ON p-FILTRATIONS OF WEYL MODULES

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ABSTRACT. This paper considers Weyl modules for a simple, simply connected algebraic group over an algebraically closed field k of positive characteristic $p \neq 2$. The main result proves, if $p \geq 2h-2$ (where h is the Coxeter number) and if the Lusztig character formula holds for all (irreducible modules with) regular restricted highest weights, then any Weyl module $\Delta(\lambda)$ has a Δ^p-filtration, namely, a filtration with sections of the form $\Delta^p(\mu_0 + p\mu_1) := L(\mu_0) \otimes \Delta(\mu_1)^{[1]}$, where μ_0 is restricted and μ_1 is arbitrary dominant. In case the highest weight λ of the Weyl module $\Delta(\lambda)$ is p-regular, the p-filtration is compatible with the G_1-radical series of the module. The problem of showing that Weyl modules have Δ^p-filtrations was first proposed as a worthwhile ("wünschenswert") problem in Jantzen's 1980 Crelle paper.

Corollary 5.3. *Assume that $p \geq 2h-2$ is an odd prime and that the LCF formula holds for all regular, restricted weights. Then for any $\gamma \in X(T)_+$, $\Delta(\gamma)$ has a $\Delta^{\mathrm{red}} = \Delta^p$-filtration.*

Theorem 5.1. *Assume that $p \geq 2h-2$ is an odd prime and that the LCF holds for any $\gamma \in X_{\mathrm{reg}}(T)_+$. Given any $\gamma \in X_{\mathrm{reg}}(T)_+$, each section $\Delta(\gamma)_{\widetilde{s}}$, $s \in \mathbb{N}$, viewed as a rational G-module has a Δ^{red}-filtration. In particular, $\Delta(\gamma)$ has a $\Delta^{\mathrm{red}} = \Delta^p$-filtration.*

Given $\lambda \in X(T)_+$, fix a highest weight vector $v^+ \in L_\zeta(\lambda)$. Then there is a unique admissible lattice $\widetilde{\Delta}^{\mathrm{red}}(\lambda)$ (resp., $\widetilde{\nabla}_{\mathrm{red}}(\lambda)$) of $L_\zeta(\lambda)$ which is minimal (resp., maximal) with respect to all admissible lattices $\widetilde{L}$ such that $\widetilde{L} \cap L_\zeta(\lambda)_\lambda = \mathcal{O}v^+$. For example, put $\widetilde{\Delta}^{\mathrm{red}}(\lambda) = \widetilde{U}_\zeta \cdot v^+$. By abuse of notation, we call $\widetilde{\Delta}^{\mathrm{red}}(\lambda)$ (resp., $\widetilde{\nabla}_{\mathrm{red}}(\lambda)$) the minimal (resp., maximal) lattice of $L_\zeta(\lambda)$. Any two "minimal" (resp., "maximal") lattices are isomorphic as $\widetilde{U}_\zeta$-modules.

quantum enveloping algebra

For $\lambda \in X(T)_+$, put

$$\Delta^{\mathrm{red}}(\lambda) := \widetilde{\Delta}^{\mathrm{red}}(\lambda)/\pi\widetilde{\Delta}^{\mathrm{red}}(\lambda) \text{ and } \nabla_{\mathrm{red}}(\lambda) := \widetilde{\nabla}_{\mathrm{red}}(\lambda)/\pi\widetilde{\nabla}_{\mathrm{red}}(\lambda).$$

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Since the LCF holds for U_ζ if $p > h$,

(due to Z. Lin) $$\operatorname{ch} \Delta^{\mathrm{red}}(\lambda) = \operatorname{ch} \nabla_{\mathrm{red}}(\lambda) = \chi_{\mathrm{KL}}(\lambda).$$

Proposition 2.4. *(a) ([19, Thm. 2.7] or [12, Prop. 1.78]) Assume that* $\mu = \mu_0 + p\mu_1 \in X(T)_+$ *for* $\mu_0 \in X_1(T)$ *and* $\mu_1 \in X(T)_+$*. Then* $\Delta^{\mathrm{red}}(\mu) \cong \Delta^{\mathrm{red}}(\mu_0) \otimes \Delta(\mu_1)^{[1]}$ *and* $\nabla_{\mathrm{red}}(\mu) \cong \nabla_{\mathrm{red}}(\mu_0) \otimes \nabla(\mu_1)^{[1]}$.

Notation

The following notation related to a p-modular system will be used throughout the paper.

(1) $(K, \mathcal{O}, k)$: p-modular system. Thus, $\mathcal{O}$ is a DVR with maximal ideal $\mathfrak{m} = (\pi)$, fraction field K, and residue field k.

(2) $\widetilde{A}$: $\mathcal{O}$-algebra which is finite and free over $\mathcal{O}$. Let $\widetilde{A}_K := K \otimes_{\mathcal{O}} \widetilde{A}$ and $A := k \otimes_{\mathcal{O}} \widetilde{A}$. More generally, if $\widetilde{M}$ is an $\widetilde{A}$-module, put $\widetilde{M}_K := K \otimes_{\mathcal{O}} \widetilde{M}$ and $M = \widetilde{M}_k := k \otimes_{\mathcal{O}} \widetilde{M}$. Sometimes, $\widetilde{M}_k$ is also denoted $\overline{\widetilde{M}}$. Often $\widetilde{M}$ will be finite and free over $\mathcal{O}$—namely, a *lattice* for $\mathcal{O}$ (or $\widetilde{A}$).

(3) Let $\widetilde{M}$ be an $\widetilde{A}$-lattice. Put $\widetilde{\operatorname{rad}}^n \widetilde{M} := \widetilde{M} \cap \operatorname{rad}^n \widetilde{M}_K$, where $\operatorname{rad}^n \widetilde{M}_K$ denotes the nth-radical of the $\widetilde{A}_K$-module $\widetilde{M}_K$. Of course, $\operatorname{rad}^n \widetilde{M}_K = (\operatorname{rad}^n \widetilde{A}_K)\widetilde{M}_K$.

Dually, let $\widetilde{\operatorname{soc}}^{-n} \widetilde{M} := \operatorname{soc}^{-n} \widetilde{M}_K \cap \widetilde{M}$, $n = 0, 1, \cdots$, where $\{\operatorname{soc}^{-n} \widetilde{M}_K\}_n$ is the socle series of $\widetilde{M}_K$.

(4) Again, let $\widetilde{M}$ is an $\widetilde{A}$-lattice. $\operatorname{gr}\widetilde{M} := \bigoplus_{n\geq 0} \widetilde{\operatorname{rad}}^n \widetilde{M}/\widetilde{\operatorname{rad}}^{n+1} \widetilde{M}$, viewed as a (positively) graded module for the $\mathcal{O}$-algebra $\operatorname{gr}\widetilde{A} := \bigoplus_{n\geq 0} \widetilde{\operatorname{rad}}^n \widetilde{A}/\widetilde{\operatorname{rad}}^{n+1} \widetilde{A}$. Notice that $\operatorname{gr}\widetilde{A}$ is finite and free over $\mathcal{O}$, and that $\operatorname{gr}\widetilde{M}$ is a $\operatorname{gr}\widetilde{A}$-lattice.

Dually, let $\operatorname{gr}^\diamond \widetilde{M} := \bigoplus_{n\geq 0} \widetilde{\operatorname{soc}}^{-n} \widetilde{M}/\widetilde{\operatorname{soc}}^{-n+1} \widetilde{M}$, regarded as a *negatively* graded $\operatorname{gr}\widetilde{A}$-lattice. Observe that, taking $\mathcal{O}$-duals, $(\operatorname{gr}^\diamond \widetilde{M})^* = (\operatorname{gr}\widetilde{M}^*)$ as $(\operatorname{gr}\widetilde{A})^{\mathrm{op}}$-lattice.

(5) We say that a $\widetilde{A}$-lattice $\widetilde{M}$ is $\widetilde{A}$-*tight* (or just *tight*, if $\widetilde{A}$ is clear from context) if

$$(\widetilde{\operatorname{rad}^n \widetilde{A}})\widetilde{M} = \widetilde{\operatorname{rad}^n \widetilde{M}}, \quad \forall n \geq 0. \tag{1.0.1}$$

Clearly, if $\widetilde{M}$ is also $\widetilde{A}$-projective, then it is tight. (We will see that many other lattices can be tight.)

(6) Now let $\widetilde{\mathfrak{a}}$ be an $\mathcal{O}$-subalgebra of $\widetilde{A}$. Then items (2)–(5) all make perfectly good sense using $\widetilde{\mathfrak{a}}$ in place of $\widetilde{A}$. If $\widetilde{M}$ is an $\widetilde{A}$-lattice, then it is an $\widetilde{\mathfrak{a}}$-lattice. In our applications later, it will usually be the case that $(\operatorname{rad}^n \widetilde{\mathfrak{a}}_K)\widetilde{A}_K = \operatorname{rad}^n \widetilde{A}_K$, for all $n \geq 0$; see (3.0.13). In that case, if $\widetilde{M}$ is an $\widetilde{A}$-lattice, then $\widetilde{\operatorname{rad}^n \widetilde{M}}$ can be constructed viewing $\widetilde{M}$ as an $\widetilde{A}$-lattice or as an $\widetilde{\mathfrak{a}}$-lattice. Both constructions lead to identical $\mathcal{O}$-modules.

Corollary 3.7. *Let* $\widetilde{A} = \widetilde{A}_\Lambda$. (current proof uses p>2h-2, lifting of PIMs)

(a) $\widetilde{A}$ *is a tight* $\widetilde{\mathfrak{a}}$*-module, i. e.,*

$$(\widetilde{\mathrm{rad}}^{\,n}\widetilde{\mathfrak{a}})\widetilde{A} = \widetilde{\mathrm{rad}}^{\,n}\widetilde{A} = \mathrm{rad}^n\,\widetilde{A}_K \cap \widetilde{A} = (\mathrm{rad}^n\,\widetilde{\mathfrak{a}}_K)\widetilde{A}_K \cap \widetilde{A}, \quad \forall n \geq 0. \tag{3.0.17}$$

Corollary 3.8. *Let $\widetilde{M}$ be an $\widetilde{A}$-lattice.* (\tilde A is \tilde A_\Lamba above)

(a) $\widetilde{M}$ is $\widetilde{A}$-tight if and only if $\widetilde{M}$ is $\widetilde{\mathfrak{a}}$-tight.

$\widetilde{\mathfrak{a}}$ is the image of the (integral) small quantum group $\widetilde{u}_\zeta$ in $\widetilde{A}_\Lambda$. Also, $\mathfrak{a} := \widetilde{\mathfrak{a}}_k$ is isomorphic (through the natural surjection) to the sum of the regular blocks in the restricted enveloping algebra u of G.

Let M be an $\mathfrak{a}$-module. For a non-negative integer r, let, in the notation of (3.0.14),

$$M_{\widetilde{r}} := (\widetilde{\mathrm{gr}}\, M)_r = \widetilde{F}^r M/\widetilde{F}^{r+1}M = (\widetilde{\mathrm{rad}}^{\,r}\widetilde{\mathfrak{a}})M/(\widetilde{\mathrm{rad}}^{\,r+1}\widetilde{\mathfrak{a}})M. \tag{4.0.21}$$

(This is a slight abuse of notation. We will not use the symbol "$\widetilde{r}$", for an integer r, except as a subscript as above.) If M is an A-module, then $M_{\widetilde{r}}$ is also an A-module (with $\widetilde{F}^{>0}A := \Sigma_{n>0}F^nA$ acting trivially).

Put $\widetilde{\mathrm{soc}}^{\,-n}M := \{x \in M \mid \widetilde{F}^n x = 0\}$, and set

$$\widetilde{\mathrm{gr}}^{\,\diamond}M := \bigoplus_{n\geq 0} \widetilde{\mathrm{soc}}^{\,-n-1}M/\widetilde{\mathrm{soc}}^{\,-n}M \tag{4.0.22}$$

with the index n giving the degree $-n$ term. (Thus, $\widetilde{\mathrm{gr}}^{\,\diamond}M$ is negatively graded.) If M is an A-module, then $\widetilde{\mathrm{gr}}^{\,\diamond}M$ is a graded $\widetilde{\mathrm{gr}}\,A$-module.

Theorem 4.5. *Suppose the A_Γ-module M has a tight lifting. Then $M_{\tilde{s}}$ has a Δ^{red}-filtration, for each $s \geq 0$, provided that*

$$\mathrm{Ext}^1_{\widetilde{\mathrm{gr}}\, A}(\widetilde{\mathrm{gr}}\, M, \widetilde{\mathrm{gr}}\,{}^{\diamond}Q^\sharp(\mu)) = 0, \quad \forall \mu \in \Gamma.$$

This is a $\widetilde{U}_\zeta$-module, and also $\widetilde{A}_\Lambda$-lattice, because of our requirements on Λ. Similarly, we can define a $\widetilde{A}_\Lambda$-lattice (or a $\widetilde{U}_{\zeta,\Lambda}$-module)

A PIM when restricted to the small integral quantum group

$$\widetilde{Q}^\sharp(\gamma) := \widetilde{Q}^\sharp(\gamma_0) \otimes \widetilde{\nabla}(\gamma_1)^{[1]} \tag{2.0.8}$$

As a consequence of this discussion, we have the following result.

Proposition 2.6. *Assume that $p \geq 2h-2$ is an odd prime. Let Γ be a finite, non-empty ideal in $X_{\mathrm{reg}}(T)_+$ and let Λ be as above.* (Lamda is generally larger than Gamma.)

(a) The modules defined in (2.0.7) and (2.0.8) satisfy

$$\begin{cases} \overline{\widetilde{P}^\sharp(\gamma)} \cong P^\sharp(\gamma); \\ \overline{\widetilde{Q}^\sharp(\gamma)} \cong Q^\sharp(\gamma) \end{cases}$$

for all $\gamma \in \Gamma$, where $Q^\sharp(\gamma)$ and $P^\sharp(\gamma)$ are defined as in (2.0.3).

Theorem 4.4. *Suppose an A_Γ-module M has an $\widetilde{A}$-tight lifting to a lattice $\widetilde{M}$ for $\widetilde{A}$. Then $M_{\widetilde{s}}$ has a Δ^{red}-filtration, for each $s \geq 0$, if and only if*

(Here \tilde A is associated to a poset Lambda, generally larger than Gamma.)

$$\mathrm{Ext}^1_{\mathrm{gr}\widetilde{A}}(\mathrm{gr}\widetilde{M}, \mathrm{gr}^{\diamond}\widetilde{Q}^{\sharp}(\mu)) = 0, \quad \forall \mu \in \Gamma. \tag{4.0.25}$$

proved in [25], see especially Remark 3.18 and Theorem 5.5 there.

For the algebras $\widetilde{A}_\Lambda$, a key step in [25] in proving that $\mathrm{gr}\widetilde{A}_\Lambda$ is a QHA is showing that each $\mathrm{gr}\widetilde{A}_\Lambda$-module $\mathrm{gr}\widetilde{\Delta}(\lambda)$, $\lambda \in \Lambda$, has a simple head. As a consequence of this fact we also record the following result, using the proof of [25, Cor. 3.15].

[25]is a PS paper"Forced gradings..."

Theorem 3.1. *Let $\widetilde{N}$ be a $\widetilde{A}_\Lambda$-lattice which has a $\widetilde{\Delta}$-filtration. Then the graded $\mathrm{gr}\widetilde{A}_\Lambda$-module $\mathrm{gr}\widetilde{N}$ has a $\mathrm{gr}\widetilde{\Delta}$-filtration. In addition, the multiplicity of $\widetilde{\Delta}(\nu)$ as a section of $\widetilde{N}$ agrees with the multiplicity of $\mathrm{gr}\widetilde{\Delta}(\nu)$ as a graded section of $\mathrm{gr}\widetilde{N}$.* (Uses p>2h-2.)

As a consequence, we obtain

Corollary 3.2. *Let $\widetilde{A}_\Lambda$ be as above and form the graded QHA $\mathrm{gr}\widetilde{A}$ with weight poset Λ. Then $\widetilde{B} := (\mathrm{gr}\widetilde{A}_\Lambda)_0$ (the term in grade 0 in $\mathrm{gr}\widetilde{A}$) is an integral QHA with weight poset Λ, standard (resp., costandard) modules $\widetilde{\Delta}^{\mathrm{red}}(\lambda)$ (resp., $\widetilde{\nabla}_{\mathrm{rad}}(\lambda)$), $\lambda \in \Lambda$.*

Theorem 4.3. *Suppose that $N \in A$–mod is annihilated by $\widetilde{\operatorname{rad}\widetilde{\mathfrak{a}}}$. Then N has a Δ^{red}-filtration if and only if*

$$\dim N = \sum_{\mu\in\Lambda} \dim \operatorname{Hom}_A(N, \nabla_{\mathrm{red}}(\mu)) \dim \Delta^{\mathrm{red}}(\mu). \tag{4.0.24}$$

Proof. By Corollary 3.2, $\widetilde{A}/\widetilde{\operatorname{rad}\widetilde{A}} = (\mathrm{gr}\widetilde{A})_0$ is an integral QHA with standard (resp., costandard) objects $\widetilde{\Delta}^{\mathrm{red}}(\lambda)$ (resp., $\widetilde{\nabla}_{\mathrm{rad}}(\lambda)$). Because $\widetilde{A}$ is $\widetilde{\mathfrak{a}}$-tight, $\overline{\widetilde{A}/\widetilde{\operatorname{rad}\widetilde{A}}} \cong A/\widetilde{\operatorname{rad}\widetilde{\mathfrak{a}}A}$ by (3.0.20) in Corollary 3.8. Now N has a semistandard filtration in the sense of [27], with the multiplicity with which a nontrivial homomorphic image of a given $\Delta^{\mathrm{red}}(\mu)$ appears equal to $\dim \operatorname{Hom}_A(N, \nabla_{\mathrm{red}}(\mu))$. Thus,

$$\dim N \leq \sum_{\mu\in\Lambda} \dim \operatorname{Hom}_A(N, \nabla_{\mathrm{red}}(\mu)) \dim \Delta^{\mathrm{red}}(\mu)$$

Equality holds if and only if each homomorphic image of a $\Delta^{\mathrm{red}}(\mu)$ appearing is actually isomorphic to $\Delta^{\mathrm{red}}(\mu)$, in which case the filtration is a Δ-filtration for the QHA $A/\widetilde{\operatorname{rad}\widetilde{\mathfrak{a}}A}$, i. e., a Δ^{red}-filtration. □

Delta^p(gamma)=Delta^red(gamma),all gamma in Gamma

Theorem 7.1. *Assume that $p \geq 2h-2$ is odd. Suppose that $\lambda \in X_{\rm reg}(T)_+$ and that the LCF holds for the poset $\Gamma := \{\gamma \in X_{\rm reg}(T)_+ \,|\, \gamma < \lambda\}$. Then each section $\Delta_{\widetilde{s}}(\lambda)$, $s \geq 0$, viewed as a rational G-module, has a $\Delta^{\rm red}$-filtration. Each standard module $\Delta(\gamma)$, with $\gamma \in X(T)_+$ satisfying $\gamma \leq \lambda$, also has a $\Delta^{\rm red}$-filtration.*

Using the following lemma, we can recast Theorem 5.1 using G_1-radical series. We continue to assume in the rest of this section that $p \geq 2h-2$ is an odd prime and that the LCF holds for all regular, restricted weights. For $\mu_0 \in X_1(T)$, let $Q_1(\mu_0) = \widehat{Q}_1(\mu_0)|_{G_1}$. Also, $u = u(\mathfrak{g})$ is the restricted enveloping algebra of the Lie algebra $\mathfrak{g}$ of G.

Lemma 5.4. *Let $\mu_0 \in X_1(T) \cap X_{\rm reg}(T)_+$. For $n \geq 0$, we have*

$$\overline{\widetilde{Q}_\zeta(\mu_0) \cap \operatorname{rad}^n_{u_\zeta} Q_\zeta(\mu_0)} \cong \operatorname{rad}^n_{u(\mathfrak{g})} Q_1(\mu_0). \tag{5.0.28}$$

For an $\mathfrak{a}$-module M, we put (for emphasis) $\mathrm{gr}_{\mathfrak{a}} M = \bigoplus_{n \geq 0} \mathrm{rad}^n M / \mathrm{rad}^{n+1} M$, where here $\mathrm{rad}^n M = (\mathrm{rad}\, \mathfrak{a})^n M$. As a consequence, we immediately obtain the following important result.

All here assuming the LCF and p>2h-2

Corollary 5.6. *For $n \in \mathbb{N}$,*

$$\overline{\widetilde{\mathrm{rad}}^{\,n} \widetilde{\mathfrak{a}}} = (\mathrm{rad}\, \mathfrak{a})^n.$$

Thus, for any $\mathfrak{a}$-module M, we have $\widetilde{\mathrm{gr}}\, M = \mathrm{gr}_{\mathfrak{a}}(M)$.

Theorem 5.7. *For any $\lambda \in X_{\mathrm{reg}}(T)_+$, each section of the G_1-radical series of $\Delta(\lambda)$ has a Δ^{red}-filtration. In particular, each section of this radical series has a Δ^p-filtration.*

Not posted. gr A here is defined by the small a radical series.The LCF is assumed, and p>2h-2.

hence isomorphisms. $\square$

Theorem 3.2. *For $\lambda, \mu \in X_{\rm reg}(T)_+$, $\mathrm{Ext}^n_{G_1}(\Delta^{\rm red}(\lambda), \nabla_{\rm red}(\mu))^{[-1]}$ has a ∇-filtration for all $n \geq 0$. (Equivalently, $\mathrm{Ext}^n_{G_1}(L(\lambda), L(\mu))^{[-1]}$ has a ∇-filtration.)*

Theorem 3.3. *Let $\lambda, \mu \in X_{\rm reg}(T)_+$. Then, for any non-negative integer n and any integer r, $\mathrm{ext}^n_{\mathrm{gr}A}(\Delta^{\rm red}(\lambda), \nabla_{\rm red}(\mu)(r)) = 0$ unless $r = n$.*

When $n = r$ in the statement of the theorem, the value of $\dim \mathrm{ext}^n_{\mathrm{gr}A}(\Delta^{\rm red}(\lambda), \nabla_{\rm red}(\mu))$ can thus be calculated in terms Kazhdan-Lusztig polynomials; see [11, Thm. 5.4], which gives the corresponding calculation of Ext^n.

Corollary 3.4. *Suppose the ideal Γ is contained in the Janzten region, then $\mathrm{gr}A_\Gamma$ is a Koszul algebra. In addition, $\mathrm{gr}\Delta(\lambda)$ is a Koszul module and $\mathrm{gr}A_\Gamma$-mod has a graded Kazhdan-Lusztig theory.*

Recall that a positively graded algebra B (taken to be finite dimensional over a field) is Koszul provided

(1) the algebra B_0 is semisimple;
(2) if L, L' are irreducible B-modules given pure grade 0, then
$$\mathrm{Ext}^n_B(L, L'(r)) \neq 0 \implies r = n.$$

Theorem 3.3 above inspires the following generalization.

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Definition 0.1. A finite dimensional positively graded algebra B with poset Λ is called Q-Koszul provided that

(1) The algebra B_0 is quasi-hereditary, with poset Λ and standard (resp., costandard) modules $\Delta^0(\lambda)$ (resp., $\nabla_0(\lambda)$), $\lambda \in \Lambda$.

(2) If $\lambda, \mu \in \Lambda$ and $\Delta^0(\lambda)$ and $\nabla_0(\mu)$ are given pure grade 0 as B-modules, then μ

$$\mathrm{Ext}^n_B(\Delta^0(\lambda), \nabla_0(\mu)(r)) \neq 0 \implies n = r.$$

This is clearly a natural definition to make provided there are interesting examples. Of course, Koszul algebras and quasi-hereditary algebras are examples of Q-Koszul algebras rather trivially. But Theorem 3.3 shows there are many interesting examples in modular representation theory of algebraic groups provided p is large enough. (Assume the weights involved are p-regular, the LCF holds and p>2h-2.)

FORCED GRADINGS IN INTEGRAL QUASI-HEREDITARY ALGEBRAS WITH APPLICATIONS TO QUANTUM GROUPS

BRIAN J. PARSHALL AND LEONARD L. SCOTT

Theorem 5.3. *Assume that $p > 2h - 2$ is an odd prime and consider the algebra $\widetilde{A} = \widetilde{U}_{\zeta,\Gamma}$, where Γ is an non-empty ideal of p-regular dominant weights. Then $\operatorname{gr} \widetilde{A}$ is a QHA over $\mathcal{O}$, with standard modules $\operatorname{gr} \widetilde{\Delta}(\lambda)$, $\lambda \in \Gamma$.*

In addition, we assume, for the rest of this section, that A has a pure subalgebra $\widetilde{\mathfrak{a}}$ and a Wedderburn complement $\widetilde{A}_{K,0}$ of $\widetilde{A}_K$. For use in the results below, we record the following conditons.

Needed to prove gr \tilde A is integral QHA. Need also that it is QHA on base change to K.

Conditions 4.1. (1) $\widetilde{\mathfrak{a}}_K$ has a tight grading $\widetilde{\mathfrak{a}}_K = \widetilde{\mathfrak{a}}_{K,0} \oplus \widetilde{\mathfrak{a}}_{K,1} \oplus \cdots$[14].

(2) $\operatorname{rad} \widetilde{A}_K = (\operatorname{rad} \widetilde{\mathfrak{a}}_K)\widetilde{A}_K = \widetilde{A}_K(\operatorname{rad} \widetilde{\mathfrak{a}}_K)$.

(3) For $\lambda \in \Lambda$, $\Delta_K(\lambda)$ has a graded $\widetilde{\mathfrak{a}}_K$-structure, and is generated as an $\widetilde{\mathfrak{a}}_K$-module by $\Delta_K(\lambda)_0$.

(4) In (3), $\Delta_K(\lambda)_0$ is $\widetilde{A}_{K,0}$-stable. Also, $\widetilde{A}_{K,0}$ contains $\widetilde{\mathfrak{a}}_{K,0}$ (defined in (1)) and all idempotents $e_\lambda, \lambda \in \Lambda$.

(5) $\widetilde{\mathfrak{a}}$ has a positive grading $\widetilde{\mathfrak{a}} = \oplus_{r \geq 0} \widetilde{\mathfrak{a}}_r$ such that $K\widetilde{\mathfrak{a}}_r = \widetilde{\mathfrak{a}}_{K,r}$, the rth grade of $\widetilde{\mathfrak{a}}_K$ in (1), for each $r \in \mathbb{N}$.

Theorem 7.1. *Assume that $p > h$ is a prime. Then the algebra $\widetilde{u}'_\zeta$ over $\mathcal{O}$ has a positive grading which base changes to the Koszul grading on u'_ζ obtained in* [2, §§17-18].

regular part of small quantum group

Parshall and Scott believe that the quasi-heredity of $\mathrm{gr}\widetilde{A}$ mentioned above can be proved at least for $p > h$ and possibly for all p at least in type A (and perhaps in other simply laced types). A step in this direction is the following conjecture, which is a forced *graded* version of the Humphreys-Verma conjecture. Part (a) was announced without proof in the PS paper, Varitions on a theme of Cline and Donkin, Algebras and Representation Theory, in press 2012; cf footnote 11, Part (b) is new.

Conjecture 2.11. In arbitrary characteristic p:

(a) Let Q be a PIM for the restricted enveloping algebra u of G. For any suitably large poset Γ of dominant weights, the graded $\mathrm{gr}u := \bigoplus_{n\geq 0} \mathrm{rad}^n u/\mathrm{rad}^{n+1} u$-module

$$\mathrm{gr}_u Q := \bigoplus (\mathrm{rad}^n u)Q/(\mathrm{rad}^{n+1} u)Q$$

is the restriction to $\mathrm{gr}\, u$ of a graded $\mathrm{gr}_u A_\Gamma$-module.

(b) Let $\widetilde{Q}$ be a PIM for the integral form $\widetilde{u}$ of the small quantum group. For any suitably large poset Γ of dominant weights, the graded $\mathrm{gr}\widetilde{u} \coloneqq \bigoplus_{n\geq 0} \widetilde{\mathrm{rad}}^{\,n}\widetilde{u}/\widetilde{\mathrm{rad}}^{\,n+1}\widetilde{u}$-module

$$\mathrm{gr}_{\widetilde{u}}\widetilde{Q} \coloneqq \bigoplus (\widetilde{\mathrm{rad}}^{\,n}\widetilde{u})\widetilde{Q}/(\widetilde{\mathrm{rad}}^{\,n+1}\widetilde{u})\widetilde{Q}$$

is the restriction to $\mathrm{gr}\,\widetilde{u}$ of a graded $\mathrm{gr}_{\widetilde{u}}\widetilde{A}_\Gamma$-module.

A SEMISIMPLE SERIES FOR q-WEYL AND q-SPECHT MODULES

BRIAN J. PARSHALL AND LEONARD L. SCOTT

Abstract. In [41], the authors studied the radical filtration of a Weyl module $\Delta_\zeta(\lambda)$ for quantum enveloping algebras $U_\zeta(\mathring{\mathfrak g})$ associated to a finite dimensional complex semisimple Lie algebra $\mathring{\mathfrak g}$. There $\zeta^2 = \sqrt[e]{1}$ and λ was, initially, required to be e-regular. Some additional restrictions on e were required—e. g., $e > h$, the Coxeter number, and e odd. Translation to a facet gave an explicit semisimple series for all quantum Weyl modules with singular, as well as regular, weights. That is, the sections of the filtration are explicit semisimple modules with computable multiplicities of irreducible constituents. However, in the singular case, the filtration conceivably might not be the radical filtration. This paper shows how a similar semisimple series result can be obtained for all positive integers e in case $\mathring{\mathfrak g}$ has type A, and for all positive integes $e \geq 3$ in type D. One application describes semisimple series (with computable multiplicities) on q-Specht modules. We also discuss an analogue for Weyl modules for classical Schur algebras and Specht modules for symmetric group algebras in positive characteristic p. Here we assume the James Conjecture and a version of the Bipartite Conjecture.

7. Quantum enveloping algebras and category equivalences

We continue to work with the indecomposable root system $\overset{\circ}{\Phi}$, and we let ℓ be a positive integer. Set $D = (\theta_l, \theta_l)/(\theta_s, \theta_s) \in \{1, 2, 3\}$. Let

$$e := \begin{cases} \ell, & \text{if } \ell \text{ is odd;} \\ \ell/2, & \text{if } \ell \text{ is even.} \end{cases} \tag{7.1}$$

Theorem 8.1.1. *Assume that K is a field containing $\mathbb{Q}(\zeta)$.*

(a) For $\lambda \in \Lambda^+(n,r)$, the q-Weyl module $\Delta_q(\lambda)$ for the q-Schur algebra $S_q(n,r)$ has a filtration $\Delta_q(\lambda) = F^0(\lambda) \supseteq F^1(\lambda) \supseteq \cdots \supseteq F^m(\lambda) = 0$ with semisimple sections $F^i(\lambda)/F^{i+1}(\lambda)$ in which, given $\nu \in \Lambda^+(r)$, the multiplicity of $L_q(\nu)$ in $F^i(\lambda)/F^{i+1}(\lambda)$ is the coefficient of $t^{l(w_{\bar\lambda})-l(w_{\bar\nu})-i}$ in the inverse Kazhdan-Lusztig polynomial $Q_{w_{\bar\nu},w_{\bar\lambda}}$ associated to the affine Weyl group W_e of type A_{r-1}.

(b) For $\lambda \in \Lambda^+(r)$, the q-Specht module S^λ for the Hecke algebra H has a filtration $0 = G^0(\lambda) \subseteq G^1(\lambda) \subseteq \cdots \subseteq G^m(\lambda) = S^\lambda$ with semisimple sections $G^{i+1}(\lambda)/G^i(\lambda)$ in which, given $\nu \in \Lambda^+_{\mathrm{res}}(r)$, the multiplicity of the irreducible H-module $D^\Psi_{\nu'}$ in the section $G^{i+1}(\lambda)/G^i(\lambda)$ is the coefficient of $t^{l(w_{\bar\lambda})-l(w_{\bar\nu})-i}$ in the inverse Kazhdan-Lusztig polynomial $Q_{w_{\bar\nu},w_{\bar\lambda}}$ associated to the affine Weyl group W_e of type A_{r-1}.

Proof. (a) is merely a translation into the language of q-Schur algebras of Theorem 7.5(a).

As for (b), we can take $n = r$. We first observe $T := T(r,r) \cong S_q(n,r)f$ for an idempotent $f \in S_q(r,r)$

Now we can prove our main result on strong linkage in the sense of [27, II, 6.1–6.11]. To avoid conflict with our notation on $\mathfrak{h}^*$, we use $\uparrow_e$ to denote the $\uparrow$ ordering in [27, II, Ch. 6]. That is, if $\mu, \nu \in \overset{\circ}{P}$, write $\mu \uparrow_e \nu$ to mean that $\mu = \nu$ or there exists a chain $\mu = \mu_0 \leq \mu_1 \leq \cdots \leq \mu_m = \nu$ in $\overset{\circ}{P}$ and reflections $s_1, \cdots, s_m$ (not necessarily fundamental) in W_e such that $s_i \cdot \mu_i = \mu_{i+1}$, for $i = 1, \cdots, m$. The order $\leq$ used is the usual dominance order on $\overset{\circ}{P}$.

Theorem 9.6. *Let $\mu, \nu \in \overset{\circ}{P}{}^+$ lie in the same orbit of W_e under the dot action. Then*

$$\mu \uparrow_e \nu \iff f(\mu) \leq_e f(\nu)$$

where $f(\mu)$ and $f(\nu)$ are the elements of W_e described above.