R(\beta_1)\oplus\cdots\oplus R(\beta_n) & \xrightarrow{\quad A \quad} & R(\alpha_1)\oplus\cdots\oplus R(\alpha_n) \\ & {\scriptstyle B}\searrow \qquad \nearrow{\scriptstyle C} & \\ & R(\lambda_1)\oplus\cdots\oplus R(\lambda_{n-1}) & \end{array}$$

Let H and $\varphi\colon F\to H$ be as before and consider the induced gradation on $k\langle x,y\rangle$.

Consider $\vartheta\colon k\langle x,y\rangle \hookrightarrow D_\varphi(H)$

- **Reutenauer's method:**
  Let $A\in\mathfrak{M}^H(k\langle x,y\rangle)$ such that A^ϑ is not invertible.

  $$A^\vartheta\colon D_\varphi(H)(\overline{\sigma})\to D_\varphi(H)(\overline{\tau}),\qquad \text{is not injective}$$

  Then there exist an identity matrix and $P,Q\in GL(k\langle x,y\rangle)$ such that $P\begin{pmatrix} A & \\ & I\end{pmatrix}Q$ has one row of zeros. Thus $\begin{pmatrix} A & \\ & I\end{pmatrix}$ is not gr-full.
- It implies that $A\in\mathfrak{M}^H(k\langle x,y\rangle)$ becomes invertible in $D_\varphi(H)$ if and only if A is gr-full.

Each $D_\varphi(H)$ is a universal localization

$A \in \mathfrak{M}(R)$ is nonfull if there exist compatible homogeneous matrices

$$\begin{array}{ccc} R(\beta_1)\oplus\cdots\oplus R(\beta_n) & \xrightarrow{\quad A \quad} & R(\alpha_1)\oplus\cdots\oplus R(\alpha_n) \\ & {\searrow}^{B} \qquad {\nearrow}^{C} & \\ & R(\lambda_1)\oplus\cdots\oplus R(\lambda_{n-1}) & \end{array}$$

Let H and $\varphi\colon F \to H$ be as before and consider the induced gradation on $k\langle x,y\rangle$.

Consider $\vartheta\colon k\langle x,y\rangle \hookrightarrow D_\varphi(H)$

- **Reutenauer's method:**
  Let $A \in \mathfrak{M}^H(k\langle x,y\rangle)$ such that A^ϑ is not invertible.

  $$A^\vartheta\colon D_\varphi(H)(\overline{\sigma}) \to D_\varphi(H)(\overline{\tau}), \qquad \text{is not injective}$$

  Then there exist an identity matrix and $P, Q \in GL(k\langle x,y\rangle)$ such that $P\begin{pmatrix} A & \\ & I\end{pmatrix}Q$ has one row of zeros. Thus $\begin{pmatrix} A & \\ & I\end{pmatrix}$ is not gr-full.
- It implies that $A \in \mathfrak{M}^H(k\langle x,y\rangle)$ becomes invertible in $D_\varphi(H)$ if and only if A is gr-full.
- Then $D_\varphi(H) = k\langle x,y\rangle_{\Phi_H}$, the universal localization of $k\langle x,y\rangle$ at Φ_H, the set of gr-full matrices.

Thank you!

gr-prime spectrum

Let $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ be a Γ-graded ring. Let Ω be normal subgroup of Γ.

- R can be regarded as a Γ/Ω-graded ring as follows

$$R = \bigoplus_{\Delta \in \Gamma/\Omega} R_\Delta, \quad \text{where } R_\Delta = \bigoplus_{\gamma \in \Delta} R_\gamma.$$

gr-prime spectrum

Let $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ be a Γ-graded ring. Let Ω be normal subgroup of Γ.

- R can be regarded as a Γ/Ω-graded ring as follows

$$R = \bigoplus_{\Delta \in \Gamma/\Omega} R_\Delta, \quad \text{where } R_\Delta = \bigoplus_{\gamma \in \Delta} R_\gamma.$$

- If $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ is a Γ-graded R-module, it can be regarded as a Γ/Ω-graded R-module: $M = \bigoplus_{\Delta \in \Gamma/\Omega} M_\Delta, \quad \text{where } M_\Delta = \bigoplus_{\gamma \in \Delta} M_\gamma$

gr-prime spectrum

Let $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ be a Γ-graded ring. Let Ω be normal subgroup of Γ.

- R can be regarded as a Γ/Ω-graded ring as follows

$$R = \bigoplus_{\Delta \in \Gamma/\Omega} R_\Delta, \quad \text{where } R_\Delta = \bigoplus_{\gamma \in \Delta} R_\gamma.$$

- If $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ is a Γ-graded R-module, it can be regarded as a Γ/Ω-graded R-module: $M = \bigoplus_{\Delta \in \Gamma/\Omega} M_\Delta, \ \text{where } M_\Delta = \bigoplus_{\gamma \in \Delta} M_\gamma$

- Since $\mathfrak{M}^\Gamma(R) \subseteq \mathfrak{M}^{\Gamma/\Omega}(R)$, we obtain by restriction

$$\left\{ \begin{array}{c} \Gamma/\Omega\text{-gr-prime matrix} \\ \text{ideals of } R \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \Gamma\text{-gr-prime matrix} \\ \text{ideals of } R \end{array} \right\}, \quad \mathcal{P} \mapsto \mathcal{P} \cap \mathfrak{M}^\Gamma(R).$$

gr-prime spectrum

Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring. Let Ω be normal subgroup of Γ.

- R can be regarded as a Γ/Ω-graded ring as follows
$$R = \bigoplus_{\Delta\in\Gamma/\Omega} R_\Delta, \quad \text{where } R_\Delta = \bigoplus_{\gamma\in\Delta} R_\gamma.$$
- If $M = \bigoplus_{\gamma\in\Gamma} M_\gamma$ is a Γ-graded R-module, it can be regarded as a Γ/Ω-graded R-module: $M = \bigoplus_{\Delta\in\Gamma/\Omega} M_\Delta, \ \text{where } M_\Delta = \bigoplus_{\gamma\in\Delta} M_\gamma$
- Since $\mathfrak{M}^\Gamma(R) \subseteq \mathfrak{M}^{\Gamma/\Omega}(R)$, we obtain by restriction
$$\left\{\begin{matrix}\Gamma/\Omega\text{-gr-prime matrix}\\ \text{ideals of } R\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\Gamma\text{-gr-prime matrix}\\ \text{ideals of } R\end{matrix}\right\}, \quad \mathcal{P} \mapsto \mathcal{P} \cap \mathfrak{M}^\Gamma(R).$$
$$\left\{\begin{matrix}\Gamma/\Omega\text{-gr-Sylvester matrix}\\ \text{rank functions for } R\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\Gamma\text{-gr-Sylvester matrix}\\ \text{rank functions for } R\end{matrix}\right\}, \quad \mathrm{r}^{\Gamma/\Omega} \mapsto \mathrm{r}^\Gamma.$$

gr-prime spectrum

Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring. Let Ω be normal subgroup of Γ.

- R can be regarded as a Γ/Ω-graded ring as follows

$$R = \bigoplus_{\Delta\in\Gamma/\Omega} R_\Delta, \quad \text{where } R_\Delta = \bigoplus_{\gamma\in\Delta} R_\gamma.$$

- If $M = \bigoplus_{\gamma\in\Gamma} M_\gamma$ is a Γ-graded R-module, it can be regarded as a Γ/Ω-graded R-module: $M = \bigoplus_{\Delta\in\Gamma/\Omega} M_\Delta, \; \text{where } M_\Delta = \bigoplus_{\gamma\in\Delta} M_\gamma$

- Since $\mathfrak{M}^\Gamma(R) \subseteq \mathfrak{M}^{\Gamma/\Omega}(R)$, we obtain by restriction

$$\left\{\begin{array}{c}\Gamma/\Omega\text{-gr-prime matrix}\\ \text{ideals of } R\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\Gamma\text{-gr-prime matrix}\\ \text{ideals of } R\end{array}\right\}, \quad \mathcal{P} \mapsto \mathcal{P}\cap\mathfrak{M}^\Gamma(R).$$

$$\left\{\begin{array}{c}\Gamma/\Omega\text{-gr-Sylvester matrix}\\ \text{rank functions for } R\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\Gamma\text{-gr-Sylvester matrix}\\ \text{rank functions for } R\end{array}\right\}, \quad \mathrm{r}^{\Gamma/\Omega} \mapsto \mathrm{r}^\Gamma.$$

$$\left\{\begin{array}{c}\Gamma/\Omega\text{-gr-Sylvester module}\\ \text{rank functions for } R\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\Gamma\text{-gr-Sylvester module}\\ \text{rank functions for } R\end{array}\right\}, \quad \mathrm{d}^{\Gamma/\Omega} \mapsto \mathrm{d}^\Gamma.$$

gr-prime spectrum

Theorem (Kawai, S.)

Let R be a Γ-graded ring. The following assertions hold true.

- *If there exists a normal subgroup Ω of Γ such that there does not exist a Γ/Ω-graded epic R-division ring (of fractions), then there does not exist an epic R-division ring (of fractions).*
- *Let $(K_{\mathcal{P}'}, \varphi_{\mathcal{P}'})$, $(K_{\mathcal{Q}'}, \varphi_{\mathcal{Q}'})$ be epic R-division rings, such that there exists a specialization from $(K_{\mathcal{P}'}, \varphi_{\mathcal{P}'})$ to $(K_{\mathcal{Q}'}, \varphi_{\mathcal{Q}'})$. Then there exists a gr-specialization between the corresponding Γ-graded epic R-division rings.*
- *If $\mathrm{Spec}(R) \to \mathrm{Spec}_\Gamma(R)$, $\mathcal{Q}' \mapsto \mathcal{Q}' \cap \mathfrak{M}^\Gamma(R)$, is surjective, the existence of a universal R-division ring, implies the existence of a universal Γ-graded epic R-division ring.*
- *If for each Γ-graded epic R-division ring there exist ring homomorphisms to division rings, then $\mathrm{Spec}(R) \to \mathrm{Spec}_\Gamma(R)$, $\mathcal{Q} \mapsto \mathcal{Q} \cap \mathfrak{M}^\Gamma(R)$, is surjective.*