Diagrammatic description of parabolic category $\mathcal{O}$ in type (D_n, A_{n-1}).

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Basic Notations

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We are then interested in $\mathcal{O}^{\mathfrak{p}}(\mathfrak{g})$, the category of $\mathfrak{g}$-modules M such that

- M is finitely generated as a $\mathfrak{g}$-module,
- $\mathfrak{p}$ acts locally nilpotent on M,
- $\mathfrak{h}$ acts semi-simple on M with integral weights.

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We have classes of modules indexed by $\lambda \in \widetilde{X}_n^{\mathfrak{p}}$:

- $M(\lambda)$ the parabolic Verma module with highest weight $\lambda - \rho$,
- $L(\lambda)$ the irreducible quotient of $M(\lambda)$,
- $P(\lambda)$ the projective cover of $L(\lambda)$.

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To λ we then associate the sequence $a(\lambda) = (a(\lambda)_i)_{i \in \mathbb{Z}_{\geq 0}}$ as follows

$$a(\lambda)_i = \begin{cases} \vee & i \in P_\vee(\lambda) \setminus P_\times(\lambda), \\ \wedge & i \in P_\wedge(\lambda) \setminus P_\times(\lambda), \\ \times & i \in P_\times(\lambda), \\ \circ & i \in P_\circ(\lambda). \end{cases}$$

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The corresponding sequence is

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The set $\Lambda_n^{\mathfrak{p}}$ should be viewed as the combinatorial analogue of Verma modules with the identification

$$M(\lambda) \longleftrightarrow a(\lambda).$$

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Looking at these linkage conditions one easily deduces that a block of $\mathcal{O}^{\mathfrak{p}}(\mathfrak{g})$ is given by

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From now on we fix such a block!

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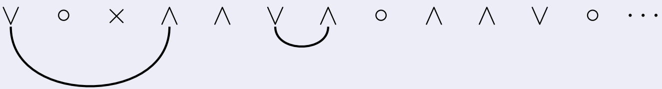

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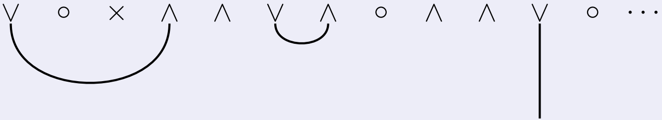

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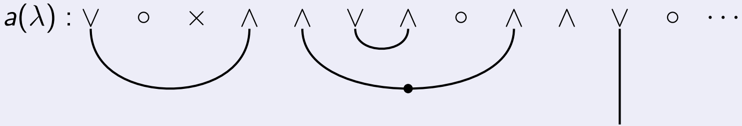

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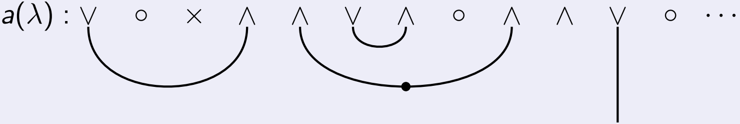

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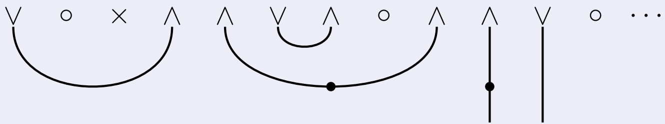

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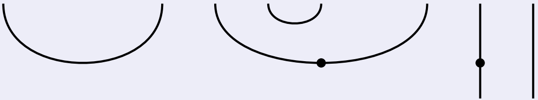

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Question:

What do we gain from this combinatorial description?

Combinatorics: Oriented Cup Diagrams

For $\mu, \eta \in X_n^{\mathfrak{p}}(\lambda)$, put $a(\eta)$ on top of $\underline{\mu}$ to obtain $\eta\underline{\mu}$.

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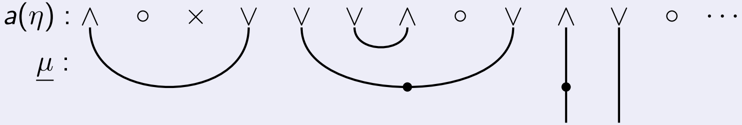

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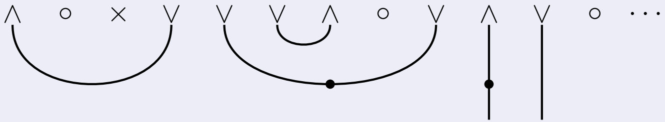

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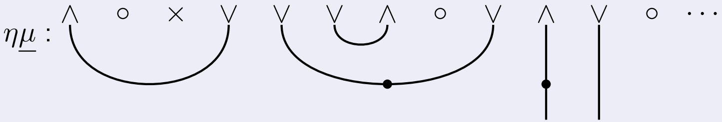

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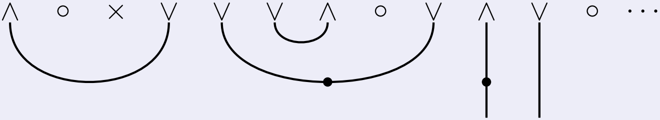
$\eta\underline{\mu}$: 

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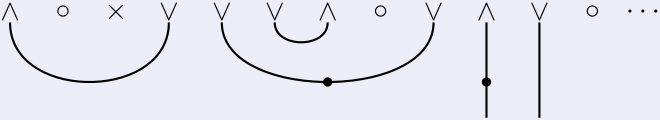

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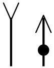
Rays: 

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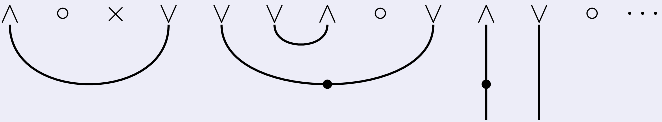
$\eta\underline{\mu}$: 

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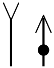
Clockwise:  Anti-Clockwise:  Rays: 

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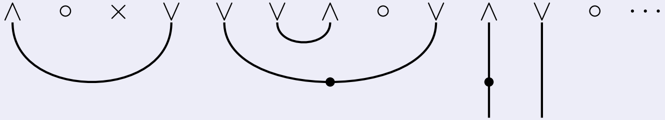
$\eta\underline{\mu}$: 

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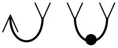
Clockwise: (deg $= 1$) 

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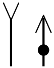
Rays: 

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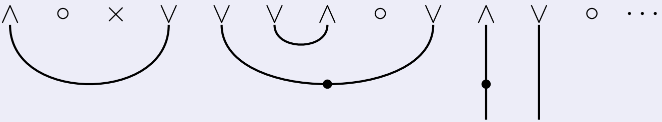
$\eta\underline{\mu}$: 

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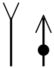
Clockwise: (deg $= 1$)  Anti-Clockwise: (deg $= 0$)  Rays: 

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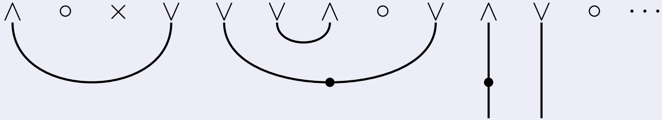
$\eta\underline{\mu}$: 

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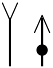

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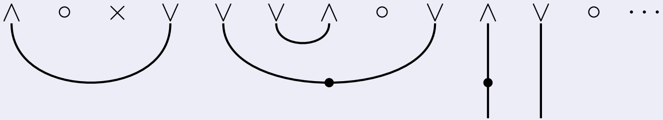

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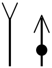

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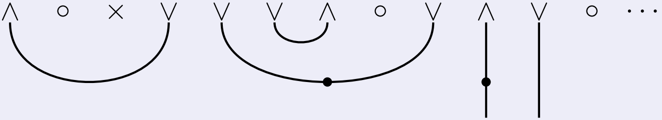

$$\deg(\eta\underline{\mu}) = 2.$$

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Combinatorics: Diagrams and Multiplicities

Can use the diagrams to determine multiplicities in Verma flags of indecomposable projectives.

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Proposition

For $\mu, \eta \in X_n^{\mathfrak{p}}(\lambda)$ it holds

$$(P(\mu) : M(\eta)) = \begin{cases} 1 & \text{if } \eta\underline{\mu} \text{ is oriented,} \\ 0 & \text{otherwise.} \end{cases}$$

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Remark

Can also work in the graded version of $\mathcal{O}^{\mathfrak{p}}(\mathfrak{g})$, then the degree of the diagram will give graded multiplicity formulas.

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Combinatorics: Diagrams and Morphisms

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&= \sum_{\theta} (P(\mu) : M(\theta)) \cdot (P(\eta) : M(\theta)) \\
&= \# \left\{ \begin{array}{c} \text{weights } \theta \text{ such that } \theta\underline{\mu} \text{ and } \theta\underline{\eta} \\ \text{are oriented} \end{array} \right\}
\end{aligned}$$

Combinatorics: (Oriented) Circle Diagrams

For $\mu, \eta, \theta \in X_n^{\mathfrak{p}}(\lambda)$

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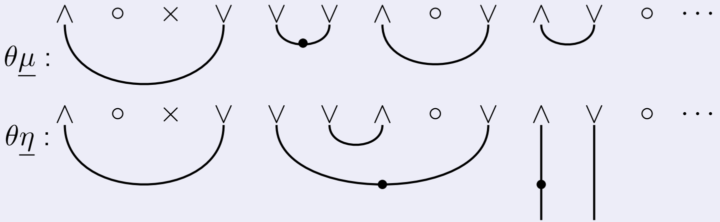
Example



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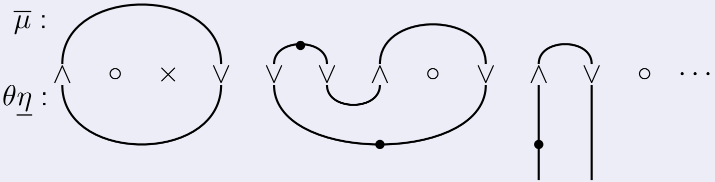

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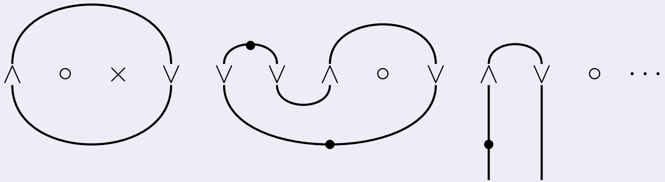

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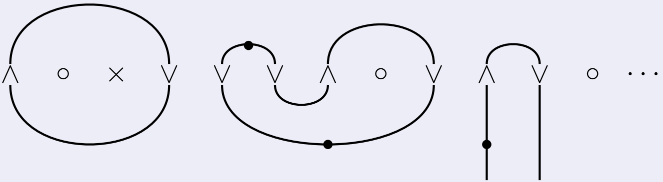

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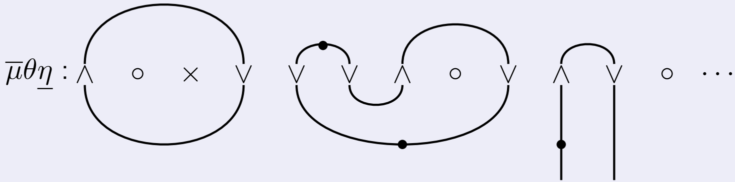

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Let $B_n^{\mathfrak{p}}(\lambda)$ denote the set of all such oriented circle diagrams obtained from weights in $X_n^{\mathfrak{p}}(\lambda)$.

Diagram Algebra: Definition

For two weights $\mu, \eta \in X_n^{\mathfrak{p}}(\lambda)$ define

$$ {}_{\mu}(\mathbb{D}_n)_{\eta} = \left\langle \text{ oriented circle diagrams } \overline{\mu}\theta\underline{\eta} \right\rangle . $$

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$\mathbb{D}_n$ can be equipped with the structure of an associative algebra.

Diagram Algebra: Multiplication

Example

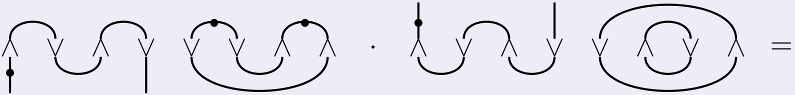

Diagram Algebra: Multiplication

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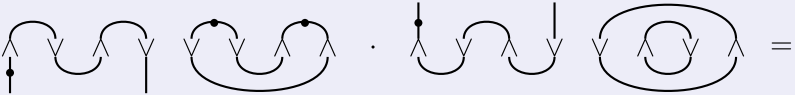
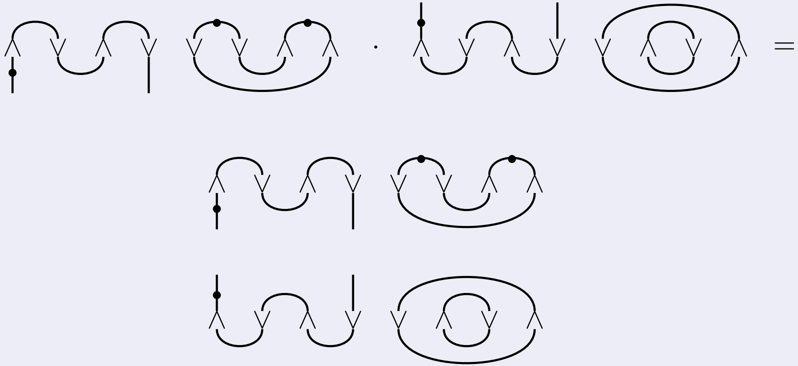


Put diagrams on top of each other.

Diagram Algebra: Multiplication

Example



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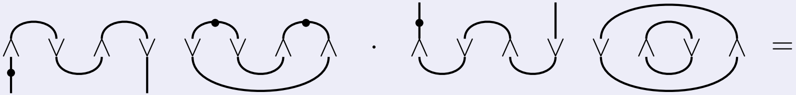

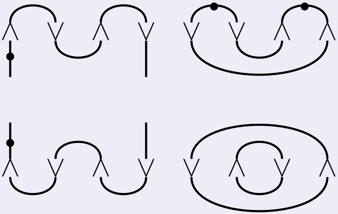

Connect the rays.

Diagram Algebra: Multiplication

Example

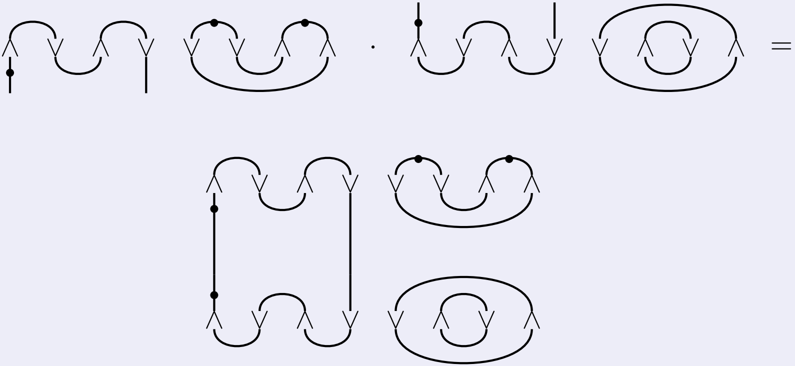

Diagram Algebra: Multiplication

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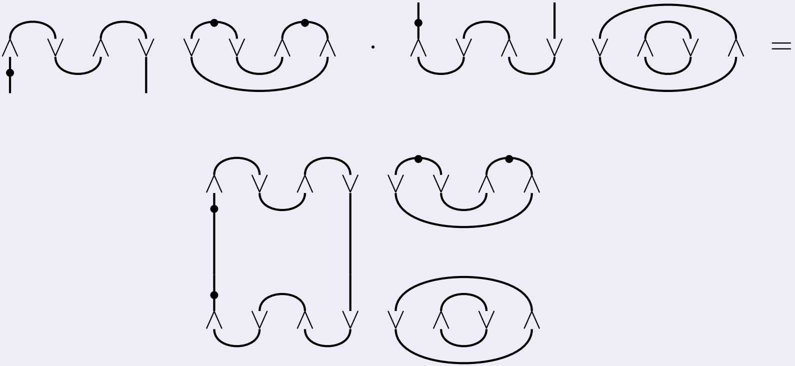
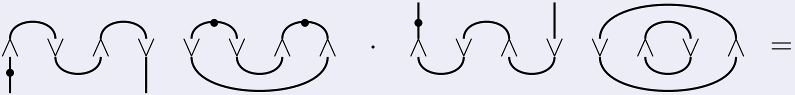
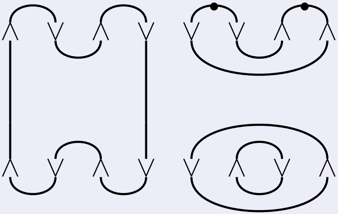


Eliminate double dots on connected rays.

Diagram Algebra: Multiplication

Example





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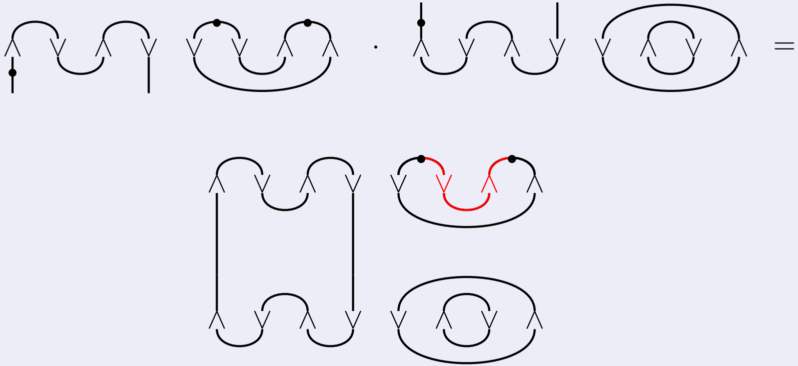
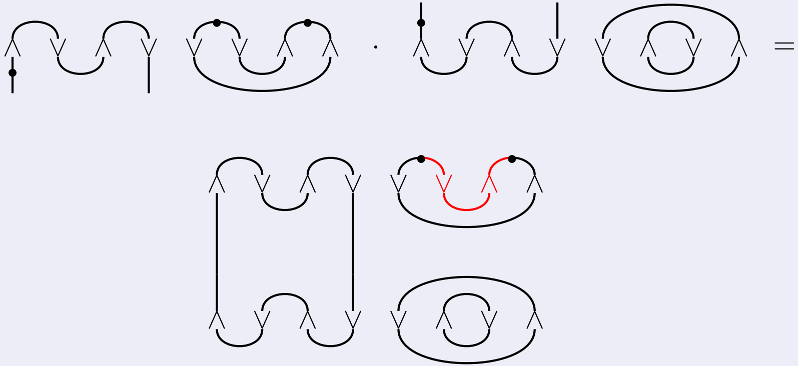


Color circles.

Diagram Algebra: Multiplication

Example

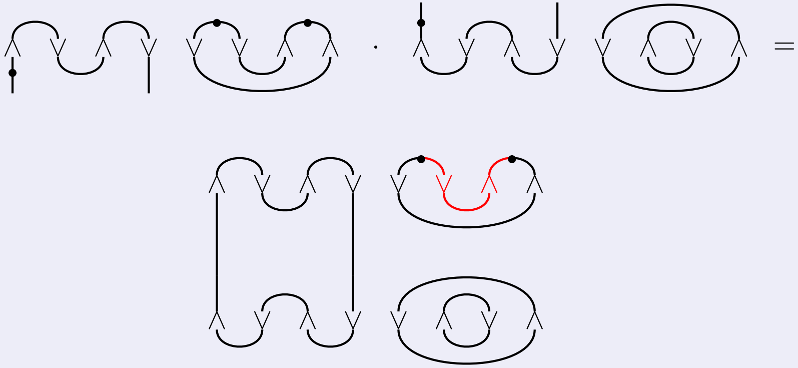


Basic Idea: View circles as elements in $\mathbb{C}[x]/(x^2)$.

anticlockwise circle $\leftrightarrow$ 1, clockwise circle $\leftrightarrow$ x.

Diagram Algebra: Multiplication

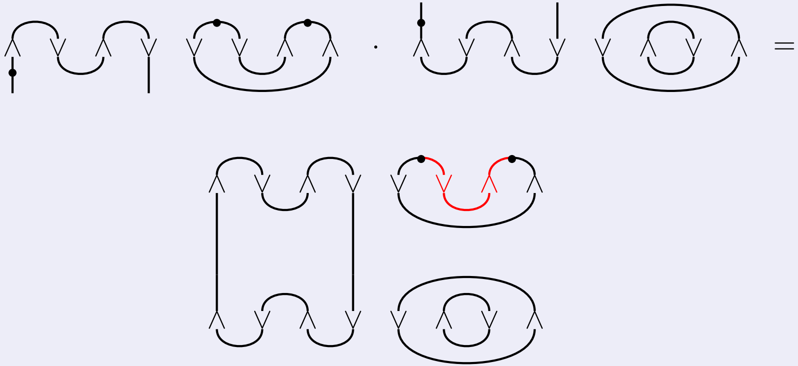
Example



Change symmetric cup/cap pairs into pairs of rays.

Diagram Algebra: Multiplication

Example

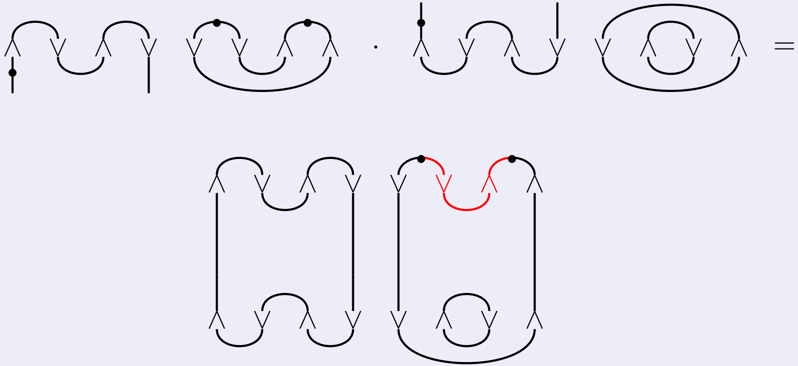


Two anticlockwise circles are combined into one anticlockwise circle.

Idea: $1 \otimes 1 \mapsto 1$.

Diagram Algebra: Multiplication

Example



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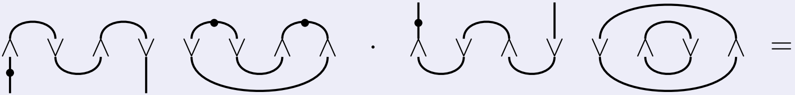
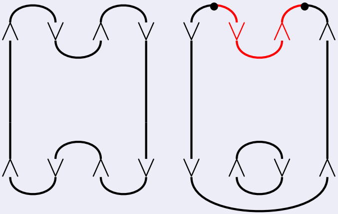
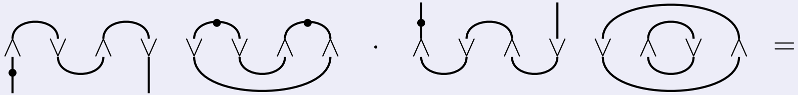


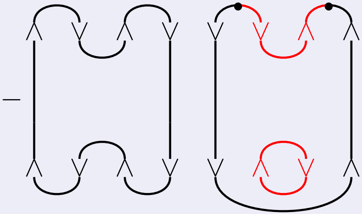


Need to change colouring if it does not match.

Diagram Algebra: Multiplication

Example



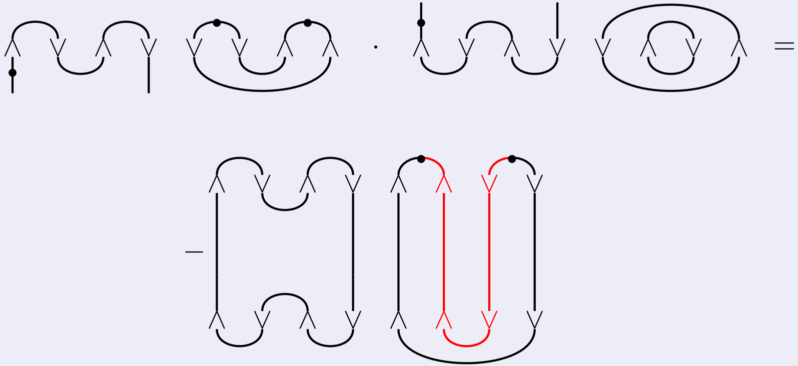


Changing colour of a clockwise circle gives a sign.

Idea: Substitution $x \mapsto -x$.

Diagram Algebra: Multiplication

Example

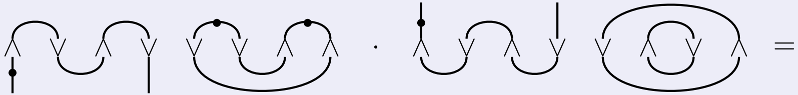


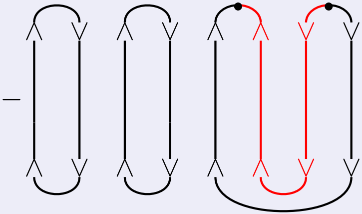
Anticlockwise and clockwise circle combine to a clockwise circle.

Idea: $x \otimes 1 \mapsto x$.

Diagram Algebra: Multiplication

Example





A clockwise circle splits into two clockwise circles.

Idea: $x \mapsto x \otimes x$.

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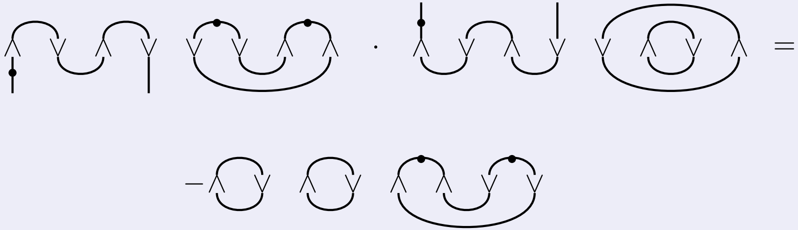

Collapse middle part of the diagram.

Diagram Algebra: Results

Remark

Can use combinatorics to show that $\mathbb{D}_n$ is cellular.

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$$\mathbb{D}_n \cong \mathrm{End}_{\mathfrak{g}}(P).$$

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Corollary

$$\mathbb{D}_n - \mathrm{mod} \cong \mathcal{O}^{\mathfrak{p}}(\mathfrak{g}, \mathbb{Z}^n)$$