Indecomposable pure-injective objects in stable categories of Gorenstein-projective modules over Gorenstein orders

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23rd Feb., 2022

Online satellite event to Homological methods in representation theory

In honour of Lidia Angeleri Hügel

<u>Aim</u> Give a Cohen-Macaulay version of :

<u>Thm</u> (Auslander-Ringel-Tachikawa + Model theory of modules)

A: an Artin algebra

TFAE

(1) A is of finite representation type.

(2) Every A-modules is a direct sum of finitely generated A-modules.

(3) Every indecomposable pure-injective A-module is finitely generated.

<u>Setup</u> $(R, \mathfrak{m}, k)$: a Gorenstein local ring , $d := \dim R$

$R \hookrightarrow \widehat{R} := \varprojlim_{n \geq 1} R/\mathfrak{m}^n$

$\mathrm{CM}\, R := \{ M \in \mathrm{mod}\, R \mid \mathrm{Ext}^i_R(k, M) = 0 \ \ \forall i \leq d \}$

A : an R-order, i.e., A : an R-algebra with str. map $R \xrightarrow{\varphi} A$ s.t. $\varphi_* A \in \mathrm{CM}\, R$

$$\begin{tikzcd} R \arrow[rr, "\varphi"] \arrow[dr] & \arrow[d, phantom, "\circlearrowright"] & A \\ & Z(A) \arrow[ur] & \end{tikzcd}$$

$(\varphi_* : \mathrm{Mod}\, A \to \mathrm{Mod}\, R)$

$\mathrm{CM}\, A := \{ M \in \mathrm{mod}\, A \mid \varphi_* M \in \mathrm{CM}\, R \} = \{$maxmal CM (right) A-modules$\}$

$\mathrm{GProj}\, A := \{$Gorenstein-projective A-modules$\}$

$\underline{\mathrm{CM}}A$ (resp $\underline{\mathrm{GProj}}A$) : the stable category of $\mathrm{CM}A$ (resp. $\mathrm{GProj}A$) modulo projectives

Assume A is Gorenstein, i.e., $\mathrm{Hom}_R(A,R) \in \mathrm{proj}\,A$.

Fact (1) $\mathrm{CM}A = \mathrm{mod}A \cap \mathrm{GProj}A$

(2)
$$\begin{tikzcd}
\underline{\mathrm{CM}}A \arrow[r, hook, "\mathrm{can}"] \arrow[dr, "\simeq"'] & \underline{\mathrm{GProj}}A \arrow[d, phantom, "\circlearrowleft"'] \\
& (\mathrm{GProj}A)^c \arrow[u, phantom, sloped, "\subset"]
\end{tikzcd}$$
: a triangulated functor

$\underline{\mathrm{GProj}}A$: compactly generated (Jørgensen, Chen)

(3) We can define pure-injectivity for objects in a comp. gen. tri. cat $\mathcal{T}$. (Beligiannis, Krause)

$\mathrm{Zg}(\mathcal{T}) := \{\text{indec. pure-inj. obj in } \mathcal{T}\}/\cong$

the Ziegler spectrum of $\mathcal{T}$ (a topological space)

<u>Thm</u> (N) R. a complete Gorenstein local ring
A. a Gorenstein R-order

TFAE

(1) A is of finite CM representation type.

(2) Every Gorenstein-projective A-module is a direct sum of maximal CM A-modules.

(3) Every indecomposable pure-injective object in $\underline{\mathrm{GProj}}\, A$ is compact.

Rem "(1) $\Leftrightarrow$ (2)" can be deduced from recent work of Psaroudakis and Rump (2022)

If A is commutative, "(1) $\Leftrightarrow$ (2)" is due to Beligiannis (2011)

(3) is new even if A is commutative

Sketch proof of Thm

Prop 1 (Laking) $\mathcal{T}$: a compactly generated triangulated category

$\mathcal{T} \xrightarrow{F} Ab$: a coherent functor

Assume : (i) Every indec. pure-inj. obj. $L \in \mathcal{T}^c$ admits an AR-triangule in $\mathcal{T}^c$ startin from L.

(ii) $(F) := \{ X \in Zg(\mathcal{T}) \mid F(X) \neq 0 \}$ contains infinitely many objects in $\mathcal{T}^c$ up to isom.

Then (F) contains a non-compact object in $\mathcal{T}$.

<u>Prop 2</u> (N)

$$\mathrm{Hom}_{\underline{\mathrm{CM}}A}(\Omega^d(A/\mathrm{rad}A), M) \neq 0, \quad 0 \neq {}^{\forall}M \in \underline{\mathrm{CM}}A$$
$(\underline{\mathrm{GProj}}A)$

Props 1 and 2 $\rightsquigarrow$ Thm. $\square$