Cyclotomic quiver Schur algebras of type A

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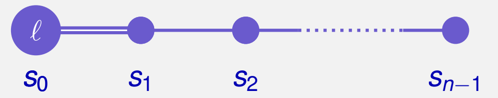
Complex reflection groups of type $G(\ell, 1, n)$

The complex reflection group $W_{\ell,n}$ of type $G(\ell, 1, n)$ is the group with presentation encoded by the Coxeter diagram



$\Longrightarrow$ $W_{\ell,n} = \langle s_0, s_1, \dots, s_{n-1} \mid s_0^\ell = s_i^2 = 1 + \text{braid relations}\rangle$

Concretely, $W_{\ell,n} = \left\langle \begin{pmatrix} \sqrt[\ell]{1} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} \right\rangle$

As an abstract group $W_{\ell,n} = \mathbb{Z}/\ell\mathbb{Z} \wr \mathfrak{S}_n = (\mathbb{Z}/\ell\mathbb{Z} \times \cdots \times \mathbb{Z}/\ell\mathbb{Z}) \rtimes \mathfrak{S}_n$

$\Longrightarrow$ the ordinary irreducible representations of $W_{\ell,n}$ are labelled by ℓ-tuples of partitions $\boldsymbol{\lambda} = (\lambda^{(1)}|\dots|\lambda^{(\ell)})$ such that $|\lambda^{(1)}| + \cdots + |\lambda^{(\ell)}| = n$.

Let $\mathcal{P}_n^\Lambda$ be the set of multipartitions of n.

Cyclotomic Hecke algebras of type $G(\ell, 1, n)$

Fix a ring R and parameters $\xi \in R^\times$ and $Q_1, \dots, Q_\ell \in R$.

The cyclotomic Hecke algebra $\mathcal{H}_n = \mathcal{H}_n(\xi; Q_1, \dots, Q_\ell\boldsymbol{\kappa})$ is the unital associative algebra generated by $T_1, \dots, T_{n-1}, L_1, \dots, L_n$ with relations

$$\prod_{l=1}^{\ell}(L_1 - Q_l[\kappa_l]) = 0, \qquad (T_r + 1)(T_r - \xi) = 0,$$
$$L_1T_1L_1T_1 = T_1L_1T_1L_1, \qquad T_sT_{s+1}T_s = T_{s+1}T_sT_{s+1}$$
$$L_rL_t = L_tL_r,$$
$$T_rL_r + 1 = L_{r+1}T_r - (\xi - 1)L_{r+1}T_rL_r + 1 = L_{r+1}T_r - (\xi - 1)L_{r+1},$$
$$T_rL_t = L_tT_r, \qquad \text{if } t \neq r, r+1,$$
$$T_rT_s = T_sT_r, \qquad \text{if } |r - s| > 1.$$

Remark $L_r' = (\xi - 1)L_r + 1$ gives the usual representation when $\xi \neq 1$.

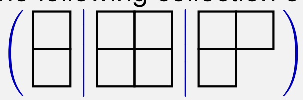
The integral case: fix a multicharge $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_\ell) \in \mathbb{Z}^\ell$ and set $Q_l = [\kappa_l]$, where for $k \in \mathbb{Z}$ we define

$$[k] = \begin{cases} 1 + \xi + \cdots + \xi^{k-1}, & \text{if } k \geq 0, \\ -(\xi^k + \xi^{k+1} + \cdots + \xi^{-1}), & \text{if } k < 0. \end{cases}$$

Tableaux combinatorics...by example

Let $\boldsymbol{\lambda} = (1^2|2^2|2, 1)$, a multipartition of 9 with $\ell = 3$.

The diagram of $\boldsymbol{\lambda}$ is the following collection of boxes in the plane:



A $\boldsymbol{\lambda}$-tableau is a filling of its diagram with the numbers $1, \dots, n$:

$$\mathfrak{t}^{\boldsymbol{\lambda}} = \left(\begin{array}{c} 1 \\ 2 \end{array} \middle| \begin{array}{cc} 3 & 4 \\ 5 & 6 \end{array} \middle| \begin{array}{cc} 7 & 8 \\ 9 & \end{array} \right)$$

A tableau is standard if its entries increase from left to right in each row and from top to bottom in each column.

If $\mathfrak{t}$ is a $\boldsymbol{\lambda}$-tableau we let $d(\mathfrak{t}) \in \mathfrak{S}_n$ be the *unique* permutation such that $\mathfrak{t} = \mathfrak{t}^{\boldsymbol{\lambda}} \cdot d(\mathfrak{t})$.

So $\mathfrak{t} = \left(\begin{array}{c} 3 \\ 7 \end{array} \middle| \begin{array}{cc} 1 & 5 \\ 2 & 8 \end{array} \middle| \begin{array}{cc} 4 & 6 \\ 9 & \end{array} \right) \Longrightarrow d(\mathfrak{t}) = (1, 3)(2, 7, 4, 5)(6, 8)$.

Let $\mathrm{Std}(\boldsymbol{\lambda})$ be the set of standard $\boldsymbol{\lambda}$-tableaux.

The Murphy basis of $\mathcal{H}_n^\Lambda$

For $(\mathsf{s},\mathsf{t}) \in \mathrm{Std}^2(\boldsymbol{\lambda})$ let $m_{\mathsf{st}} = T_{d(\mathsf{s})^{-1}} m_{\boldsymbol{\lambda}} T_{d(\mathsf{t})}$, where $m_{\boldsymbol{\lambda}} = u_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}}$ and

$$u_{\boldsymbol{\lambda}} = \prod_{1\le k<l\le \ell} \prod_{(k,r,c)\in\boldsymbol{\lambda}} \xi^{-\kappa_l}(L_k - [\kappa_l]) \quad \text{and} \quad x_{\boldsymbol{\lambda}} = \sum_{w\in\mathfrak{S}_{\boldsymbol{\lambda}}} T_w.$$

Set $\mathrm{Std}^2(\mathcal{P}_n^\Lambda) = \bigsqcup_{\boldsymbol{\lambda}} \mathrm{Std}^2(\boldsymbol{\lambda})$.

Theorem (Dipper-James-M.)

The basis $\{\, m_{\mathsf{st}} : (\mathsf{s},\mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}_n^\Lambda) \,\}$ is a cellular basis of $\mathcal{H}_n^\Lambda$.

The whole point of constructing a cellular basis is that it gives, for free, a collection of Specht modules, or cell modules. Using these we quickly obtain a complete set of simple $\mathcal{H}_n^\Lambda$-modules.

Cellular algebras ... (Graham–Lehrer)

Cellular bases can be thought of as approximations to the Wedderburn basis which are defined over rings where the algebra is not semisimple.

(C1) The map $* : m_{\mathsf{st}} \mapsto m_{\mathsf{ts}}$ is an anti–isomorphism.

(C2) Given t and $h \in \mathcal{H}_n^\Lambda$ there exist $r_{\mathsf{tv}}^h \in R$ such that

$$m_{\mathsf{st}} h \equiv \sum_{\mathsf{v}\in\mathrm{Std}(\boldsymbol{\lambda})} r_{\mathsf{tv}}^h m_{\mathsf{sv}} \quad \text{(mod higher terms)}$$

Importantly, the scalar r_{tv}^h is independent of s !

(C1) and (C2) combined imply:

(C2)′ $$h m_{\mathsf{st}} \equiv \sum_{\mathsf{v}\in\mathrm{Std}(\boldsymbol{\lambda})} r_{\mathsf{sv}}^h m_{\mathsf{vt}} \quad \text{(mod higher terms)}$$

Specht modules

The Specht module $\underline{S}^{\boldsymbol{\lambda}}$ is the free R–module with basis $\{\, m_{\mathsf{t}} : \mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda}) \,\}$ and with $\mathcal{H}_n^\Lambda$-action:

$$m_{\mathsf{t}} h = \sum_{\mathsf{v}\in\mathrm{Std}(\boldsymbol{\lambda})} r_{\mathsf{tv}}^h m_{\mathsf{v}}.$$

Compare with: $m_{\mathsf{st}} h \equiv \sum_{\mathsf{v}} r_{\mathsf{tv}}^h m_{\mathsf{sv}}$ (mod higher terms)

Importantly, $\underline{S}^{\boldsymbol{\lambda}}$ has a natural bilinear form $\langle\ ,\ \rangle$.

To define $\langle\ ,\ \rangle$ it is enough to specify $\langle m_{\mathsf{t}}, m_{\mathsf{u}} \rangle$:

Consider: $m_{\mathsf{st}} m_{\mathsf{uv}} = \langle m_{\mathsf{t}}, m_{\mathsf{u}} \rangle m_{\mathsf{sv}}$

$\Longrightarrow$ $\mathrm{rad}\, \underline{S}^{\boldsymbol{\lambda}} = \{\, x \in \underline{S}^{\boldsymbol{\lambda}} : \langle x, y\rangle = 0 \text{ for all } y \in \underline{S}^{\boldsymbol{\lambda}} \,\}$ is an $\mathcal{H}_n^\Lambda$-submodule of $\underline{S}^{\boldsymbol{\lambda}}$ as $\langle xh, y\rangle = \langle x, yh^*\rangle$

Define $\underline{D}^{\boldsymbol{\lambda}} = \underline{S}^{\boldsymbol{\lambda}} / \mathrm{rad}\, \underline{S}^{\boldsymbol{\lambda}}$

Theorem (Graham-Lehrer)

Over a field, the non-zero $\underline{D}^{\boldsymbol{\lambda}}$ give a complete set of pairwise non-isomorphic irreducible $\mathcal{H}_n^\Lambda$-modules.

Cyclotomic Schur algebras

The algebra $\mathcal{H}_n^\Lambda$ has a quasi-hereditary cover

$$\mathcal{S}_n^{\mathrm{DJM}} = \mathrm{End}_{\mathcal{H}_n^\Lambda}\big(\textstyle\bigoplus_{\boldsymbol{\mu}} M^{\boldsymbol{\mu}}\big), \qquad \text{where } M^{\boldsymbol{\mu}} = m_{\mathsf{t}^{\boldsymbol{\mu}}\mathsf{t}^{\boldsymbol{\mu}}}\mathcal{H}_n^\Lambda,$$

called the cyclotomic Schur algebra.

The algebra $\mathcal{S}_n^{\mathrm{DJM}}$ is the Dipper-James q-Schur algebra when $\Lambda = \Lambda_0$. It is the classical Schur algebra when $\Lambda = \Lambda_0$ and $\xi = 1$.

The algebra $\mathcal{S}_n^{\mathrm{DJM}}$ is cellular, in the sense of Graham-Lehrer, with

- Weyl modules $\underline{\Delta}^{\boldsymbol{\lambda}}$, for λ a multipartition
- Simple modules $\underline{L}^{\boldsymbol{\mu}} = \underline{\Delta}^{\mu} / \mathrm{rad}\, \underline{\Delta}^{\mu} \neq 0$

There is an exact Schur functor $\mathrm{F}_n : \mathcal{S}_n^{\mathrm{DJM}}\text{-Mod} \longrightarrow \mathcal{H}_n^\Lambda\text{-Mod}$ such that

- $\mathrm{F}_n(\underline{\Delta}^{\boldsymbol{\lambda}}) = \underline{S}^{\boldsymbol{\lambda}}$
- $\mathrm{F}_n(\underline{L}^{\boldsymbol{\mu}}) = \underline{D}^{\boldsymbol{\mu}}$, which is 0 is $\boldsymbol{\mu}$ is not restricted

Therefore, $[\underline{\Delta}^{\boldsymbol{\lambda}} : \underline{L}^{\boldsymbol{\mu}}] = [\underline{S}^{\boldsymbol{\lambda}} : \underline{D}^{\boldsymbol{\mu}}]$ if $\underline{D}^{\boldsymbol{\mu}} \neq 0$.

Brundan-Kleshchev's graded isomorphism theorem

Theorem (Brundan and Kleshchev)

Suppose that K is a field. Then $\mathcal{H}_n^\Lambda \cong \mathcal{R}_n^\Lambda$ is a $\mathbb{Z}$-graded algebra.

This is proved by giving a new homogeneous set of generators and relations for $\mathcal{H}_n^\Lambda$. The degree function on $\mathcal{H}_n^\Lambda$ is determined by the Cartan matrix of quiver of $\mathbb{Z}/e\mathbb{Z}$ and Λ is a dominant for the corresponding Kac-Moody algebra.

We want to put a grading on the Schur algebra $\mathcal{S}_n^\Lambda$.

By Brundan-Kleshchev-Wang there is a graded lift S^λ of the ungraded Specht module $\underline{S}^\lambda$. Is there a graded lift of M^λ?

Theorem (Ariki, Brundan-Stroppel, Hu-M., Stroppel-Webster)

There is a $\mathbb{Z}$-grading on $\mathcal{S}_n^\Lambda$ which is compatible with the grading on $\mathcal{H}_n^\Lambda$.

Cyclotomic quiver Schur algebras

As $\mathcal{H}_n^\Lambda$ is graded one can show directly that $M = \bigoplus_\mu M^\mu$ is graded.

By definition, $M^\mu = m_{\mathfrak{t}^\mu\mathfrak{t}^\mu}\mathcal{H}_n^\Lambda$

Write $m_{\mathfrak{t}^\mu\mathfrak{t}^\mu} = \sum_{k\in\mathbb{Z}} m_k^\mu$, with m_k^μ homogeneous of degree k

Theorem (Hu-M.)

Suppose that $\mu \in \mathcal{P}_n^\Lambda$ and let $d_\mu = 2\deg \mathfrak{t}^\mu$. Then:

- $m_k^\mu \neq 0$ *only if* $k \geq d_\mu$.
- $m_{d_\mu}^\mu \equiv m^\mu \mod (\mathcal{H}_n^\Lambda)^{\rhd\mu}$ *and if* $k > d_\mu$ *then* $m_k^\mu \in (\mathcal{H}_n^\Lambda)^{\rhd\mu}$.
- $e_\beta M^\mu = e_\beta m_{d_\mu}^\mu \mathcal{H}_n^\Lambda$, *for a known central idempotent* e_β.

The main point is that $e_\beta M^\mu$ is generated by the homogeneous component of $e_\beta m^\mu$ which is of minimal degree.

Corollary

The cyclotomic Schur algebra $\mathcal{S}_n^{DJM} = \operatorname{End}_{\mathcal{H}_n^\Lambda}(M)$ inherits a $\mathbb{Z}$-grading from $\mathcal{H}_n^\Lambda$. Consequently, $\mathcal{S}_n^{DJM}$ is a graded cellular algebra.

A graded cellular basis of $\mathcal{H}_n^\Lambda$

The KLR generators of $\mathcal{H}_n^\Lambda$, which induce its grading, are

$$\psi_1,\dots,\psi_{n-1}, \quad y_1,\dots,y_n, \quad e(\mathbf{i}), \qquad \text{for } \mathbf{i} \in I^n = (\mathbb{Z}/e\mathbb{Z})^n.$$

Theorem (Hu-M.)

Suppose that K is a field, Then $\mathcal{H}_n^\Lambda$ is a graded cellular algebra with graded cellular basis $\{\,\psi_{\mathfrak{st}} : \mathfrak{s},\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ and $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda\,\}$.

Example Take $e = 3$, $\Lambda = 2\Lambda_0 + \Lambda_2$ and $\boldsymbol{\lambda} = (4,2|1|1^2)$.
The initial tableau $\mathfrak{t}^\lambda$ and the residues in $\boldsymbol{\lambda}$ are:

$$\mathfrak{t}^\lambda = \left(\begin{array}{cccc} 1&2&3&4\\ 5&6 \end{array}\;\middle|\;\begin{array}{c}7\end{array}\;\middle|\;\begin{array}{c}8\\9\end{array}\right) \text{ and } \left(\begin{array}{cccc} 0&1&2&0\\ 2&0 \end{array}\;\middle|\;\begin{array}{c}0\end{array}\;\middle|\;\begin{array}{c}2\\1\end{array}\right)$$

Then $\psi_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} = e(\mathbf{i}^\lambda)y^\lambda$, where $\mathbf{i}^\lambda = \operatorname{res}(\mathfrak{t}^\lambda) = (0,1,2,0,2,0,0,2,1)$,
The element y_λ is defined by "reading" along $\mathfrak{t}^\lambda$:

$$y_\lambda = y_1 y_3^2 y_4 y_5 y_6$$

In general, $\psi_{\mathfrak{st}} = \psi_{d(\mathfrak{s})^{-1}} e(\mathbf{i}^\lambda) y^\lambda \psi_{d(\mathfrak{t})}$, where $\mathfrak{s} = \mathfrak{t}^\lambda d(\mathfrak{s})$ and $\mathfrak{t} = \mathfrak{t}^\lambda d(\mathfrak{t})$.

Cyclotomic quiver Schur algebras when $e = 0$ or $e \geq n$

If $e = 0$ or $e \geq n$ then $e(\beta)m^\mu$ has a particularly nice form:

$$e(\beta)m^\mu = c\psi_{\mathfrak{t}^\mu\mathfrak{t}^\mu} + \text{terms of strictly higher degree}$$

Define the graded permutation module $G^\mu = \psi_{\mathfrak{t}^\mu\mathfrak{t}^\mu}\mathcal{H}_n^\Lambda\langle -\deg \mathfrak{t}^\mu\rangle$.

$\implies$ G^μ is a graded $\mathcal{H}_n^\Lambda$-module with basis

$$\{\,\psi_{\mathfrak{st}} : \mathfrak{s} \in \operatorname{Std}^\mu(\boldsymbol{\lambda}) \text{ and } \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})\,\},$$

where $\operatorname{Std}^\mu(\boldsymbol{\lambda}) = \{\,\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda}) : \mathfrak{s} \trianglerighteq \mathfrak{t}^\mu \text{ and } \operatorname{res}(\mathfrak{s}) = \operatorname{res}(\mathfrak{t}^\mu)\,\}$.

$\implies$ If $\mathfrak{s} \in \operatorname{Std}^\mu(\boldsymbol{\lambda})$ and $\mathfrak{t} \in \operatorname{Std}^\nu(\boldsymbol{\lambda})$ then $\Psi_{\mathfrak{st}}^{\mu\nu} \in \operatorname{Hom}_{\mathcal{H}_n^\Lambda}(G^\nu, G^\mu)$,
where $\Psi_{\mathfrak{st}}^{\mu\nu}(\psi_{\mathfrak{t}^\nu\mathfrak{t}^\nu}h) = \psi_{\mathfrak{st}}h$, for $h \in \mathcal{H}_n^\Lambda$.

Theorem (Hu-M.)

Suppose that $e = 0$ or $e \geq n$. Then the algebra $\mathcal{S}_n^\Lambda = \operatorname{End}_{\mathcal{H}_n^\Lambda}(\bigoplus_\mu G^\mu)$ is a quasi-hereditary graded cellular algebra with graded cellular basis

$$\{\,\Psi_{\mathfrak{st}}^{\mu\nu} : \mathfrak{s} \in \operatorname{Std}^\mu(\boldsymbol{\lambda}) \text{ and } \mathfrak{t} \in \operatorname{Std}^\nu(\boldsymbol{\lambda})\,\}$$

with $\deg \Psi_{\mathfrak{st}}^{\mu\nu} = (\deg \mathfrak{s} - \deg \mathfrak{t}^\mu) + (\deg \mathfrak{t} - \deg \mathfrak{t}^\nu)$.

Weyl modules, blocks and a graded Schur functor

The algebras $\mathcal{S}_n^\Lambda$ are in many respects nicer than the cyclotomic q-Schur algebras. For example, $\mathcal{S}_n^\Lambda = \bigoplus_\beta \mathcal{S}_\beta^\Lambda$ with each block $\mathcal{S}_\beta^\Lambda$ being a quasi-hereditary graded cellular algebra.

There are graded Weyl modules Δ^λ, graded simple modules $L^\mu = \Delta^\mu / \operatorname{rad} \Delta^\mu$, and graded decomposition numbers

$$[\Delta^\lambda : L^\mu]_q = \sum_{k\in\mathbb{Z}} [\Delta^\lambda : L^\mu\langle k\rangle]\, q^k \in \mathbb{N}[q, q^{-1}].$$

We obtain a graded Schur-Weyl duality and a graded Schur functor $\mathsf{F}_n^\Lambda : \mathcal{S}_n^\Lambda\text{-GrMod} \longrightarrow \mathcal{H}_n^\Lambda\text{-GrMod}$. Hence, we have that

$$[\Delta^\lambda : L^\mu]_q = [S^\mu : D^\mu]_q, \text{ whenever } D^\mu \neq 0.$$

A slightly harder fact is that G^μ is a direct summand of M^μ

$\Longrightarrow$ $\mathcal{S}_n^\Lambda$ is a (graded) subalgebra of $\mathcal{S}_n^{\mathsf{DJM}}$

$\Longrightarrow$ $\mathcal{S}_n^\Lambda$-Mod and $\mathcal{S}_n^{\mathsf{DJM}}$-Mod are (graded) Morita equivalent

Graded higher Schur-Weyl duality

Brundan and Kleshchev have shown that each block $\mathcal{S}_\beta^{\mathsf{DJM}}$ of $\mathcal{S}_n^{\mathsf{DJM}}$ is Morita equivalent to a block $\mathcal{O}_\beta^\Lambda$ of parabolic category $\mathcal{O}$ for $\mathfrak{gl}_N$.

Theorem (Hu-M.)

Suppose $e = 0$ and $K = \mathbb{C}$. Then there is graded equivalence of categories $\mathsf{E}_n^\Lambda : \mathcal{O}_\beta^\Lambda \longrightarrow \mathcal{S}_\beta^\Lambda$-Mod such that the diagram

$$\begin{array}{ccc} \mathcal{O}_\beta^\Lambda & \xrightarrow{\mathsf{E}_\beta^\Lambda} & \mathcal{S}_\beta^\Lambda\text{-Mod} \\ & \searrow^{\mathsf{F}_\beta^{\mathcal{O}}} & \downarrow \mathsf{F}_\beta^\Lambda \\ & & \mathcal{H}_\beta^\Lambda\text{-Mod} \end{array}$$

commutes. Consequently, $\mathcal{S}_n^\Lambda$-Mod $= \bigoplus_\beta \mathcal{S}_\beta^\Lambda$-Mod is Koszul.

In particular, the following hold:

1. $[\Delta^\lambda : L^\mu]_q = [\Delta_{\mathcal{O}}^\lambda : L_{\mathcal{O}}^\mu]_q$ are polynomials in $\mathbb{N}[q]$.
2. These polynomials describe the Jantzen and grading filtrations of Δ^λ and S^λ — which coincide with the radical filtrations for $\mathcal{S}_n^\Lambda$.

Sketch of the proof

We first show that the gradings on $\mathcal{H}_n^\Lambda$ induced by $\mathcal{R}_n^\Lambda$ and by parabolic category $\mathcal{O}$ are the "same" $=$ graded Morita equivalent.

- As category $\mathcal{O}$ is Koszul, the PIM $P_{\mathcal{O}}^\mu$ is rigid whenever $D^\mu \neq 0$: that is, the socle, radical and grading filtrations of $P_{\mathcal{O}}^\mu$ coincide.
- The polynomials $[\Delta_{\mathcal{O}}^\lambda : L_{\mathcal{O}}^\mu]_q = [S^\lambda : D^\mu]_q$ describe these filtrations and these polynomials agree for the two gradings on $\mathcal{H}_n^\Lambda$ by BK.
- By Higher Schur-Weyl duality, there is an isomorphism $\underline{S}_\beta^\Lambda \cong \underline{S}_\beta^{\mathcal{O}}$ of ungraded algebras. Fix an ungraded isomorphism $\Xi : \mathcal{S}_\beta^\Lambda \to \mathcal{S}_\beta^{\mathcal{O}}$.

$$\Longrightarrow \quad \Xi(\operatorname{rad}^s \underline{P}_{\mathcal{O}}^\mu) \cong \operatorname{rad}^s \underline{P}^\mu, \text{ for } s \geq 0$$

We manufacture a positive grading on P^μ by defining, for $f \geq 0$,

$$P^\mu(f) = \textstyle\sum_{\theta : P^\nu \to P^\mu} \operatorname{im}\theta, \quad \text{where in the sum } \deg\theta \geq f.$$

$\Longrightarrow$ $[P^\mu(f) : L^\nu\langle s\rangle] \neq 0$ only if $s \geq f$, whenever $D^\mu, D^\nu \neq 0$

$\Longrightarrow$ $[\operatorname{rad}^s P^\mu : L^\nu]_q = [\operatorname{rad}^s P_{\mathcal{O}}^\mu : L_{\mathcal{O}}^\nu]_q$, if $s \geq 0$ and $D^\mu, D^\nu \neq 0$

$\Longrightarrow$ the two gradings on $\mathcal{H}_n^\Lambda$ are graded Morita equivalent

$\Longrightarrow$ By looking at Young modules, the algebras $\mathcal{S}_\beta^{\mathcal{O}}$ and $\mathcal{S}_\beta^\Lambda$ are graded Morita equivalent $\Longrightarrow$ $\mathcal{S}_n^\Lambda$ is positively graded and Koszul.

Distinguished bases of Fock spaces and dualities

Let $\mathfrak{F}^\Lambda = \bigoplus_{n\geq 0} Rep(\mathcal{S}_n^\Lambda)$, the direct sum of Grothendieck groups. We consider $\mathfrak{F}^\Lambda$ as a $\mathbb{Z}[q, q^{-1}]$-module by setting $[M\langle 1\rangle] = q[M]$.

The Fock space $\mathfrak{F}^\Lambda$ has many natural bases including:

- Irreducible modules $\{\, [L^\mu] : \mu \in \mathcal{P}^\Lambda \,\}$.
- Standard modules $\{\, [\Delta^\mu] : \mu \in \mathcal{P}^\Lambda \,\}$.
- Projective indecomposable modules $\{\, [P^\mu] : \mu \in \mathcal{P}^\Lambda \,\}$.
- Twisted tilting modules $\{\, [T_\mu] : \mu \in \mathcal{P}^\Lambda \,\}$.

The Fock space $\mathfrak{F}^\Lambda$ comes equipped with two dualities:

Easy lemma

Define involutions by $[M]^\circledast := [M^\circledast]$ and $[M]^\# := [\operatorname{Hom}_{\mathcal{S}_n^\Lambda}(M, \mathcal{S}_n^\Lambda)]$. Then

$$[\Delta^\lambda]^\circledast = [\Delta^\lambda] + \textstyle\sum_{\lambda \rhd \mu} f_{\lambda\mu}(q)[\Delta^\mu],$$
$$[\Delta^\lambda]^\# = [\Delta^\lambda] + \textstyle\sum_{\mu \rhd \lambda} g_{\lambda\mu}(q)[\Delta^\mu],$$

for some Laurent polynomials $f_{\lambda\mu}(q), g_{\lambda\mu}(q) \in \mathbb{Z}[q, q^{-1}]$.

Canonical bases and decomposition numbers

Lusztig's Lemma, and the positivity property $d_{\lambda\mu}(q) \in \mathbb{N}[q]$, now imply:

Theorem (Hu-M.)

Suppose that $e = 0$ and $K = \mathbb{C}$. Then the three bases

$$\{\,[P^\mu] : \mu \in \mathcal{P}^\Lambda\,\}, \quad \{\,[L^\mu] : \mu \in \mathcal{P}^\Lambda\,\} \quad \text{and} \quad \{\,[T_\mu] : \mu \in \mathcal{P}^\Lambda\,\}$$

are "canonical bases" of $\mathfrak{F}^\Lambda$ which are uniquely determined by:

1. $[P^\mu]^\# = [P^\mu]$ *and* $[P^\mu] \equiv [\Delta^\mu] \pmod{q\mathfrak{F}^\Lambda_+}$
2. $[L^\mu]^\circledast = [L^\mu]$ *and* $[L^\mu] \equiv [\Delta^\mu] \pmod{q\mathfrak{F}^\Lambda_+}$
3. $[T_\mu]^\circledast = [T_\mu]$ *and* $[T_\mu] \equiv [\Delta^\mu] \pmod{q^{-1}\mathfrak{F}^\Lambda_-}$,

where $\mathfrak{F}^\Lambda_\pm$ is the $\mathbb{Z}[q^{\pm 1}]$-lattice spanned by $\{\,[\Delta^\mu] : \mu \in \mathcal{P}^\Lambda\,\}$.

In cases 1 and 3 the transition matrix is the graded decomposition matrix and in case 2 it is the inverse graded decomposition matrix.

This result is a purely formal consequence of the fact that the $[\Delta^\lambda : L^\mu]_q$ are polynomials, rather than Laurent polynomials.

In fact, this result holds if and only if $[\Delta^\lambda : L^\mu]_q \in \mathbb{N}[q]$, for all $\lambda, \mu \in \mathcal{P}^\Lambda$.

A cyclotomic LLT algorithm

Let $\iota^\mu : G^\mu \longrightarrow G^\mu$ be the identity map on G^μ

$\Longrightarrow$ ι^μ is an idempotent in $\mathcal{S}^\Lambda_n$

$\Longrightarrow$ $Z^\mu = \iota^\mu \mathcal{S}^\Lambda_n$ is a projective $\mathcal{S}^\Lambda_n$-module

$\Longrightarrow$ $Z^\mu = P^\mu \oplus \bigoplus_{\lambda \rhd \mu} p_{\lambda\mu}(q) P^\lambda$, for $p_{\lambda\mu}(q) \in \mathbb{N}[q, q^{-1}]$

Since Z^μ is a direct summand of $\mathcal{S}^\Lambda_n$, $(Z^\mu)^\# = Z^\mu$

$\Longrightarrow$ $p_{\lambda\mu}(q) = \overline{p_{\lambda\mu}(q)} = p_{\lambda\mu}(q^{-1})$

Using the cellular basis of $\mathcal{S}^\Lambda_n$,

$$[Z^\mu] = [\Delta^\mu] + \sum_{\nu \rhd \mu} \sum_{\mathsf{s} \in \mathrm{Std}^\mu(\nu)} q^{\deg \mathsf{s} - \deg \mathsf{t}^\mu} [\Delta^\nu]$$

$$= \sum_{\nu \trianglerighteq \mu} z_{\nu\mu}(q, q^{-1}) [\Delta^\nu]$$

Now find λ such that $z_{\lambda\mu} = a_{-k} q^{-k} + a_{1-k} q^{1-k} + \dots$ with k maximal such that $a_{-k} \neq 0$

$\Longrightarrow$ $Z^\mu = Z' \oplus a_{-k}(P^\lambda\langle -k\rangle \oplus P^\lambda\langle k\rangle)$, for Z' projective

$\Longrightarrow$ Continuing in this way we can compute P^μ and hence the graded decomposition numbers $[\Delta^\lambda : L^\mu]_q$, for all λ, μ

The cyclotomic LLT algorithm – two examples

Example (Level 2) $\Lambda = \Lambda_i + \Lambda_j$ and $e = 0$ or $e \geq n$.

In this case, $|\operatorname{Std}^\mu(\lambda)| \leq 1 \quad \Longrightarrow \quad [\Delta^\lambda : L^\mu]_q = q^k$ for $k > 0$ if $\lambda \neq \mu$

$\Longrightarrow$ $Y^\mu = G^\mu$ and $P^\mu = Z^\mu$ and we have explicit bases for both

In this case, $\mathcal{S}^\Lambda_n$ is a positively graded basic Koszul algebra.

Example Suppose that $\Lambda = 3\Lambda_0$ and $\mu = (1|2, 1|2^2)$.

$$\Longrightarrow [Z^\mu] = [1|2, 1|2^2] + v[1|2^2|2, 1] + v[1^2|2|2^2] + (v^2 + 1)[1^2|2^2|2]$$
$$+ v[2|1^2|2^2] + (v^2 + 1)[2|2^2|1^2] + v^2[2, 1|1|2^2] + v^2[2^2|1|2, 1]$$
$$+ (v^3 + 3v + v^{-1})[2, 1|2^2|1] + (v^3 + v)[2^2|1^2|2] + (v^3 + v)[2^2|2|1^2]$$
$$+ (v^4 + 3v^2 + 1)[2^2|2, 1|1] + (v^3 + v)[2^2|2^2|0]$$
$$= (v + v^{-1})[P(2, 1|2^2|1)] + [1|2, 1|2^2] + v[1|2^2|2, 1] + v[1^2|2|2^2]$$
$$+ (v^2 + 1)[1^2|2^2|2] + v[2|1^2|2^2] + (v^2 + 1)[2|2^2|1^2] + v^2[2, 1|1|2^2]$$
$$+ (v^3 + 2v)[2, 1|2^2|1] + v^2[2^2|1|2, 1] + (v^3 + v)[2^2|1^2|2]$$
$$+ (v^3 + v)[2^2|2|1^2] + (v^4 + 2v^2)[2^2|2, 1|1]$$
$$= (v + v^{-1})[P(2, 1|2^2|1)] + [P(1^2|2^2|2)] + [P(2|2^2|1^2)]$$

This implies that, as an $\mathcal{S}^\Lambda_n$-module, we have the decomposition

$$Z^\mu = (v + v^{-1})P(2, 1|2^2|1) \oplus P(1^2|2^2|2) \oplus P(2|2^2|1^2) \oplus P(1|2, 1|2^2)$$

A graded decomposition matrix

Example Suppose that $e = 0$, $\beta = \alpha_{-1} + 3\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$ and $\Lambda = 3\Lambda_0$. The graded decomposition matrix of $\mathcal{S}^\Lambda_\beta$ is:

$(0\|\,1\,\|4,2)$	1														
$(0\|4,2\|1)$	q	1													
$(1\|\,0\,\|4,2)$	q	.	1												
$(1\|\,1\,\|4,1)$	q^2	.	q	1											
$(1\|\,1^2\,\|4)$	.	.	.	q	1										
$(1\|\,4\,\|1^2)$	.	.	.	q	.	1									
$(1\|4,1\|1)$	q^2	q	q	q^2	q	q	1								
$(1\|4,2\|0)$	q^3	q^2	q^2	.	.	.	q	1							
$(1^2\|\,1\,\|4)$	.	.	q	q^2	q	.	.	.	1						
$(1^2\|\,4\,\|1)$	.	.	q^2	q^3	q^2	q^2	q	.	q	1					
$(4\|\,1\,\|1^2)$	.	.	q	q^2	.	q	.	.	.	.	1				
$(4\|\,1^2\,\|1)$	.	.	q^2	q^3	q^2	q^2	q	.	.	.	q	1			
$(4,1\|\,1\,\|1)$	q^2	q	q^3+q	q^4	q^3	q^3	q^2	.	q^2	q	q^2	q	1		
$(4,2\|\,0\,\|1)$	q^3	q^2	q^2	.	.	.	.	.	.	.	.	.	q	1	
$(4,2\|\,1\,\|0)$	q^4	q^3	q^3	.	.	.	q^2	q	.	q	.	q	q^2	q	1