Cellularity of Wreath Product Algebras

Geetha Thangavelu

The Institute of Mathematical Sciences
CIT Campus, Chennai, India

Workshop Grading and Decomposition Numbers

Stuttgart

September 26, 2012

- The main goal of this work is to study cellular structure of the wreath product algebras $A \wr \mathfrak{S}_{\mathfrak{n}}$. For this we introduce a variant of the notion of cellularity called cyclic cellularity: A cellular algebra A is called *cyclic cellular* if all of its cell modules are cyclic A–modules.
- Although it seems to be stronger than cellularity, it includes most of the important classes of cellular algebras appearing already in the literature.
  For example, Hecke algebras of type A, q–Schur algebras, Brauer algebras and BMW algebras are cyclic cellular.

- The main goal of this work is to study cellular structure of the wreath product algebras $A \wr \mathfrak{S}_{\mathfrak{n}}$. For this we introduce a variant of the notion of cellularity called cyclic cellularity: A cellular algebra A is called *cyclic cellular* if all of its cell modules are cyclic A–modules.
- Although it seems to be stronger than cellularity, it includes most of the important classes of cellular algebras appearing already in the literature.
  For example, Hecke algebras of type A, q–Schur algebras, Brauer algebras and BMW algebras are cyclic cellular.

Definition

Let A a unital R–algebra over an integral domain R. A *cell datum* for A consists of an R–linear algebra involution $a \mapsto a^*$; a finite poset $(\Gamma, \geq)$; for each $\gamma \in \Gamma$ a finite index set $\mathcal{T}(\gamma)$; and a subset

$$\mathcal{C} = \{c^{\gamma}_{s,t} : \gamma \in \Gamma \text{ and } s, t \in \mathcal{T}(\gamma)\} \subseteq A$$

with the following properties:

(1) $\mathcal{C}$ is an R–basis of A.

(2) For each $\gamma \in \Gamma$, let $\bar{A}^{\gamma}$ be the span of the $c^{\mu}_{s,t}$ with $\mu > \gamma$.

Definition

Let A a unital R–algebra over an integral domain R. A *cell datum* for A consists of an R–linear algebra involution $a \mapsto a^*$; a finite poset $(\Gamma, \geq)$; for each $\gamma \in \Gamma$ a finite index set $\mathcal{T}(\gamma)$; and a subset

$$\mathcal{C} = \{c^{\gamma}_{s,t} : \gamma \in \Gamma \text{ and } s, t \in \mathcal{T}(\gamma)\} \subseteq A$$

with the following properties:

(1) $\mathcal{C}$ is an R–basis of A.

(2) For each $\gamma \in \Gamma$, let $\bar{A}^{\gamma}$ be the span of the $c^{\mu}_{s,t}$ with $\mu > \gamma$.

Definition

Let A a unital R–algebra over an integral domain R. A *cell datum* for A consists of an R–linear algebra involution $a \mapsto a^*$; a finite poset $(\Gamma, \geq)$; for each $\gamma \in \Gamma$ a finite index set $\mathcal{T}(\gamma)$; and a subset

$$\mathcal{C} = \{c^{\gamma}_{s,t} : \gamma \in \Gamma \text{ and } s, t \in \mathcal{T}(\gamma)\} \subseteq A$$

with the following properties:

(1) $\mathcal{C}$ is an R–basis of A.

(2) For each $\gamma \in \Gamma$, let $\bar{A}^{\gamma}$ be the span of the $c^{\mu}_{s,t}$ with $\mu > \gamma$.

Definition

(defn. contd.) for $a \in A$

$$ac^{\gamma}_{s,t} \equiv \sum_{v} r^{s}_{v}(a) c^{\gamma}_{v,t} \mod \bar{A}^{\gamma}.$$

where the co-efficients in the expansion are independent of t.

(3) $(c^{\gamma}_{s,t})^* \equiv c^{\gamma}_{t,s} \mod \bar{A}^{\gamma}$ for all $\gamma \in \Gamma$ and, $s, t \in \mathcal{T}(\gamma)$.

The original definition of Graham and Lehrer includes a stronger version of condition (3), as follows:

(3′) $(c^{\gamma}_{s,t})^* = c^{\gamma}_{t,s}$ for all $\gamma \in \Gamma$ and $s, t \in \mathcal{T}(\gamma)$.

For brevity, $(\mathcal{C}, \Gamma)$ is a cellular basis of A and $(A, *, \Gamma, \geq, \mathcal{T}, \mathcal{C})$ is a cell datum for A.

Definition

(defn. contd.) for $a \in A$

$$ac^{\gamma}_{s,t} \equiv \sum_{v} r^{s}_{v}(a) c^{\gamma}_{v,t} \mod \bar{A}^{\gamma}.$$

where the co-efficients in the expansion are independent of t.

(3) $(c^{\gamma}_{s,t})^{*} \equiv c^{\gamma}_{t,s} \mod \bar{A}^{\gamma}$ for all $\gamma \in \Gamma$ and, $s, t \in \mathcal{T}(\gamma)$.

The original definition of Graham and Lehrer includes a stronger version of condition (3), as follows:

(3′) $(c^{\gamma}_{s,t})^{*} = c^{\gamma}_{t,s}$ for all $\gamma \in \Gamma$ and $s, t \in \mathcal{T}(\gamma)$.

For brevity, $(\mathcal{C}, \Gamma)$ is a cellular basis of A and $(A, *, \Gamma, \geq, \mathcal{T}, \mathcal{C})$ is a cell datum for A.

Given $\gamma \in \Gamma$, let A^γ *cell ideal of A* denote the span of the $c^\mu_{s,t}$ with $\mu \geq \gamma$.

For $\gamma \in \Gamma$, the *left cell module* Δ^γ is defined as follows:

1. as an R–module, Δ^γ is free with basis indexed by $\mathcal{T}(\gamma)$, say $\{c^\gamma_s : s \in \mathcal{T}(\gamma)\}$;
2. for each $a \in A$, the action of a on Δ^γ is defined by $ac^\gamma_s = \sum_v r^s_v(a)c^\gamma_v$ where the elements $r^s_v(a) \in R$ are the coefficients in the definition.

Given $\gamma \in \Gamma$, let A^γ *cell ideal of A* denote the span of the $c^\mu_{s,t}$ with $\mu \geq \gamma$.

For $\gamma \in \Gamma$, the *left cell module* Δ^γ is defined as follows:

1. as an R–module, Δ^γ is free with basis indexed by $\mathcal{T}(\gamma)$, say $\{c^\gamma_s : s \in \mathcal{T}(\gamma)\}$;
2. for each $a \in A$, the action of a on Δ^γ is defined by $ac^\gamma_s = \sum_v r^s_v(a)c^\gamma_v$ where the elements $r^s_v(a) \in R$ are the coefficients in the definition.

Let A be a cellular algebra with cell datum $(A, *, \Gamma, \geq, \mathcal{T}, \mathcal{C})$. We say that a cellular basis

$$\mathcal{B} = \{b^{\gamma}_{s,t} : \gamma \in \Gamma \text{ and } s, t \in \mathcal{T}(\gamma)\}$$

is *equivalent* to the original cellular basis $\mathcal{C}$ if it determines the same ideals A^{γ} and the same cell modules as does $\mathcal{C}$. More precisely, the requirement is that

1. for all $\gamma \in \Gamma$,

$$A^{\gamma} = \text{span}\{b^{\gamma'}_{s,t} : \gamma' \geq \gamma \text{ and } s, t \in \mathcal{T}(\gamma')\}, \text{ and}$$

2. for all $\gamma \in \Gamma$ and $t \in \mathcal{T}(\gamma)$,

$$\text{span}\{b^{\gamma}_{s,t} + \bar{A}^{\gamma} : s \in \mathcal{T}(\gamma)\} \cong \Delta^{\gamma}, \text{ as } A\text{–modules.}$$

Let A be a cellular algebra with cell datum $(A, *, \Gamma, \geq, \mathcal{T}, \mathcal{C})$. We say that a cellular basis

$$\mathcal{B} = \{b^{\gamma}_{s,t} : \gamma \in \Gamma \text{ and } s, t \in \mathcal{T}(\gamma)\}$$

is *equivalent* to the original cellular basis $\mathcal{C}$ if it determines the same ideals A^{γ} and the same cell modules as does $\mathcal{C}$. More precisely, the requirement is that

1. for all $\gamma \in \Gamma$,

$$A^{\gamma} = \text{span}\{b^{\gamma'}_{s,t} : \gamma' \geq \gamma \text{ and } s, t \in \mathcal{T}(\gamma')\}, \text{ and}$$

2. for all $\gamma \in \Gamma$ and $t \in \mathcal{T}(\gamma)$,

$$\text{span}\{b^{\gamma}_{s,t} + \bar{A}^{\gamma} : s \in \mathcal{T}(\gamma)\} \cong \Delta^{\gamma}, \text{ as } A\text{–modules.}$$

A cellular algebra A always admits many different cellular basis. In fact, any choice of an R–basis in each cell module can be globalized to a cellular basis of A as follows.

Lemma

Let A be a cellular algebra with cell datum $(A, *, \Gamma, \geq, \mathcal{T}, \mathcal{C})$. For each $\gamma \in \Gamma$, fix an A–A bimodule isomorphism $\beta_\gamma : A^\gamma/\bar{A}^\gamma \to \Delta^\gamma \otimes_R (\Delta^\gamma)^*$ satisfying $* \circ \beta_\gamma = \beta_\gamma \circ *$, and let $\{b_t : t \in \mathcal{T}(\gamma)\}$ be an R–basis of Δ^γ. Finally, for each $\gamma \in \Gamma$ and each pair $s, t \in \mathcal{T}(\gamma)$, let $b^\gamma_{s,t}$ be an arbitrary lifting of $\beta_\gamma^{-1}(b_s \otimes b_t^*)$ in A^γ. Then

$$\mathcal{B} = \{b^\gamma_{s,t} : \gamma \in \Gamma \text{ and } s, t \in \mathcal{T}(\gamma)\}$$

is a cellular basis of A equivalent to the original cellular basis $\mathcal{C}$

Definition

A cellular algebra is said to be *cyclic cellular* if every cell module of A is cyclic.

A cellular basis is cyclic cellular if the cell modules defined via this basis are cyclic.

Lemma

Let A be a cellular algebra over an integral domain R with cell datum $(A, *, \Gamma, \geq, \mathcal{T}, \mathcal{C})$. The following are equivalent:

1. *A is cyclic cellular.*
2. *For each $\gamma \in \Gamma$, there exists an element $y_\gamma \in A^\gamma$ with the properties:*
   1. $y_\gamma \equiv y_\gamma^* \mod \bar{A}^\gamma$.
   2. $A^\gamma = Ay_\gamma A + \bar{A}^\gamma$.
   3. $(Ay_\gamma + \bar{A}^\gamma)/\bar{A}^\gamma \cong \Delta^\gamma$, *as A–modules.*

Lemma

Let A be a cellular algebra over an integral domain R with cell datum $(A, *, \Gamma, \geq, \mathcal{T}, \mathcal{C})$. The following are equivalent:

1. *A is cyclic cellular.*
2. *For each $\gamma \in \Gamma$, there exists an element $y_\gamma \in A^\gamma$ with the properties:*
   1. $y_\gamma \equiv y_\gamma^* \mod \bar{A}^\gamma$.
   2. $A^\gamma = Ay_\gamma A + \bar{A}^\gamma$.
   3. $(Ay_\gamma + \bar{A}^\gamma)/\bar{A}^\gamma \cong \Delta^\gamma$, *as A–modules.*

Lemma

Let A be a cellular algebra over an integral domain R with cell datum $(A, *, \Gamma, \geq, \mathcal{T}, \mathcal{C})$. The following are equivalent:

1. *A is cyclic cellular.*
2. *For each $\gamma \in \Gamma$, there exists an element $y_\gamma \in A^\gamma$ with the properties:*
   1. $y_\gamma \equiv y_\gamma^* \mod \bar{A}^\gamma$.
   2. $A^\gamma = Ay_\gamma A + \bar{A}^\gamma$.
   3. $(Ay_\gamma + \bar{A}^\gamma)/\bar{A}^\gamma \cong \Delta^\gamma$, *as A–modules.*

Lemma

Let A be a cellular algebra over an integral domain R with cell datum $(A, *, \Gamma, \geq, \mathcal{T}, \mathcal{C})$. The following are equivalent:

1. *A is cyclic cellular.*
2. *For each $\gamma \in \Gamma$, there exists an element $y_\gamma \in A^\gamma$ with the properties:*
    1. $y_\gamma \equiv y_\gamma^* \mod \bar{A}^\gamma$.
    2. $A^\gamma = Ay_\gamma A + \bar{A}^\gamma$.
    3. $(Ay_\gamma + \bar{A}^\gamma)/\bar{A}^\gamma \cong \Delta^\gamma$, *as A–modules.*

Lemma

Let A be a cellular algebra over an integral domain R with cell datum $(A, *, \Gamma, \geq, \mathcal{T}, \mathcal{C})$. The following are equivalent:

1. *A is cyclic cellular.*
2. *For each $\gamma \in \Gamma$, there exists an element $y_\gamma \in A^\gamma$ with the properties:*
   1. $y_\gamma \equiv y_\gamma^* \mod \bar{A}^\gamma$.
   2. $A^\gamma = Ay_\gamma A + \bar{A}^\gamma$.
   3. $(Ay_\gamma + \bar{A}^\gamma)/\bar{A}^\gamma \cong \Delta^\gamma$, *as A–modules.*

Lemma

Let A be a cellular algebra over an integral domain R with cell datum $(A, *, \Gamma, \geq, \mathcal{T}, \mathcal{C})$. The following are equivalent:

1. *A is cyclic cellular.*
2. *For each $\gamma \in \Gamma$, there exists an element $y_\gamma \in A^\gamma$ with the properties:*
    1. $y_\gamma \equiv y_\gamma^* \mod \bar{A}^\gamma$.
    2. $A^\gamma = Ay_\gamma A + \bar{A}^\gamma$.
    3. $(Ay_\gamma + \bar{A}^\gamma)/\bar{A}^\gamma \cong \Delta^\gamma$, *as A–modules.*

1. For each $\gamma \in \Gamma$, let δ^γ be a generator of the cell module Δ^γ, and let y_γ be a lifting in A^γ of $\alpha_\gamma^{-1}(\delta^\gamma \otimes (\delta^\gamma)^*)$.
2. Let $\{c_{\mathfrak{t}}^\gamma : \mathfrak{t} \in \mathcal{T}(\gamma)\}$ be the standard basis of the cell module Δ^γ derived from the cellular basis $\mathcal{C}$ of A. Since Δ^γ is cyclic, there exist elements $v_{\mathfrak{t}} \in A$ such that $c_{\mathfrak{t}}^\gamma = v_{\mathfrak{t}}\delta^\gamma$. We denote

$$V^\gamma = \{v_{\mathfrak{t}} : \mathfrak{t} \in \mathcal{T}(\gamma)\}.$$

1. For each $\gamma \in \Gamma$, let δ^γ be a generator of the cell module Δ^γ, and let y_γ be a lifting in A^γ of $\alpha_\gamma^{-1}(\delta^\gamma \otimes (\delta^\gamma)^*)$.
2. Let $\{c_{\mathfrak{t}}^\gamma : \mathfrak{t} \in \mathcal{T}(\gamma)\}$ be the standard basis of the cell module Δ^γ derived from the cellular basis $\mathcal{C}$ of A. Since Δ^γ is cyclic, there exist elements $v_{\mathfrak{t}} \in A$ such that $c_{\mathfrak{t}}^\gamma = v_{\mathfrak{t}}\delta^\gamma$. We denote

$$V^\gamma = \{v_{\mathfrak{t}} : \mathfrak{t} \in \mathcal{T}(\gamma)\}.$$

Lemma

For each $\gamma \in \Gamma$, let $\{b_\mathfrak{t} : \mathfrak{t} \in \mathcal{T}(\gamma)\}$ be an R–basis of the cell module Δ^γ. For $\mathfrak{t} \in \mathcal{T}(\gamma)$, choose $v'_\mathfrak{t} \in A$ such that $b_\mathfrak{t} = v'_\mathfrak{t}\,\delta^\gamma$. For $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\gamma)$, let $b^\gamma_{\mathfrak{s},\mathfrak{t}} = v'_\mathfrak{s} y_\gamma (v'_\mathfrak{t})^*$ Then $\mathcal{B} = \{b^\gamma_{\mathfrak{s},\mathfrak{t}} : \gamma \in \Gamma \text{ and } \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\gamma)\}$ is a cellular basis of A equivalent to the original cellular basis $\mathcal{C}$.

$R\mathfrak{S}_n$ with Murphy basis is an example of a cyclic cellular algebra.

1. Λ is set of partitions of n with dominance order, $\mathcal{T}(\lambda)$ is the set of standard λ–tableaux and $*$ is such that $\pi^* = \pi^{-1}$ for $\pi \in \mathfrak{S}_n$.
2. For $\lambda \in \Lambda$, $x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} w$, where $\mathfrak{S}_\lambda$ is the row stabilizer of $\mathfrak{t}^\lambda$. Let $d(\mathfrak{t})$ be the unique permutation such that $\mathfrak{t} = d(\mathfrak{t})\mathfrak{t}^\lambda$.
3. For $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)$, define
$$m^\lambda_{\mathfrak{s},\mathfrak{t}} = d(\mathfrak{s}) x_\lambda d(\mathfrak{t})^*.$$
4. The cell module Δ^λ is spanned by $\{d(\mathfrak{s})x_\lambda + \overline{R\mathfrak{S}_n}^\lambda : \mathfrak{s} \in \mathcal{T}(\lambda)\}$. The cell module Δ^λ is evidently cyclic with generator $x_\lambda + \overline{R\mathfrak{S}_n}^\lambda$.

$R\mathfrak{S}_n$ with Murphy basis is an example of a cyclic cellular algebra.

1. Λ is set of partitions of n with dominance order, $\mathcal{T}(\lambda)$ is the set of standard λ–tableaux and $*$ is such that $\pi^* = \pi^{-1}$ for $\pi \in \mathfrak{S}_n$.
2. For $\lambda \in \Lambda$, $x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} w$, where $\mathfrak{S}_\lambda$ is the row stabilizer of $\mathfrak{t}^\lambda$. Let $d(\mathfrak{t})$ be the unique permutation such that $\mathfrak{t} = d(\mathfrak{t})\mathfrak{t}^\lambda$.
3. For $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)$, define
$$m^\lambda_{\mathfrak{s},\mathfrak{t}} = d(\mathfrak{s}) x_\lambda d(\mathfrak{t})^*.$$
4. The cell module Δ^λ is spanned by $\{d(\mathfrak{s})x_\lambda + \overline{R\mathfrak{S}_n}^\lambda : \mathfrak{s} \in \mathcal{T}(\lambda)\}$. The cell module Δ^λ is evidently cyclic with generator $x_\lambda + \overline{R\mathfrak{S}_n}^\lambda$.

$R\mathfrak{S}_n$ with Murphy basis is an example of a cyclic cellular algebra.

1. Λ is set of partitions of n with dominance order, $\mathcal{T}(\lambda)$ is the set of standard λ–tableaux and $*$ is such that $\pi^* = \pi^{-1}$ for $\pi \in \mathfrak{S}_n$.
2. For $\lambda \in \Lambda$, $x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} w$, where $\mathfrak{S}_\lambda$ is the row stabilizer of $\mathfrak{t}^\lambda$. Let $d(\mathfrak{t})$ be the unique permutation such that $\mathfrak{t} = d(\mathfrak{t})\mathfrak{t}^\lambda$.
3. For $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)$, define

$$m^\lambda_{\mathfrak{s},\mathfrak{t}} = d(\mathfrak{s}) x_\lambda d(\mathfrak{t})^*.$$

4. The cell module Δ^λ is spanned by $\{d(\mathfrak{s})x_\lambda + \overline{R\mathfrak{S}_n}^\lambda : \mathfrak{s} \in \mathcal{T}(\lambda)\}$. The cell module Δ^λ is evidently cyclic with generator $x_\lambda + \overline{R\mathfrak{S}_n}^\lambda$.

$R\mathfrak{S}_n$ with Murphy basis is an example of a cyclic cellular algebra.

1. Λ is set of partitions of n with dominance order, $\mathcal{T}(\lambda)$ is the set of standard λ–tableaux and $*$ is such that $\pi^* = \pi^{-1}$ for $\pi \in \mathfrak{S}_n$.
2. For $\lambda \in \Lambda$, $x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} w$, where $\mathfrak{S}_\lambda$ is the row stabilizer of $\mathfrak{t}^\lambda$. Let $d(\mathfrak{t})$ be the unique permutation such that $\mathfrak{t} = d(\mathfrak{t})\mathfrak{t}^\lambda$.
3. For $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)$, define
$$m^\lambda_{\mathfrak{s},\mathfrak{t}} = d(\mathfrak{s}) x_\lambda d(\mathfrak{t})^*.$$
4. The cell module Δ^λ is spanned by $\{d(\mathfrak{s})x_\lambda + \overline{R\mathfrak{S}_n}^\lambda : \mathfrak{s} \in \mathcal{T}(\lambda)\}$. The cell module Δ^λ is evidently cyclic with generator $x_\lambda + \overline{R\mathfrak{S}_n}^\lambda$

Let A be an R–algebra. The wreath product is $A \wr \mathfrak{S}_n = A^{\otimes n} \rtimes \mathfrak{S}_n$ where $\mathfrak{S}_n$ acts on $A^{\otimes n}$ by place permutations. If A is an algebra with involution $*$, then $\mathfrak{S}_n$ acts by $*$–preserving automorphisms and the wreath product is also an algebra with involution determined by

$$\begin{aligned}((a_1 \otimes \cdots \otimes a_n)\pi)^* &= \pi^{-1}\,(a_1^* \otimes \cdots \otimes a_n^*) \\ &= {}^{\pi^{-1}}(a_1^* \otimes \cdots \otimes a_n^*)\,\pi^{-1}.\end{aligned}$$

Theorem

Let A be a cyclic cellular algebra. Then for all $n \geq 1$, the wreath product algebra $A \wr \mathfrak{S}_n$ is a cyclic cellular algebra.

Let Λ_n^Γ denote the set of maps λ from Γ to the set of partitions such that $\sum_{\gamma \in \Gamma} |\lambda(\gamma)| = n$.
By fixing a listing of Γ consistent with it's partial order, in the sense that

$$\gamma(i) \geq \gamma(j) \implies i \leq j.$$

, we can identify Λ_n^Γ with multi-partitions via

$$\lambda^{(i)} = \lambda(\gamma(i)) \quad (1 \leq i \leq r).$$

Then for each $\lambda \in \Lambda_n^\Gamma$, we have analogues of elements $x_\lambda, \mathfrak{s}, \mathfrak{t}^\lambda$ and $d(\mathfrak{s})$ as mentioned earlier.

We generalize the dominance order $\trianglerighteq$ on multi-partitions, called Γ-dominance order $\trianglerighteq_\Gamma$: if for all $\gamma \in \Gamma$, and for all $j \geq 0$,

$$\sum_{\gamma' > \gamma} |\lambda(\gamma')| + \sum_{i \leq j} \lambda(\gamma)_i \geq \sum_{\gamma' > \gamma} |\mu(\gamma')| + \sum_{i \leq j} \mu(\gamma)_i.$$

which takes into an account of the partial order on Γ also.
Let A be cyclic cellular with basis $(\mathcal{C}, \Gamma)$ with r elements in it's poset.
For $\lambda = (\lambda(\gamma_1), \cdots, \lambda(\gamma_r))$ let $\alpha(\lambda) = (|\lambda(\gamma_1)|, \cdots, |\lambda(\gamma_r)|)$.

Let V^α be the set of simple tensors in $A^{\otimes n}$ whose first α_1 tensorands belong to $V^{\gamma(1)}$, the next α_2 tensorands belong to $V^{\gamma(2)}$, and so forth. Set

$$y^\alpha = y_{\gamma(1)}^{\otimes \alpha_1} \otimes \cdots \otimes y_{\gamma(r)}^{\otimes \alpha_r}.$$

For $\lambda \in \Lambda_n^\Gamma$, let $\mathcal{T}(\lambda)$ denote the set of pairs $(\mathfrak{s}, v)$, where $\mathfrak{s}$ is a standard λ–tableau and $v \in V^{\alpha(\lambda)}$.

Define

$$m^\lambda_{(\mathfrak{s},v),\,(\mathfrak{t},w)} = d(\mathfrak{s}) v\, y^\alpha x_\lambda\, w^* d(\mathfrak{t})^*,$$

where $\lambda \in \Lambda_n^\Gamma$, $\alpha = \alpha(\lambda)$, $\mathfrak{s}, \mathfrak{t}$ are row standard λ–tableaux, and $v, w \in V^\alpha$.

Cell modules of $A \wr \mathfrak{S}_n$

Let $E_1, \ldots, E_r$ be a collection of A–modules. For a multipartition $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of total size n with r parts, $\alpha = \alpha(\lambda)$. Then

$$\Delta_R^{\lambda} = \Delta_R^{\lambda^{(1)}} \otimes \cdots \otimes \Delta_R^{\lambda^{(r)}},$$

is a cell module for $R\mathfrak{S}_\alpha \cong R\mathfrak{S}_{\alpha_1} \otimes \cdots \otimes R\mathfrak{S}_{\alpha_r}$. Let $E^\alpha = E_1^{\otimes \alpha_1} \otimes \cdots \otimes E_r^{\otimes \alpha_r}$. Then E^α is an $A \wr \mathfrak{S}_\alpha$–module, with $A^{\otimes n}$ acting by the tensor product action and $\mathfrak{S}_\alpha$ acting by place permutations. Moreover, $E^\alpha \otimes \Delta_R^\lambda$ is also an $A \wr \mathfrak{S}_\alpha$–module, with $a(v \otimes m) = av \otimes m$ and $\pi(v \otimes m) = {}^\pi v \otimes \pi m$, for $a \in A^{\otimes n}$, $\pi \in \mathfrak{S}_\alpha$, $v \in E^\alpha$ and $m \in \Delta_R^\lambda$.

We obtain an $A \wr \mathfrak{S}_n$–module by

$$\mathrm{Ind}_{A\wr\mathfrak{S}_\alpha}^{A\wr\mathfrak{S}_n}(E^\alpha \otimes \Delta_R^\lambda) = (A \wr \mathfrak{S}_n) \otimes_{A\wr\mathfrak{S}_\alpha} (E^\alpha \otimes \Delta_R^\lambda).$$

When this construction is applied to the simple modules of A, one obtains the simple modules of the wreath product $A \wr \mathfrak{S}_n$.

Theorem

Let $\lambda \in \Lambda_n^\Gamma$ and let $\alpha = \alpha(\lambda)$. The cell module C^λ of $A \wr \mathfrak{S}_n$ satisfies

$$C^\lambda \cong \mathrm{Ind}_{A\wr\mathfrak{S}_\alpha}^{A\wr\mathfrak{S}_n}(E^\alpha \otimes \Delta_R^\lambda).$$