# Homological Methods in Representation Theory
# A conference in honour of Lidia Angeleri Hügel

## Abstracts

**Silvana Bazzoni** (University of Padova)
*Minimal Approximations and Generalisations of Bass' Theorem*

Bass in 1960 in his famous Theorem P, characterised the rings over which every module has a projective cover. These rings are called perfect and their characterisation is in term of many equivalent conditions both from the ring theoretic and the homological point of view.

The condition of the closure under direct limits of the class of projective modules gives a first important instance of the validity of Enochs Conjecture, namely the question whether a covering class is necessarily closed under direct limits.

We explore Enochs Conjecture in particular for the class $\mathcal{P}_1$ of modules of projective dimension at most one looking for properties of the rings over which $\mathcal{P}_1$ is a covering class.

One of Bass' characterisation of perfect rings is in terms of the vanishing of finitistic dimensions.

We show cases for which the class $\mathcal{P}_1$ is covering if and only the value of the finitistic dimension is at most one.

This is work in progress with Giovanna Le Gros.

**Asmae Ben Yassine** (Charles University Prague)
*Flat relative Mittag-Leffler modules and approximations*

The classes $\mathcal{D}_\mathcal{Q}$ of relative flat Mittag-Leffler modules are sandwiched between the class $\mathcal{FM}$ of all (absolute) flat Mittag-Leffler modules, and the class $\mathcal{F}$ of all flat modules. Building on the works of Angeleri Hügel, Herbera, and Šaroch, we give a characterization of relative flat Mittag-Leffler modules in terms of their local structure, and also show that Enochs' Conjecture holds for all the classes $\mathcal{D}_\mathcal{Q}$. In the final section, we apply these results to the particular case of f-projective modules.

**Isaac Bird** (Charles University Prague)
*Duality pairs in compactly generated tensor triangulated categories*

Duality pairs of modules, as originally introduced by Holm and Jørgensen, provide an intuitive way to determine whether classes of modules are closed under pure subobjects and quotients, as well as to determine the existence of approximations. In this talk I will present joint work with Jordan Williamson on how to define a similar construction in suitable tensor triangulated categories by combining Brown representability and the internal Hom induced by the monoidal structure. Analogues to Holm and Jørgensen's results will be provided, and an emphasis will be put on definability and duality; this will include describing the Auslander-Gruson-Jensen duality on coherent functors and establishing elementary duality for definable classes in said triangulated categories. Ultimately, I will provide a brief comment on how the necessity for a symmetric monoidal structure can be removed in the common setting of the derived category of a ring.

**Simion Breaz** (Babes-Bolyai University)
*Transfer of homological properties along some canonical functors*

Given a ring homomorphism $\lambda : R \to S$, I will discuss when the derived of the induction and coinduction functors associated to $\lambda$ preserve or reflect some homological properties in the corresponding derived categories. In particular, I am interested in properties that are used in various characterizations of (co-)silting complexes. Under some reasonable hypotheses (e.g. $\lambda$ is faithful flat and the rings are commutative or $\lambda$ is Frobenius) the derived induction functor preserves and reflects the silting property of some complexes. Moreover, in the case of compact silting complexes, we can transfer the Frobenius property of $\lambda$ via the derived equivalences associated with compact silting complexes to the corresponding morphism between the associated (dg-)endomorphism algebras.

**Aslak Bakke Buan** (Norwegian University of Science and Technology, NTNU, Trodnheim)
*On wide subcategories* (joint work with Eric Hanson and with Bethany Marsh)

In earlier joint work with Marsh, we defined a categorical structure on the set of all wide subcategories of a fixed module category, where maps correspond to *tau*-rigid objects. Our construction worked for *tau*-rigid finite algebras, and was motivated by Igusa-Todorovs definition of such a category in the hereditary case.

In recent joint work with Hanson, we considered the general case of finite dimensional algebras. One then needs to restrict to a certain subclass of functorially finite wide subcategories, the *tau*-perpendicular categories. These were introduced by Jasso, generalizing Geigle-Lenzing perpendicular categories in the hereditary setting. We characterize such categories, in particular in view of corresponding torsion pairs, inspired by work of Marks-Stoviceck/Ingalls-Thomas.

**Dolors Herbera** (Autonomous University of Barcelona)
*Mittag-Leffler conditions, covering classes and enveloping classes*

The first part of the talk will be devoted to make a survey of the so called *Mittag-Leffler conditions for modules* and its applications. The starting point will be the joint paper with Lidia Angeleri [AH] and how this lead to a solution of the Baer splitting problem in [ABH]. Then, we will move quickly to the world of covering and enveloping classes.

In general, it is a difficult and challenging problem to give intrinsic conditions on a class of modules $\mathcal{C}$ that ensure that any module has an envelope and/or a cover in the class $\mathcal{C}$. We recall that there is a celebrated conjecture by Enochs suggesting that covering classes must be closed under direct limits.

In the case of a cotorsion pair there are plenty of interesting and striking results around this topic. One of the nicest one is due to Angeleri, Saroch and Trilifaj [AST], and verifies Enochs conjecture for the left hand class of an $n$-tilting cotorsion pair. More precisely, if $C = (\mathcal{A}, \mathcal{B})$ is an $n$-tilting cotorsion pair then $\mathcal{A}$ is covering if and only if $\mathcal{A}$ is closed under direct limits and, in this case, the $n$-tilting class $\mathcal{B}$ provides also for envelopes. We will stress the role that Mittag-Leffler conditions have in the proof of this result.

It is easy to find examples showing that if the $n$-tilting class $\mathcal{B}$ is enveloping, then $\mathcal{A}$ may not be covering. Bazzoni and Le Gros have recently characteried when a 1-tilting class over a commutative ring $R$ is enveloping [BG]. In this talk we will present some results for general $n$-tilting classes over commutative rings. The basic tool is the wonderful characterization of $n$-tilting cotorsion pairs over commutative rings due to Hrbek and Šťovíček [HS]. Our results show that, for a commutative ring $R$, having certain enveloping $n$-tilting classes is closely related to $R$ being *Cohen-Maculay* but in the setting of general commutative rings.

The last part of the talk is based on work in progress with Giovanna Le Gros.

[ABH] L. Angeleri Hügel, S. Bazzoni and D. Herbera, *A Solution to the Baer Splitting Problem* Trans. Amer. Math. Soc. **360**, (2008), 2409–2421.

[AH] L. Angeleri Hügel and D. Herbera, *Mittag-Leffler conditions for modules*, Indiana Univ. Math. J. **57**, (2008) 2459–2517.

[AST] L. Angeleri Hügel, J. Saroch, J. Trlifaj, *Approximations and Mittag-Leffler conditions–the applications*, Israel J. Math. **226**(2018), 757–780.

[BG] Silvana Bazzoni and Giovanna Le Gros. *Enveloping classes over commutative rings.* To appear in J. of Pure and Appl. Alg., arXiv:1901.07921, 2019.

[HS] Michal Hrbek and Jan Šťovíček, *Tilting classes over commutative rings*, Forum Math. **32**(2019), 235–267.

**Michal Hrbek** (Czech Academy of Sciences, Prague)
*Derived equivalences induced by codimension functions on Spec(R)*

In the setting of a commutative noetherian ring, the silting (and dually, cosilting) t-structures in the derived category admit a tidy classification in terms of certain filtrations of the Zariski spectrum. However, an explicit construction of the silting complexes seems a much harder task, and only rather specific examples have been available. On the other hand, an explicit construction often allows one to determine whether the silting object is in fact tilting, that is, when the realization functor from the heart of the t-structure is a derived equivalence. We consider a natural class of filtrations of $Spec(R)$ called the slice filtrations, and show that the induced (co)silting objects can be always constructed in a very explicit way. If the base ring admits a dualizing complex, we show that these silting objects are tilting if and only if they are induced by a codimension function on $Spec(R)$. In the absence of a dualizing complex, the situation is more complicated: the tilting property of the t-structure induced by a codimension function is tied to certain good geometric properties of the ring. This is a report on a joint work with Tsutomu Nakamura and Jan Šťovíček.

**Henning Krause** (University of Bielefeld)
*The category of finite strings*

Strings are considered to be one of the most basic combinatorial structures arising in representation theory. My talk will focus on a category of finite strings. The category is closely related to the augmented simplex category, and it models categories of linear representations via an explicit equivalence. Each lattice of non-crossing partitions (of type A and B) arises naturally as a lattice of subobjects, and this can be used to complete the classification of all thick subcategories for representations of any tame hereditary algebras, including the ones which are not generated by exceptional sequences.

**Dirk Kussin** (Technical University of Berlin)
*Irrational slopes in case of tubular/elliptic curves*

It is well-known that in the category of quasicoherent sheaves over a tubular or an elliptic curve each indecomposable has a well-defined slope $w$, which is a real number or $\infty$. (Essentially due to Reiten-Ringel.) Each slope $w$ yields a (cotilting) torsion pair and the heart of the corresponding HRS-tilted $t$-structure, which is a locally coherent Grothendieck category with (generalised) Serre duality. Whereas the hearts are well-understood for rational $w$, we will discuss several interesting questions and problems and some results in case $w$ is irrational. Several properties of these categories have been studied e.g. by Šťovíček, Rapa, Prest, and in joint work with Angeleri Hügel and with Laking.

**Rosanna Laking** (University of Verona)
*Mutation in large silting theory*

In this talk we will consider a large class of t-structures in compactly generated triangulated categories called cosilting t-structures. In particular, we will consider those determined by pure-injective cosilting objects, which have the desirable property that the heart is a Grothendieck abelian category. Examples of such t-structures in the derived category of a finite-dimensional algebra include those induced by derived equivalences with Grothendieck categories and those 'lifted' from bounded t-structures in the bounded derived category (in the sense of Marks-Zvonareva). In this talk we will define and explore a mutation operation on such t-structures that naturally extends the mutation operation on silting t-structures in the bounded derived category of a finite-dimensional algebra. This is a report on joint work with Lidia Angeleri Hügel, Jan Šťovíček and Jorge Vitória.

**Lorenzo Martini** (University of Verona)
*Local Coherence of Hearts Associated with Thomason filtrations*

Any Thomason filtration of a commutative ring yields (at least) two t-structures in the derived category of the ring, one of which is compactly generated [Hrb18,HHZ21]. We study the hearts of these two t-structures and prove that they coincide in case of a weakly bounded below filtration. Prompted by [SS20], in which it is proved that the heart of a compactly generated t-structure in a triangulated category with coproduct is a locally finitely presented Grothendieck category, we study the local coherence of the hearts associated with a weakly bounded below Thomason filtration, achieving a useful recursive characterisation in case of a finite length filtration. Low length cases involve hereditary torsion classes of finite type of the ring, and even their Happel-Reiten-Smalo hearts; in these cases, the relevant characterisations are given by few module-theoretic conditions.

Joint work with Carlos Parra

[Hrb18] M. Hrbek, Compactly generated t-structures in the derived category of a commutative ring, Math. Z. 295 (2020).
[HHZ21] M. Hrbek, J. Hu, R. Zhu, Gluing compactly generated t-structure over stalks of affine schemes, arXiv:2101.09966
[SS20] M. Saorín, J. Šťovíček, t-Structures with Grothendieck hearts via functor categories, arXiv:2003.01401v2

**Kaveh Mousavand** (Queen's University, Kingston)
*Classification of Minimal $\tau$-tilting Infinite Biserial algebras*

To give a novel interpretation of $\tau$-tilting finiteness of algebras in the algebro-geometric setting, we treat a new conjecture on the behavior of bricks (Schur representations). In particular, we state a Brauer-Thrall type conjecture, and to systematically approach it, we study "minimal $\tau$-tilting infinite algebras" as the modern counterpart of minimal representation-infinite algebras.

In this talk, after an introduction to minimal $\tau$-tilting infinite algebras and their fundamental properties, we focus on the biserial algebras. In particular, for those minimal $\tau$-tilting infinite algebras which are biserial, we give an explicit classification of them in terms of quivers and relations. From this, we immediately verify the conjecture for all biserial algebras. Our results could be viewed as the modern analogue of the classification of minimal representation-infinite special biserial algebras given by Ringel.

This is based on my joint work with Charles Paquette.

**Zahra Nazemian** (University of Graz)
*Periodicity of strong tilting iteration*

Auslander and Reiten proved that, for an Artin algebra $\Lambda$, contravariant finiteness in $\Lambda-\text{mod}$ of the subcategory $\mathcal{P}^{<\infty}(\Lambda-\text{mod})$ of modules of finite projective dimension is equivalent to the existence of a strong tilting module, and these equivalent conditions imply the verfication of both finitistic dimension conjectures, namely, that the big and the small finitistic dimension of $\Lambda$ coincide and they are finite.

Assuming the equivalent conditions mentioned above and choosing the (unique up to isomorphism) basic strong tilting left $\Lambda$-module $T$, one may ask when the endomorphism algebra $\tilde{\Lambda} = \text{End}({}_\Lambda T)^{op}$ has the property that $\mathcal{P}^{<\infty}(\text{mod}-\tilde{\Lambda})$ is also contravariantly finite in $\text{mod}-\tilde{\Lambda}$. Even more, one may ask when this strong tilting process may be iterated indefinitely. In this talk we will show that if $\mathcal{P}^{<\infty}(\text{mod}-\tilde{\Lambda})$ is contravariantly finite in $\text{mod}-\tilde{\Lambda}$, then the strong tilting process can be iterated indefinitely and it is periodic.

This talk is based on joint work with Manuel Saorín and Birge Huisgen-Zimmermann.

**Sebastian Opper** (Charles University Prague)
*On the derived equivalence classification of Brauer graph algebras*

Brauer graph algebras, originating in the representation theory of finite groups, can be defined for any suitably decorated graph on an oriented surface $\Sigma$. In this talk I will recall their connection with gentle algebras and partially wrapped Fukaya categories of surfaces and explain how this leads to a natural class of $A_\infty$-algebras which encompasses Brauer graph algebras. I will then discuss the proof of the classification and describe how previously known derived invariants, first defined by Antipov, can be reinterpreted as orbit invariants of a line field under the action of the mapping class group of $\Sigma$.

This is joint work with Alexandra Zvonareva.

**David Pauksztello** (University of Lancaster)
*Complements for pre-silting objects*

A famous counterexample of Rickard and Schofield showed that a given a pre-tilting module $T$, there need not be a pre-tilting module $S$ such that $T \oplus S$ is tilting. We say that pre-tilting modules do not necessarily admit complements. However, work of Angeleri Hügel and Coelho showed that if one permits the complement to be infinite dimensional, then any pre-tilting module admits a complement.

Silting objects are a generalisation of tilting objects and it is natural, therefore, to ask whether a pre-silting object admits a complement. Presently, this question is still open. In this talk, we will discuss a criterion to detect existence of relative complements using averaging of co-t-structures which yields a silting analogue of Angeleri Hügel and Coelho's result and opens a question on how to define silting objects in small and large settings in a compatible way. We will also discuss other contexts in which existence of complements is know, e.g. the hereditary and silting-discrete settings.

This talk will be based on joint work with Lidia Angeleri Hügel and Jorge Vitória.

**Sergio Pavon** (University of Padova)
*Coderived equivalences* (based on ongoing joint work with M. Hrbek)

Let $R$ be a commutative noetherian ring, and consider a $t$-structure $\mathbb{T}$ in $\mathsf{D}(R)$, with heart $\mathcal{H}$. If $\mathbb{T}$ is intermediate and *restrictable*, i.e. it restricts to a $t$-structure on $\mathsf{D}^b(\mathsf{mod}(R))$, then $\mathcal{H}$ is a locally coherent Grothendieck category, whose finitely presented objects are $\mathsf{fp}(\mathcal{H}) = \mathcal{H} \cap \mathsf{D}^b(\mathsf{mod}(R))$. In [PV21], it was shown that for such $\mathbb{T}$ there is a derived equivalence $\mathsf{D}(\mathcal{H}) \simeq \mathsf{D}(R)$, which in particular restricts to an equivalence $\mathsf{D}^b(\mathsf{fp}(\mathcal{H})) \simeq \mathsf{D}^b(\mathsf{mod}(R))$.

For the commutative noetherian ring $R$, Krause [K05] constructed a recollement

$$\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(R)) \Lleftarrow\!\Rrightarrow \mathsf{K}(\mathsf{Inj}(R)) \Lleftarrow\!\Rrightarrow \mathsf{D}(R)$$

The homotopy category of injectives $\mathsf{K}(\mathsf{Inj}(R))$, which is equivalent to the coderived category of $R$ in the sense of Positselski, is compactly generated, by the image of $\mathsf{D}^b(\mathsf{mod}(R))$ under the right adjoint of $\mathsf{K}(\mathsf{Inj}(R)) \to \mathsf{D}(R)$.

In joint work with M. Hrbek, we proved the existence of a Krause-type recollement for any heart $\mathcal{H}$ as above (slightly generalising work by Stovicek [S14], which uses different methods). Moreover, we showed that the equivalence $\mathsf{D}(\mathcal{H}) \simeq \mathsf{D}(R)$ induces an equivalence of recollements

$$\begin{tikzcd}
\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(\mathcal{H})) \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[d, "\simeq"] & \mathsf{K}(\mathsf{Inj}(\mathcal{H})) \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[d, "\simeq"] & \mathsf{D}(\mathcal{H}) \arrow[d, "\simeq"] \\
\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}(R)) \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & \mathsf{K}(\mathsf{Inj}(R)) \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & \mathsf{D}(R).
\end{tikzcd} \qquad (*)$$

If time allows, we will cover an application, when $R$ admits a strongly pointwise dualising complex. In this situation, there exists an intermediate restrictable $t$-structure $\mathbb{T}_{CM}$ whose heart $\mathcal{H}_{CM}$ has the property that every local cohomology functor $\mathbb{R}\Gamma_V$ restricts to a (exact) functor $\mathcal{H}_{CM} \to \mathcal{H}_{CM}$. This gives a recollement of abelian categories $\mathcal{C}_V \leftrightarrows \mathcal{H}_{CM} \leftrightarrows \mathcal{T}_V$, which lifts to the (co)derived level, and therefore to a recollement for the (co)derived category of $R$, using the equivalence (**??**).

[K05] H.Krause, *The stable derived category of a Noetherian scheme*, Compos. Math. **141**(5):1128–1162 (2005).

[PV21] S.Pavon and J.Vitória, *Hearts for commutative noetherian rings: torsion pairs and derived equivalences*, to appear on Doc. Math. **26** (2021). `arXiv:2009.08763`.

[S14] J.Šťovíček, *On purity and applications to coderived and singularity categories*, `arXiv:1412.1615` (2014)

**Mike Prest** (University of Manchester)
*Strongly atomic modules in definable categories*

A definable category $\mathcal{D}$ might have no finitely presented objects, other than 0, but it is a result of Makkai that it does have a generating set of objects which are strongly $\mathcal{D}$-atomic (that is, strongly Mittag-Leffler with respect to the dual definable category). These, by definition, share a key property with finitely presented modules. Makkai proved this in a very general context and using category-theoretic model theory but a more direct proof (joint with Philipp Rothmaler) can be given in our context.

I will give some general results on these modules and say something about how they lie in the category of modules of some irrational slope over a tubular algebra.

**Chrysostomos Psaroudakis** (University of Thessaloniki)
*Lifting recollements of abelian categories and model structures*

Recollements of derived module categories have been extensively studied in representation theory. For instance, in a series of papers Angeleri Hügel, Koenig, Liu and Yang investigated which homological invariants can be computed inductively along a recollement of derived module categories, when recollements lift to various levels of derived categories and when a derived version of the Jordan-Hölder theorem holds. One basic source of such recollements are recollements of module categories that lift to the associated derived categories. The aim of this talk is to present a systematic method, using Quillen model structures, to lift recollements of hereditary abelian model categories to recollements of their associated homotopy categories.

In applications, we recover known results on lifting recollements of abelian categories to their derived counterpart, and we also show lifting of recollements in other contexts, such as lifting to homotopy categories that provide models for stable categories of Gorenstein projective modules and related categories. This is joint work with Georgios Dalezios.

**Alessandro Rapa** (University of Verona)
*On cosilting hearts over the Kronecker algebra*

In this talk we focus on the different hearts arising from torsion pairs of finite type in the category of modules over the Kronecker algebra. First of all, we characterize their simple objects, using a criterion that links these simple objects to peculiar modules over the Kronecker algebra. Afterwards, we compute the atom spectrum and the Gabriel dimension of these hearts.

**Lleonard Rubio Y Degrassi** (University of Verona)
*Maximal tori in $HH^1$ and the fundamental group*

Hochschild cohomology is a fascinating invariant of an associative algebra which possesses a rich structure. In particular, the first Hochschild cohomology group $HH^1(A)$ of an algebra $A$ is a Lie algebra, which is a derived invariant of $A$ and, among self-injective algebras, an invariant under stable equivalences of Morita type. Therefore, within representation theory, this structure is often used in classification problems. More recently, by studying the closely related algebraic group of outer automorphisms of $A$, maximal tori have been used to obtain combinatorial derived invariants for gentle algebras and Brauer graph algebras. Beyond this, however, fine Lie theoretic properties of $HH^1(A)$ are not often used.

In this talk, I will present joint work with Benjamin Briggs where we provide a number of results in this direction. In particular, I will show that the maximal tori of $HH^1(A)$ can be used to deduce information about the shape of the Gabriel quiver of $A$. In addition, I will prove that every maximal torus in $HH^1(A)$ arises as the dual of some fundamental group of $A$. By combining this, with known invariance results for Hochschild cohomology, I will deduce that (in rough terms) the largest rank of a fundamental group of $A$ is a derived invariant quantity, and among self-injective algebras, an invariant under stable equivalences of Morita type. Time permitting, I will also provide various applications to semimonomial and simply connected algebras.

**Manuel Saorín** (University of Murcia)
*Idempotent reduction for $\mathcal{P}^{<\infty}$-contravariant finiteness and strong tilting iteration*

In this talk we present a method, using 'idempotent reduction', to decide, for an Artin algebra $\Lambda$, whether the subcategory $\mathcal{P}^{<\infty}(\Lambda-\mathrm{mod})$ of modules of finite projective dimension is contravariantly finite in $\Lambda-\mathrm{mod}$. As shown by Auslander and Reiten, this last condition is equivalent to the existence of a (unique up to isomorphism) basic strong tilting left $\Lambda$-module $T$. Calling in such case $\tilde{\Lambda} = \mathrm{End}({}_\Lambda T)^{op}$ the strong tilt of $\Lambda$, we show that our idempotent reduction technique is also very efficient to determine whether the strong tilting process can be iterated indefinitely. Several examples where the techniques apply will be presented.

This talk is based on joint work with Birge Huisgen-Zimmermann and Zahra Nazemian.

**Jan Šťovíček** (Charles University Prague)
*Commutative silting ring epimorphisms*

A relation between tilting theory and ring epimorphisms has been at least implicitly recognized and used for a while: be it in the form of relations between representations of different finite acyclic quivers of the same representation type or tau-tilting reduction in the realm of finite dimensional algebra, or as a method of constructing and classifying infinitely generated tilting modules by Angeleri Hügel, Archetti and Sànchez. The general fact that every (infinitely generated) partial tilting module yields a natural ring epimorphism was observed by Colpi, Tonolo and Trlifaj, and a (broader) class of so-called silting epimorphisms have only recently been studied systematically by Angeleri Hügel, Marks, Vitória and the speaker. It still remains a somewhat mysterious class of ring epimorphisms which is known to be wider (strictly in general) than the class of universal localizations in the sense of Cohn and Schofield.

In this talk we focus on commutative rings. Building on recent results by several authors, we prove that in this case the class of silting epimorphisms coincides with the very classical class of flat ring epimorphisms.

**Jan Trlifaj** (Charles University Prague)
*Closure properties of $\varinjlim \mathcal{C}$*

Let $\mathcal{C}$ be a class of (right $R$-) modules closed under finite direct sums. If $\mathcal{C}$ consists of finitely presented modules, then the class $\varinjlim \mathcal{C}$ of all direct limits of modules from $\mathcal{C}$ is well-known to enjoy a number of closure properties. Moreover, if $R \in \mathcal{C}$, $\mathcal{C}$ consists of $\mathrm{FP}_2$-modules, and $\mathcal{C}$ is closed under extensions and direct summands, then $\varinjlim \mathcal{C}$ can be described homologically: $\varinjlim \mathcal{C}$ is the 'double perp' of $\mathcal{C}$ with respect to the $\mathrm{Tor}_1^R$ bifunctor [1].

Things change completely when $\mathcal{C}$ is allowed to contain infinitely generated modules: $\varinjlim \mathcal{C}$ then need not even be closed under direct limits. After presenting some positive general results (and their constraints), we will concentrate on two particular cases: $\mathcal{C} = \operatorname{add} M$ and $\mathcal{C} = \operatorname{Add} M$, for an arbitrary module $M$. We will prove that if $S = \operatorname{End} M$ and $\mathcal{F}$ is the class of all flat right $S$-modules, then $\varinjlim \operatorname{add} M = \{F \otimes_S M \mid F \in \mathcal{F}\}$. For $\varinjlim \operatorname{Add} M$, we will have a similar formula, involving the contratensor product $\odot_S$ and direct limits of projective right $S$-contramodules (for $S$ endowed with the finite topology). We will also show that for various classes of modules $\mathcal{D}$, if $M \in \mathcal{D}$ then $\varinjlim \operatorname{add} M = \varinjlim \operatorname{Add} M$. However, the equality remains open in general, even for (infinitely generated) projective modules.

The talk is based on my recent joint work [2] with Leonid Positselski.

[1] L.Angeleri Hügel, J. Trlifaj: *Direct limits of modules of finite p rojective dimension*, in Rings, Modules, Algebras, and Abelian Groups, LNPAM 236, M.Dekker, New York 2004, 27-44.

[2] L.Positelski, J.Trlifaj: *Closure properties of* $\varinjlim \mathcal{C}$, preprint.

**Simone Virili**
*Length functions in Grothendieck categories*

Given a Grothendieck category $\mathfrak{G}$, a *length function* $L\colon \mathfrak{G} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is a non-negative real invariant (that may attain infinity) of the objects of $\mathfrak{G}$ with two essential properties: it is *additive* on short exact sequences and it is *continuous* on direct unions. Classical examples of length functions are the dimension of vector spaces, the rank of modules over Ore domains, or the composition length, but also genuinely non-discrete examples exist. The problem of classification of all the length functions on a given Grothendieck category $\mathfrak{G}$ seems hopeless in general but, in some cases, one can write all the length functions on $\mathfrak{G}$ as linear combinations of a family of *indecomposable length functions*. One such example is the following: if $\mathfrak{G}$ has Gabriel dimension (e.g., $\mathfrak{G}$ is locally Noetherian) then the (equivalence classes of) indecomposable length functions are in bijection with the (isomorphism classes of) indecomposable injective objects, and every other length function is a linear combination of those. One can even go one step further and prove that, in general, the equivalence classes of *discrete* (i.e., taking values in a subset of $\mathbb{R}_{\geq 0} \cup \{\infty\}$ which is order isomorphic to $\mathbb{N} \cup \{\infty\}$) indecomposable length functions are in bijection with the so-called *atom spectrum* of $\mathfrak{G}$. Hence, the family of all the indecomposable length functions (so including also the non-discrete ones) can be considered as an extension of this spectrum. In this talk, I will summarize these classification results and I will discuss some open problems and possible future directions of this investigation.

**Alexandra Zvonareva** (University of Stuttgart)
*Functorial approach to rank functions*

For a skeletally small triangulated category $C$ Chuang and Lazarev introduced the notion of a rank function on $C$. Such functions are closely related to functors into simple triangulated categories. In this talk, I will discuss the connection between rank functions on $C$ and translation-invariant additive functions on its abelianization mod-$C$. This connection allows to relate rank functions to endofinite cohomological functors on $C$ and, in the case when $C$ is the subcategory of compact objects in a compactly generated triangulated category $T$, to endofinite objects and to the Ziegler spectrum of $T$. This is based on a joint work in progress with Teresa Conde, Mikhail Gorsky, and Frederik Marks.