On the U-module Structure of the Unipotent Specht Modules for Finite General Linear Groups

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(joint work with Richard Dipper)

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Basic setting

- q : a fixed power of some prime p.
- $\mathbb{F}_q$: finite field with q elements.
- K: a field such that $\mathrm{char}(K) \neq p$ and $\sqrt[p]{1} \in K$.
- $G = GL_n(q)$: group of invertible $n \times n$ matrices over $\mathbb{F}_q$, where $n \in \mathbb{N}$.

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- Let $B \subseteq P_\lambda$ be the corresponding parabolic subgroup, K_{P_λ} the trivial KP_λ-module.
- Let $M_K(\lambda) = \mathrm{Ind}_{P_\lambda}^G(K_{P_\lambda})$, the corresponding permutation module.
- The unipotent Specht module $S_K(\lambda)$ is a submodule of $M_K(\lambda)$ and for $K = \mathbb{C}$, $\{S_\mathbb{C}(\lambda) \mid \lambda \vdash n\}$ are precisely the irreducible constituents of $\mathrm{Ind}_B^G(\mathbb{C}_B)$.

Motivation

If we set $q \rightsquigarrow 1$, $S_K(\lambda) = S^\lambda$, the Specht module for the symmetric group $\mathfrak{S}_n$.

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Goal

Reprove DJ's conjecture for 2-part partitions by using a method tightly connected to representation theory.

Normal form of an $(m \times n)$-matrix

Let $\lambda = (n-m, m) \vdash n$. Choose an m-dimensional $\mathbb{F}_q$-vector space V_1 in $V = \mathbb{F}_q^n$. List a basis of V_1 as $m \times n$-matrix and then row reduce it to a unique normal form.

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Example

$$L = \begin{array}{cc} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} & \\ \begin{pmatrix} * & * & 1 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 1 & 0 & 0 \\ * & * & 0 & * & 0 & 1 & 0 \end{pmatrix} & \begin{array}{c} 3 \\ 5 \\ 6 \end{array} \end{array} .$$

We label the rows by column indices of "last 1's". Write

$$\operatorname{tab}(L) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & 6 \\ \cline{1-3} \end{array} \in \mathrm{RStd}(\lambda) \text{ where } \lambda = (4,3).$$

Different description of a basis of $M_K(\lambda)$

Definition

$$\begin{tikzcd}
\mathfrak{X}_{m,n} = \{\textit{row reduced}\, m \times n - \textit{matrices}\} \arrow[d, leftrightarrow, "\textit{1-1}"] \\
\mathcal{F}(\lambda) = \{0 \subseteq V_1 \subseteq V = \mathbb{F}_q^n \mid \dim_{\mathbb{F}_q} V_1 = m\}
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G acts on $\mathfrak{X}_{m,n}$ by setting $L \circ g$ for $L \in \mathfrak{X}_{m,n}, g \in G$ to be the row reduced matrix obtained from Lg. The resulting G-permutation module is exactly $M_K(\lambda) = \operatorname{Ind}_{P_\lambda}^{G} K_{P_\lambda}$.

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Remark

- *$\mathfrak{X}_{m,n}$ is a basis of $M_K(\lambda)$.*

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Remark

- *$\mathfrak{X}_{m,n}$ is a basis of $M_K(\lambda)$.*
- *For $\lambda \vdash n$ arbitrary there is a similar description of a basis of $M_K(\lambda)$ by row reduced matrices.*

Strategy

Define $\Phi_m : M_K(\lambda) \to M_K(\mu) : U \mapsto \sum_{\substack{X \subseteq U \\ \dim X = m-1}} X$, where $\dim U = m$ and $\mu = (n-m+1, m-1)$.

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Lemma

For $K = \mathbb{C}$, we have $S_{\mathbb{C}}(\lambda) = \ker \Phi_m$.

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Advantage:

$|U| = p$*-power,* $\mathrm{char}(K) \neq p \Rightarrow KU$ *is semisimple.*

$\mathfrak{t}$-batch $\mathfrak{M}_{\mathfrak{t}}$ of $M_K(\lambda)$

Note that each row reduced $m \times n$-matrix determines an unique row standard λ-tableau.

Definition

For $\mathfrak{t} \in \mathrm{RStd}(\lambda)$, *denote the set of all the row reduced matrices which determine the same row standard* λ*-tableau* $\mathfrak{t}$ *by* $\mathfrak{X}_{\mathfrak{t}} = \{L \in \mathfrak{X}_{m,n} \mid \mathrm{tab}(L) = \mathfrak{t}\}$ *and set* $\mathfrak{M}_{\mathfrak{t}} = K\mathfrak{X}_{\mathfrak{t}}$.

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The $\mathfrak{M}_{\mathfrak{t}}$ comes up naturally:

Lemma

$$\mathrm{Res}_U^G M_K(\lambda) \overset{\textit{Mackey}}{\cong} \bigoplus_{w \in D_\lambda} Ind_{p_\lambda^w \cap U}^U K \xlongequal{\mathfrak{t}=\mathfrak{t}^\lambda w} \bigoplus_{\mathfrak{t} \in \mathrm{RStd}(\lambda)} \mathfrak{M}_{\mathfrak{t}}.$$

$\mathfrak{M}_{\mathfrak{t}}$ is called the $\mathfrak{t}$-batch of $M_K(\lambda)$.

Idempotent basis of $\mathfrak{M}_{\mathfrak{t}}$

Make $\mathfrak{X}_{\mathfrak{t}}$ into an abelian p-group $(\mathfrak{X}_{\mathfrak{t}}, \diamond)$ by defining $L_1 \diamond L_2$ as adding L_1 and L_2 in all columns except those, which contain a last one, keeping those unchanged. $K(\mathfrak{X}_{\mathfrak{t}}, \diamond)$ is commutative and semisimple, since $\mathrm{char}(K) \neq p$ and $|\mathfrak{X}_{\mathfrak{t}}|$=power of q. So:

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$\mathfrak{M}_{\mathfrak{t}} = K\mathfrak{X}_{\mathfrak{t}}$ *has a* K*-basis* $\mathcal{E}_{\mathfrak{t}}$ *consisting of primitive orthogonal idempotents* $e_{_L}$ *where* $L \in \mathfrak{X}_{\mathfrak{t}}$*.*

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Advantage:

This new idempotent basis $\mathcal{E}_{\mathfrak{t}}$ *is more adaptable to the* U*-module structure.*

Monomial action of $U^w \cap U$ on $\mathcal{E}_{\mathfrak{t}}$

Proposition

Let $\mathfrak{t} = \mathfrak{t}^{\lambda} w \in \mathrm{RStd}(\lambda)$*. Then* $U^w \cap U$ *acts monomially on the idempotent basis* $\mathcal{E}_{\mathfrak{t}} = \{e_{_L} \mid L \in \mathfrak{X}_{\mathfrak{t}}\}$*.*

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Theorem

Let $\mathcal{O}$ *be an* $U^w \cap U$*-orbit of* $\mathcal{E}_\mathfrak{t}$. *Then the orbit module* $M_\mathcal{O} = K\mathcal{O}$ *is an irreducible* $(U^w \cap U)$*-module. Moreover* $M_\mathcal{O}$ *is* U*-invariant, hence it is an irreducible* U*-module.*

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Remark

We can classify these orbits and describe their sizes by "combinatorial data".

Conjecture

Using those “combinatorial data”, we can prove the following conjecture for 2-part partitions:

Conjecture (Dipper, G. 2011)

Let $\lambda \vdash n$*. Then there exists for each* $0 \leqslant c \in \mathbb{Z}$ *a polynomial* $\ell_{c,\lambda}(t) \in \mathbb{Z}[t]$ *depending only on* λ, *not on* q *such that* $\ell_{c,\lambda}(q)$ *is the number of irreducible direct summands of* $\mathrm{Res}_U^G(S_K(\lambda))$ *of dimension* q^c*.*

Main result for $\lambda = (n-m, m)$

In order to prove this conjecture for 2-part partitions we use Φ_m to carry over from the permutation module $M_K(\lambda)$ to $S_K(\lambda)$. This works since the following holds:

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For $\lambda = (n - m, m) \vdash n$, *we construct an integral standard basis for* $S_K(\lambda)$, *reproving DJ's conjecture for 2-part partitions.*

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Hope:

This method works for arbitrary partition $\lambda \vdash n$*. Of course more tools are needed in general.*

Supercharacter

Applied to $\lambda = (1^n), \mathfrak{s} = \mathfrak{t}^{\lambda}$, the unique standard λ-tableau. Similarly we can prove that U_n acts monomially on the idempotent basis corresponding to this.

Example: Idempotent in the $\mathfrak{s}$-batch of U_n:

$$e_L = \begin{array}{ccccc|c} 1 & 2 & & & n & \\ \hline 1 & & & & & 1 \\ \lambda_{21} & 1 & & & & 2 \\ \lambda_{31} & \lambda_{32} & 1 & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ \lambda_{n1} & \lambda_{n2} & \cdots & \cdots & 1 & n \end{array} \in \mathcal{E}_{\mathfrak{s}}.$$

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Facts: Supercharacters are also classified by “combinatorial datas” similarly.

Related conjectures

Note that the unipotent Specht module $S_K(1^n)$ is the Steinberg module and its restriction to U_n is the regular representation of U_n. Using number theory we obtain: Our conjecture contains as special case the following longstanding conjectures:

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Conjecture (Lehrer 1974)

The number of distinct irreducible complex characters of degree q^c of U_n is a polynomial in q with integral coefficients depending only on n not on q.

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Conjecture (Lehrer 1974)

The number of distinct irreducible complex characters of degree q^c of U_n is a polynomial in q with integral coefficients depending only on n not on q.

Conjecture (Higman 1960)

The number of conjugacy classes of U_n is a polynomial in q with integral coefficients depending only on n not on q.

Connections

Obviously, the problem of decomposing the orbit modules of the action of U on the idempotent basis of $\mathrm{Res}_U^G(S_K(1^n))$ turns into the problem of decomposing so called “supercharacters” of U into irreducibles.

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Proposition (Dipper,G. 2012)

For $\lambda = (n-m, m) \vdash n, \mathfrak{t} \in \mathrm{RStd}(\lambda), \mu = (1^n), \mathfrak{s} = \mathfrak{t}^\mu$. Let $\mathcal{O} \subseteq \mathcal{E}_\mathfrak{t}, \tilde{\mathcal{O}} \subseteq \mathcal{E}_\mathfrak{s}$ be some orbits in $M_K(\lambda)$ and $M_K(\mu)$ respectively. Then the irreducible $\mathbb{C}U$-module $\mathbb{C}\mathcal{O}$ occurs as a constituent of the orbit module $\mathbb{C}\tilde{\mathcal{O}}$ if these two orbits have the same "combinatorial data".

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Not a surprise, but difficult to prove! This splits off one specific irreducible constituent from a supercharacter of U.

Examples of “combinatorial data” for 2-part partitions

The orbits and their sizes (by example):

Let $\lambda = (2, 2), \mathfrak{t} = \begin{smallmatrix}1 & 3\\ 2 & 4\end{smallmatrix}$.

The irreducible orbit module modules occurring in the batch $\mathfrak{M}_{\mathfrak{t}}$ can be classified by the following idempotents:

z	1		
			1

	1		
		z	1

y	1		
		z	1

	1		
z			1

where $0 \neq z, y \in \mathbb{F}_q$.

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$*$	1		
z		$*$	1

where $0 \neq z, y \in \mathbb{F}_q$.

Monomial action = putting arbitrary values of $\mathbb{F}_q$ to $*$-places. Hence we obtain the dimension of the corresponding orbit respectively by:

$$1 \qquad\qquad 1 \qquad\qquad 1 \qquad\qquad q^2$$

Example

$$\lambda = (4,3), \mathfrak{s} = \begin{smallmatrix} 1 & 2 & 4 & 7 \\ 3 & 5 & 6 & \end{smallmatrix} \in \mathrm{RStd}(\lambda), 0 \neq \alpha, \beta, \gamma \in \mathbb{F}_q$$

$$\begin{array}{|c|c|c|c|c|c|c|c}
\multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{4} & \multicolumn{1}{c}{5} & \multicolumn{1}{c}{6} & \multicolumn{1}{c}{7} & \\
\hline
 & \alpha & 1 & & & & & 3 \\
\hline
\beta & & & & 1 & & & 5 \\
\hline
 & & & \gamma & & 1 & & 6 \\
\hline
\end{array}$$

$$e_{_L}$$

$$\begin{array}{|c|c|c|c|c|c|c|c}
\multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{4} & \multicolumn{1}{c}{5} & \multicolumn{1}{c}{6} & \multicolumn{1}{c}{7} & \\
\hline
1 & & & & & & & 1 \\
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 & & & 1 & & & & 4 \\
\hline
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\hline
 & & & & & 1 & & 6 \\
\hline
 & & & \gamma & & & 1 & 7 \\
\hline
\end{array}$$

$$e_{_{\hat{L}}}$$

Then $e_{_L}KU \leqslant e_{_{\hat{L}}}KU$ (actually occurring with multiplicity 1).