DELOOPING LEVELS VERSUS FINITISTIC DIMENSIONS

Notation. • $\Lambda = KQ/I$ a path algebra modulo relations, K an algebraically closed field. Identify the vertex set of the quiver Q with a complete set of primitive idempotents of Λ: $e_1, \dots, e_n$.

• $J =$ the Jacobson radical of Λ.

• $S_i = \Lambda e_i / J e_i$ the simples in Λ-mod.

• $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ (resp., $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$) is the full subcategory of Λ-mod (resp., Λ-Mod) having as objects the modules of finite projective dimension.

• $\operatorname{l.findim} \Lambda = \sup\{\operatorname{p dim} M \mid M \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})\}$, and

• $\operatorname{l.Findim} \Lambda = \sup\{\operatorname{p dim} M \mid M \in \mathcal{P}^{<\infty}(\Lambda\text{-Mod})\}$.

1991 *Mathematics Subject Classification.* .

1. A new homological invariant of Λ: the delooping level

Definition. [Gelinas]

- For $M \in \Lambda$-mod, the *delooping level of M* is

$$\operatorname{dell} M := \inf\{k \in \mathbb{N}_0 \mid \Omega^k(M) \text{ is a direct summand of } \Omega^{k+1}(N) \oplus P \text{ for some } N \in \Lambda\text{-mod and } P \text{ proj.}\}.$$

- The *left delooping level of Λ* is

$$\operatorname{l.dell} \Lambda := \sup\{\operatorname{dell} S_i \mid 1 \le i \le n\}.$$

Comments. All of the following remarks can be found in Gelinas's paper, either ex- or im-plicitly.

(1) For indecomposable $M \in \Lambda$-mod,

$$\operatorname{dell} M = 0 \iff M \subseteq JP \text{ or } M = P \text{ for some projective } P.$$

In particular: If ${}_\Lambda S$ is simple, then $\operatorname{dell} S = 0$ if and only if $S \subseteq \operatorname{soc}({}_\Lambda\Lambda)$.

Indeed, if $S \subseteq \operatorname{soc}({}_\Lambda\Lambda)$ is non-projective, then $S \subseteq {}_\Lambda J$, and hence $S \cong \Omega^1({}_\Lambda(\Lambda/S))$ has delooping level 0.

Conversely, suppose $\operatorname{dell} S = 0$, i.e., S is a direct summand of $\Omega^1(N) \oplus P$ for some $N \in \Lambda$-mod and projective P. If S is not projective, then S is contained in $\Omega^1(N)$ and hence $S \subseteq Je_i$ for some i.

(2) [an old result of Bass re-encountered]

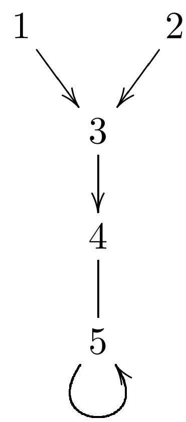
$$\operatorname{r.Findim} \Lambda = 0 \iff {}_\Lambda(\Lambda/J) \text{ embeds into } \operatorname{soc}({}_\Lambda\Lambda) \iff \operatorname{l.dell} \Lambda = 0.$$

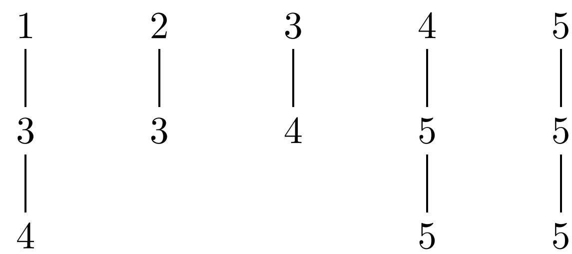
(3) If $\operatorname{gl\,dim} \Lambda < \infty$, then $\operatorname{gl\,dim} \Lambda = \operatorname{l.dell} \Lambda = \operatorname{r.dell} \Lambda$.

(4) If $\Lambda = KQ/I$ is a monomial algebra, then $\operatorname{l.dell} \Lambda$ and $\operatorname{r.dell} \Lambda$ are finite (different in general).

Indeed, for any $M \in \Lambda$-Mod and $k \ge 2$, $\Omega^k(M)$ is a direct sum of copies of cyclic modules isomorphic to Λp for nontrivial paths p in $KQ \setminus I$. In particular, the category $\Omega^2(\Lambda\text{-Mod})$ consisting of all second syzygies of Λ-modules has finite representation type. Hence there exists $N \in \mathbb{N}$ such that $\Omega^N(\Lambda\text{-Mod}) = \Omega^n(\Lambda\text{-Mod})$ for all $n \ge N$. In particular, every N-th syzygy is a higher syzygy, whence $\operatorname{l.dell} \Lambda \le N$.

First specific example.





Claim: $\operatorname{l.dell}\Lambda = 3$.

Why? The simples S_i in Λ-mod with $\operatorname{dell} S_i = 0$ are S_3, S_4, S_5.

Moreover S_1 is the only simple with delooping level 1: The delooping level is at least 1 because S_1 is not contained in the left socle of Λ; it is at most 1, because $\Omega^1(S_1)$ is projective.

The sequence of successive syzygies of S_2 is

$$\Omega^1(S_2) = S_3,\ \Omega^2(S_2) = S_4,\ \Omega^3(S_2) = Je_4 = Je_5,\ \Omega^4(S_2) = S_5,\ \Omega^5(S_2) = \Omega^3(S_2).$$

Hence $\operatorname{dell} S_2 \leq 3$.

In general, bounding delooping levels from below is the tough part, but here it's easy. Note:

- $\Omega^1(\Lambda\text{-mod})$ consists of modules of Loewy length ≤ 2 with tops in $\operatorname{add}(S_3 \oplus S_4 \oplus S_5)$;
- $\Omega^2(\Lambda\text{-mod})$ consists of modules of Loewy length ≤ 2 with tops in $\operatorname{add}(S_4 \oplus S_5)$;
- $\Omega^3(\Lambda\text{-mod})$ consists of modules of Loewy length ≤ 2 with tops in $\operatorname{add}(S_5)$.

Thus $\Omega^2(S_2) = S_4$ is not a third syzygy, and hence $\operatorname{dell} S_2 \geq 3$, which yields the claim.

2. Inequalities for finite dimensional algebras (artinian rings)

Theorem. [Gelinas] $\mathrm{l.\,depth}\,\Lambda \le \mathrm{r.\,findim}\,\Lambda \le \mathrm{r.\,Findim}\,\Lambda \le \mathrm{l.\,dell}\,\Lambda$.

Remarks. 1. The "depth" is carried over from commutative algebra: $\mathrm{l.\,depth}\,\Lambda = \sup\{\mathrm{grade}\, S_i \mid 1 \le i \le n\}$, where $\mathrm{grade}\, M = \inf\{i \in \mathbb{N}_0 \mid \mathrm{Ext}^i_\Lambda(M, \Lambda) \neq 0\}$. In fact, Gelinas considers the delooping level for arbitrary semiperfect notherian rings and shows that in this far more general scenario, the inequalities $\mathrm{l.\,depth} \le \mathrm{r.\,findim}\,\Lambda \le \mathrm{l.\,dell}\,\Lambda$ still hold.

2. Whenever we can show that, for a certain class of algebras Λ, the delooping levels are finite, this confirms the strong version of the second finitistic dimension conjecture, namely that the big finitistic dimensions attained by the algebras in this class are finite.

3. [Gelinas / Angeleri-Huegel-Herbera-Trlifaj] If Λ is Gorenstein, then the left and right delooping levels of Λ are finite and coincide; in this case the delooping levels also coincide with all the finitistic dimensions of Λ.

4. Next to Gorenstein and monomial algebras, Gelinas exhibited some further classes of algebras for which the delooping levels are finite; in essence, these are algebras with somewhat weaker forms of repetitiveness occurring in the syzygy categories. For all of these classes, the big finitistic dimensions are already known to be finite. But, also in these cases, the theorem is very useful. It helps in bounding or computing the homological dimensions.

At the end of today's talk, I'll present one class (of modest size) for which finiteness of the big finitistic dimension is so far only confirmed by way of Gelinas's theorem.

Brief return to the example we just computed and more commentary. It is straightforward that $\mathrm{r.\,Findim}\,\Lambda \ge 3$ for this algebra Λ. Thus the inequality $\mathrm{l.\,dell}\,\Lambda \le 3$ which we just proved in a fashion that is not readily replicable in more complex examples could have been obtained from Gelinas's theorem.

More generally: In determining big finitistic dimensions, it is typically much easier to obtain lower bounds than upper bounds, since every module of finite projective dimension obviously provides a lower bound. Even in cases in which the finitistic dimensions are known to be finite, finding tight upper bounds for specific examples, let alone precise values, remains problematic. The obvious reason: a priori one does not know on which finite sets of modules to focus to ensure the occurrence of maximal finite values of $\operatorname{p\,dim} M$.

This difficulty is mirrored by that of finding lower bounds for $\operatorname{dell} \Lambda$: In testing delooping levels, one does not have a convenient test class of modules to check whether a given m-th syzygy of a simple module actually occurs as a higher syzygy of a different object.

To spell out the obvious: In case a specific upper bound for $\operatorname{l.\,dell} \Lambda$ coincides with a specific lower bound for $\operatorname{r.\,Findim} \Lambda$, Gelinas's Theorem provides us with the desired bounds in the reverse direction; in that case, we conclude that $\operatorname{l.\,dell} \Lambda = \operatorname{r.\,Findim} \Lambda$ is equal to the mentioned bounds.

3. Comparing $\mathrm{l.\,dell}\,\Lambda$ with $\mathrm{r.\,Findim}\,\Lambda$

In his paper, Gelinas raised the following question:

$$\mathrm{l.\,dell}\,\Lambda = \mathrm{r.\,Findim}\,\Lambda \ \text{ for all finite dimensional algebras } \Lambda?$$

We'll answer it in the negative.

Note that an example where $\mathrm{l.\,dell}\,\Lambda - \mathrm{r.\,Findim}\,\Lambda = 1$ suffices to show that such differences may be arbitrarily large. This is due the following observation of Rickard: On tensor products, $\Lambda_1 \otimes_K \Lambda_2$, the finitistic dimensions and the delooping levels behave additively.

So far, delooping levels are still poorly understood. In order to build or refute expectations, I suggest to start by exploring delooping levels in depth for particularly well-behaved algebras.

In this spirit, I will in the following focus on algebras for which the delooping levels and finitistic dimensions are reasonably accessible. Left serial algebras and special biserial algebras are among the best understood: For left serial algebras, I'll outline a very simple algorithm for computing delooping levels (one knows them to be finite in this case). For special biserial algebras, such an algorithm has been implemented by Joe Allen, a student of Rickard's; to be effective, it presupposes finiteness of the delooping levels, however.

A. Left serial algebras (all indecomposable projectives in Λ-mod are uniserial)

Suppose that Λ is left serial. Then Λ is homologically quite distinguished, in that the left and right $\mathcal{P}^{<\infty}$-categories – i.e., $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ *and* $\mathcal{P}^{<\infty}(\text{mod-}\Lambda)$ – are contravariantly finite in the ambient categories of finitely generated modules. (The lefthand part of this statement is due to Burgess-HZ, the analogue for right modules was recently derived by Saorín, Nazemian and myself.) In particular, due to results of Auslander-Reiten, and Smalø-HZ, $\mathrm{l.\,findim}\,\Lambda = \mathrm{l.\,Findim}\,\Lambda$ is the max of the projective dimensions of the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of the simples S_i; analogously on the right. Apparently, this is not enough to guarantee equality of $\mathrm{l.\,dell}\,\Lambda$ and $\mathrm{r.\,Findim}\,\Lambda$ however. On the other hand:

Theorem. [Ringel] If Λ is also right serial, i.e., if Λ is a Nakayama algebra, then

$$\mathrm{l.\,Findim}\,\Lambda = \mathrm{r.\,Findim}\,\Lambda = \mathrm{l.\,dell}\,\Lambda = \mathrm{r.\,dell}\,\Lambda.$$

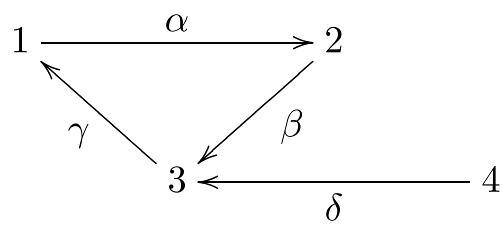
This result collapses as one moves from Nakayama algebras to one-sided serial algebras. So even contravariant finiteness of the $\mathcal{P}^{<\infty}$-categories of left and right modules does not quite take us to homological utopia.

On the other hand, Gelinas pointed out to me that Ringel's result *is* based on a strengthened form of contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{P}^{<\infty}(\text{mod-}\Lambda)$, which is satisfied by Nakayama algebras: If the left and right $\mathcal{P}^{<\infty}$-categories of Λ are contravariantly finite, there exist strong tilting modules ${}_\Lambda T$ and T'_Λ. If both of these are also *strong* tilting modules over their respective endomorphism rings, one obtains $\mathrm{l.\,Findim}\,\Lambda = \mathrm{r.\,Findim}\,\Lambda = \mathrm{l.\,dell}\,\Lambda = \mathrm{r.\,dell}\,\Lambda$. (On the side: This strong contravariant finiteness condition is also responsible for the equally good behavior of Gorenstein algebras which I cited earlier.)

Example. [Barei, Goodearl, HZ] For each $n \geq 3$, there exists a left serial algebra Λ with

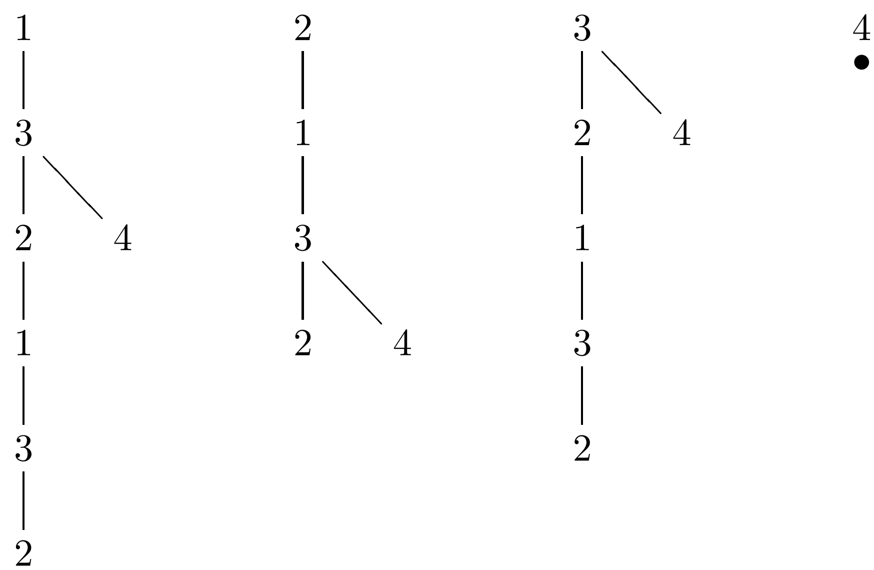
$$\mathrm{r.\,Findim}\,\Lambda = n-1 \quad \text{and} \quad \mathrm{l.\,dell}\,\Lambda = n.$$

I'll give detail for the case $n = 3$. Consider the algebra $\Lambda = KQ/\langle \alpha\gamma\beta\alpha, \beta\alpha\gamma\delta \rangle$, where Q is the quiver



The graphs of the indecomposable projectives in mod-Λ are as follows:

Indecomp. proj. in mod-Λ

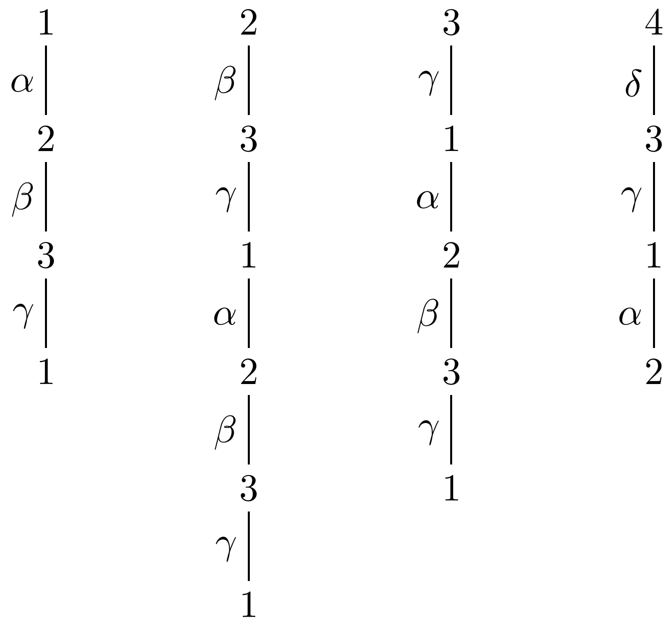


Claim 1. $\mathrm{r.Findim}\,\Lambda = 2$.

The techniques of my 1991 paper "Predicting syzygies ..." allow to compute this. Another way to go is to use contravariant finiteness of $\mathcal{P}^{<\infty}(\text{mod-}\Lambda)$ in mod-Λ and compute the minimal $\mathcal{P}^{<\infty}(\text{mod-}\Lambda)$-approximations of the simples in mod-Λ; then one obtains the big right finitistic dimension of Λ as the supremum of the projective dimensions of these approximations.

Next we move to a computation of the left delooping level of Λ. For that part, there are no existing procedures waiting to be used.

Indecomp. proj. in Λ-mod



Claim 2. $\operatorname{l.dell}\Lambda = 3$.

On my pad, I'll show $\operatorname{l.dell}\Lambda \le 3$. The argument for the reverse inequality is based on the following lemma for left serial Λ:

Lemma. [HZ] Let $N \in \Lambda$-Mod, and suppose that $N \subseteq \bigoplus_{i\in I} U_i$, where all U_i are uniserial. Then N is in turn a direct sum of uniserials and, in fact, $N = \bigoplus_{l\in L} V_l$, where each V_l is a submodule of some U_i.

Since, for any $M \in \Lambda$-Mod, we have $\Omega^1(M) \subseteq \bigoplus_{i\in I} Je(i)$ for suitable primitive idempotents $e(i)$, we find that $\Omega^1(M) \cong \bigoplus_{l\in L} V_l$, where each V_l is a submodule of some $Je(i)$. In particular, $V_l = \Omega^1(\Lambda e(i)/V_l)$ is a syzygy.

Corollary. For each $M \in \Lambda$-Mod and $k \ge 1$,

$$\Omega^k(M) = \bigoplus_{i\in I} \Omega^k(V_i), \quad \text{where each } V_i \text{ is uniserial.}$$

Next, I'll briefly describe the resulting algorithm for obtaining $\operatorname{l.dell}\Lambda$ when Λ is left serial. In light of the Corollary, all delooping levels of indecomposable left Λ-modules arise as delooping levels of uniserial modules;

keep in mind that syzygies of uniserials are again uniserial. Clearly, Λ has only finitely many isomorphism classes of uniserials. List all of those which do not arise as submodules of the left regular module ${}_\Lambda\Lambda$ (those in ${}_\Lambda\Lambda$ are either projective or submodules of ${}_\Lambda J$ and hence have delooping level 0 by an early remark): Say these are $U_1, \dots U_m$. Then all nonzero delooping levels attained in Λ-mod can be gleaned from the following tableau:

$$\begin{array}{lllll} U_1 & \Omega^1(U_1) & \Omega^2(U_1) & \Omega^3(U_1) & \dots \\ U_2 & \Omega^1(U_2) & \Omega^2(U_2) & \Omega^3(U_2) & \dots \\ . & . & . & . & \\ . & . & . & . & \\ U_m & \Omega^1(U_m) & \Omega^2(U_m) & \Omega^3(U_m) & \dots \end{array}$$

All of the above rows turn periodic after a certain number of steps. Any row may be terminated when the first repeat occurs; this termination point, in the row with index i say, only provides an upper bound for $\operatorname{dell} U_i$ however. To determine the delooping level of U_i, one needs to check for the first column index j in the i-th row such that the entry in position (i, j) occurs in some column with an index $k > j$. Then $\operatorname{dell} U_i = j$.

Concluding questions re the left serial case: Can differences

$$\text{l. dell}\,\Lambda - \text{r. Findim}\,\Lambda > 1$$

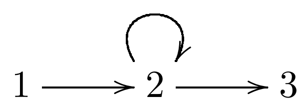
be realized for left serial Λ? Are there "structural criteria" which indicate equality $\text{l. dell}\,\Lambda = \text{r. Findim}\,\Lambda$?

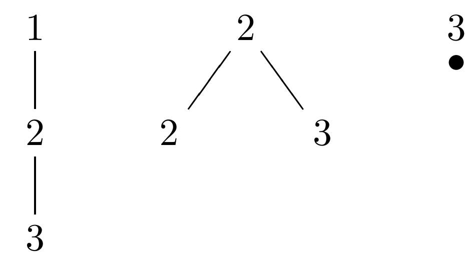
B. Special biserial algebras

The next example shows that not even representation-theoretically transparent string algebras Λ satisfy $\text{l.}\operatorname{dell}\Lambda = \text{r.}\operatorname{Findim}\Lambda$ in general. The example I'll exhibit has finite representation type and vanishing radical cube. This negative outcome contrasts the following positive result of Gelinas:

Theorem. [Gelinas] If $J^2 = 0$, then $\text{l.}\operatorname{dell}\Lambda = \text{r.}\operatorname{Findim}\Lambda$.

Example. [Goodearl-HZ] Consider the following special biserial algebra:





Clearly Λ is special biserial and monomial (i.e. a string algebra) with $J^3 = 0$. Since the alphabet of Λ does not allow for a primitive word, Λ has finite representation type. In particular, $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{P}^{<\infty}(\text{mod-}\Lambda)$ are contravariantly finite. However, $\text{l.}\operatorname{dell}\Lambda = \text{r.}\operatorname{dell}\Lambda = 2$, while $\text{l.}\operatorname{Findim}\Lambda = \text{r.}\operatorname{Findim}\Lambda = 1$.

To see that $\text{l.}\operatorname{dell}\Lambda = 2$, note that $\operatorname{dell} S_2 = \operatorname{dell} S_3 = 0$. The sequence of successive syzygies of S_1 is

$$\Omega^1(S_1) = Je_1,\ \Omega^2(S_1) = S_2,\ \Omega^3(S_1) = S_2 \oplus S_3,\ \dots,$$

and hence $\operatorname{dell} S_1 \leq 2$,

For the reverse inequality, observe that every object in $\Omega^1(\Lambda\text{-mod})$ has Loewy length ≤ 2 and a projective cover in $\operatorname{add}(\Lambda e_2 \oplus \Lambda e_3)$. Since these projectives have Loewy length 2, all second syzygies of Λ-modules are semisimple. In particular $\Omega^1(S_1)$ is not a direct summand of a second syzygy.

Again the question arises: Is the set of differences $\text{l.}\operatorname{dell}\Lambda - \text{r.}\operatorname{Findim}\Lambda$ for string algebras Λ uniformly bounded, and if so, what is the max attained?

4. Are the delooping levels of special biserial algebras always finite?

I venture the conjecture that the answer is YES.

If that were confirmed, it would show that also the big finitistic dimensions of special biserial algebras are finite. So far, this is known only for the little finitistic dimensions, due to Erdmann-Holm-Iyama-Schröer.

Here is a first positive result pointing in the "right" direction.

Theorem. [Goodearl-HZ] If Λ is special biserial with $J^3 = 0$, then the delooping levels of Λ are finite.

Our proof (probably overcomplicated so far) is based on a study of "anti-syzygies": Suppose $M \in \Lambda$-mod is contained in the radical of a projective module. Call a Λ-module N an *anti-syzygy of* M if $\Omega^1(N) \cong M$ and N has no projective direct summands. In general, modules over special biserial algebras may have infinitely many anti-syzygies.