Representation theory of diagram algebras

September 25, 2012

Diagram algebras and their geometries:

- Diagram algebras and Schur–Weyl dualities
- [BCD] Brauer algebras of type $G(m, p, n)$

Application to the symmetric group:

- [BDO] A closed formula for the decomposition of a tensor product of Specht modules
- [B] Positive characteristic

Schur–Weyl dualities

We have embeddings of the Weyl group Σ_n *and the symplectic group* $\mathrm{Sp}_{2n}(k)$ *in the general linear group* $\mathrm{GL}_n(k)$*. On the other side of the Schur–Weyl dualities we get*

$$\begin{array}{rcl} \Sigma_r \circlearrowright & E^{\otimes r} \circlearrowleft & \mathrm{GL}_n(k) \\ B_r(n) \circlearrowright & E^{\otimes r} \circlearrowleft & \mathrm{Sp}_{2n}(k) \\ P_r(n) \circlearrowright & E^{\otimes r} \circlearrowleft & \Sigma_n \end{array}$$

the partition, classical Brauer, and symmetric group algebras. A diagram algebra can appear in several such dualities, for example

$$\begin{array}{rcl} \textit{classical Brauer algebra} & \leftrightarrow & \textit{ortho-symplectic supergroup} \\ \textit{walled Brauer algebra} & \leftrightarrow & \textit{general linear supergroup} \end{array}$$

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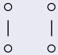
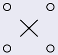
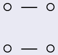
The Brauer algebra $B_2(n)$

This algebra has basis given by:

$e =$  $(12) =$  $\alpha =$ 

and multiplication given by concatenation

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(12)	(12)	e	α
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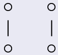
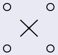
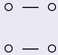
$e =$  $(12) =$  $\alpha =$ 

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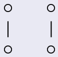
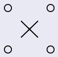
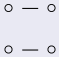
$e =$  $(12) =$  $\alpha =$ 

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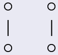
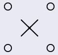
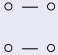
$e =$  $(12) =$  $\alpha =$ 

and character table:

	e	(12)	α
$\Delta(2)$	1	1	0
$\Delta(1^2)$	1	-1	0
$\Delta(\emptyset)$	1	1	n

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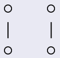
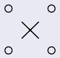
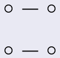
$e =$  $(12) =$  $\alpha =$ 

and one non-split extension:

$$0 \to \Delta(\emptyset) \to P(2) \to \Delta(2) \to 0$$

The cyclotomic Brauer algebra of type $G(m, 1, r)$ is the wreath product $(\mathbb{Z}/m\mathbb{Z}) \wr B_r(n)$.

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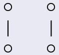
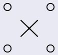
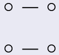
$e =$  $(12) =$  $\alpha =$ 

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A diagram algebra, $A_r(n)$,

- has multiplication defined by concatenation of diagrams
- this multiplication is specified by the parameter n
- is semisimple (and stable) for large n
- is cellular
- has a stratification by smaller cellular algebras (e.g. symmetric groups, cyclotomic Hecke algebras) [KX] and [HHKP]
- is (usually) quasi-hereditary over the complex numbers

Remark

The combinatorics of the diagram algebra and the smaller 'input algebras' are connected.

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Characteristic-zero Lie theory of diagram algebras

- *These algebras often form a 'tower' with 'translation functors'*
- *They exhibit geometries:*

$$\begin{array}{rcl} B_r(n) & \overset{CDM}{\longleftrightarrow} & A \subseteq D \\ WB_r(n) & \overset{CDDM}{\longleftrightarrow} & A \times A \subseteq A \\ P_r(n) & \overset{Cox}{\longleftrightarrow} & A \subseteq \widetilde{A} \end{array}$$

[CDM] *give the block structure, decomposition numbers, and higher extension groups of the algebras using the internal geometry.*

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The cyclotomic Brauer algebra [BCD]

The cyclotomic Brauer algebra of type $G(m,1,r)$ has a geometry given by a product of those controlling the classical Brauer ($A \subset D$) and the walled Brauer algebra ($A \times A \subset A$). For example the geometry controlling $B(6,1,n)$ is as follows

$$\leftrightarrow \quad \begin{array}{ccccccccccc} D & & & & A & & & & A & & D \\ \cup & & & & \cup & & & & \cup & & \cup \\ A & \times & (A & \times & A) & \times & (A & \times & A) & \times & A \end{array}$$

where each pair of roots of unity corresponds to either an $A \times A \subset A$ if $\xi = \overline{\xi}$ or $A \subset D$ if $\xi \neq \overline{\xi}$.

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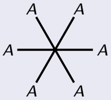

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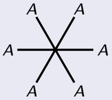

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Part 2

Calculating the decomposition of a tensor product of Specht modules of the symmetric group in characteristic zero.

The Littlewood–Richardson rule computes

$$\Sigma_r \quad \circlearrowleft \quad E^{\otimes r} \quad \circlearrowright \quad GL_n(k)$$

$$[\mathcal{S}(\nu)\downarrow_{\Sigma_{r_1,r_2}} : \mathcal{S}(\lambda) \otimes \mathcal{S}(\mu)] = [\Delta(\lambda) \otimes \Delta(\mu) : \Delta(\nu)]$$

the restriction of a Specht module to a Young subgroup of Σ_r and hence, through Schur–Weyl duality, the decompositions of tensor products of Weyl modules of $\mathrm{GL}_n(k)$.

There is no such formula for decomposing the tensor products of Specht modules.

We have that

$$P_r(n) \quad ↺ \quad E^{\otimes r} \quad ↻ \quad \Sigma_n$$

the image under the Schur functor is given by removing the first row of the partition and so

$$[L(\nu_{>1})\downarrow_{P_{r_1,r_2}(n)}: L(\lambda_{>1})\otimes L(\mu_{>1})] = [\mathcal{S}(\lambda)\otimes\mathcal{S}(\mu) : \mathcal{S}(\nu)] = g^{\nu}_{\lambda,\mu}$$

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$$g^{\nu}_{\lambda,\mu} = [L(\nu_{>1})\downarrow_{P_{r_1,r_2}(n)} : L(\lambda_{>1}) \otimes L(\mu_{>1})]$$

stabilise for large values of n to be given by:

$$\overline{g}^{\nu}_{\lambda,\mu} = [\Delta(\nu_{>1})\downarrow_{P_{r_1,r_2}(n)} : \Delta(\lambda_{>1}) \otimes \Delta(\mu_{>1})].$$

This results in a wonderful stability property in the tensor products of Specht modules.

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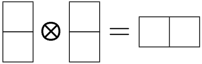

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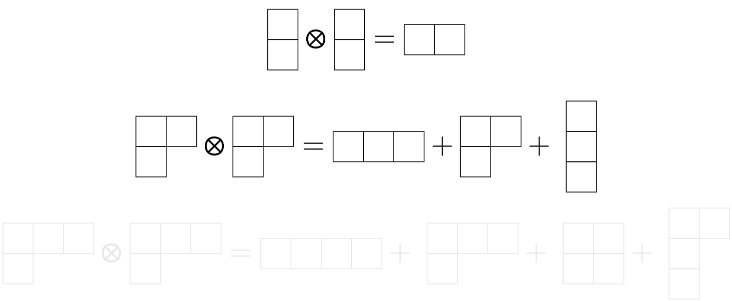

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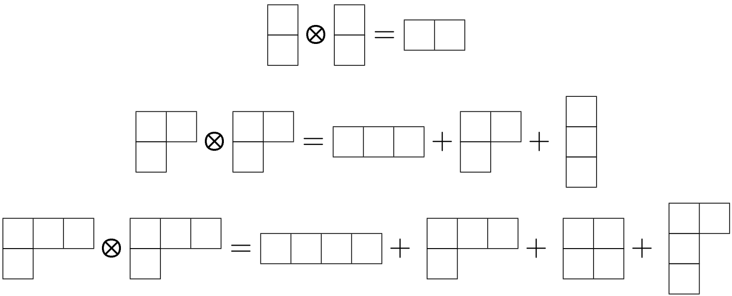

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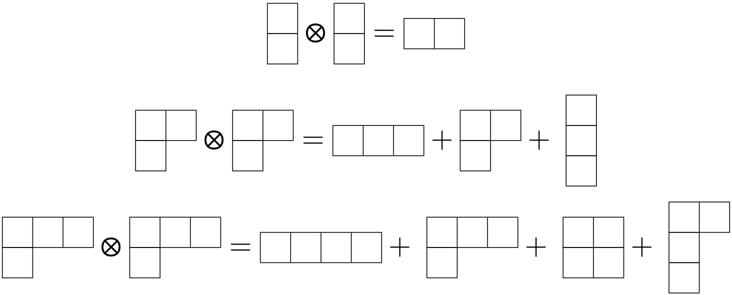

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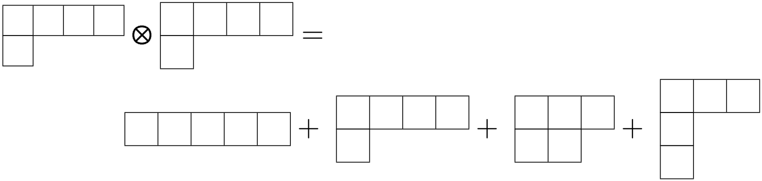

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Theorem (BDO)

Using the **stratification** *of the partition algebra we can show that the generic value of these coefficients for large n is given by:*

$$\overline{g}^{\nu}_{\lambda,\mu} = \sum_{\nu_{>1} \vdash m-(l_1+2l_2)} \sum_{\substack{\xi_1 \vdash r-l_1-l_2 \\ \xi_2 \vdash s-l_1-l_2}} \sum_{\substack{\xi_3 \vdash l_1 \\ \xi_4 \vdash l_2}} c^{\nu_{>1}}_{\xi_1,\xi_2,\xi_3} c^{\lambda_{>1}}_{\xi_1,\xi_3,\xi_4} c^{\mu_{>1}}_{\xi_2,\xi_3,\xi_4}$$

where $m = |\lambda_{>1}| + |\mu_{>1}|$.

This is entirely in terms of Littlewood–Richardson coefficents! In fact,

Corollary (BDO)

If $|\nu_{>1}| = |\lambda_{>1}| + |\mu_{>1}|$, *then*

$$\overline{g}^{\nu}_{\lambda,\mu} = c^{\nu_{>1}}_{\lambda_{>1},\mu_{>1}}$$

Theorem (BDO)

Using the **stratification** *of the partition algebra we can show that the generic value of these coefficients for large n is given by:*

$$\overline{g}^{\nu}_{\lambda,\mu} = \sum_{\nu_{>1} \vdash m-(l_1+2l_2)} \sum_{\substack{\xi_1 \vdash r-l_1-l_2 \\ \xi_2 \vdash s-l_1-l_2}} \sum_{\substack{\xi_3 \vdash l_1 \\ \xi_4 \vdash l_2}} c^{\nu_{>1}}_{\xi_1,\xi_2,\xi_3} c^{\lambda_{>1}}_{\xi_1,\xi_3,\xi_4} c^{\mu_{>1}}_{\xi_2,\xi_3,\xi_4}$$

where $m = |\lambda_{>1}| + |\mu_{>1}|$.

This is entirely in terms of Littlewood–Richardson coefficents! In fact,

Corollary (BDO)

If $|\nu_{>1}| = |\lambda_{>1}| + |\mu_{>1}|$, *then*

$$\overline{g}^{\nu}_{\lambda,\mu} = c^{\nu_{>1}}_{\lambda_{>1},\mu_{>1}}$$

Finally, we can solve the problem for small n. Using the **geometry** of the partition algebra.

$$[L(\nu)] = \sum p^n_{\xi,\nu}(-1)[\Delta(\xi)]$$

where the $p^n_{\mu,\lambda}$ are the KL polynomials of type $A \subset \tilde{A}$ and so

$$[S(\lambda) \otimes S(\mu) : S(\nu)] = [L(\nu_{>1})\downarrow_{P_r(n)\otimes P_s(n)}; L(\lambda_{>1}) \otimes L(\mu_{>1})]$$
$$= \sum_{\xi} p^n_{\xi,\nu}(-1)\overline{g}^{\xi}_{\lambda,\mu}$$

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Thus we have

- Shown that the coefficients $g^{\nu}_{\lambda,\mu} \xrightarrow{n\to\infty} \overline{g}^{\nu}_{\lambda,\mu}$
- Given a concrete representation theoretic meaning to the limiting coefficients, $\overline{g}^{\nu}_{\lambda,\mu}$.
- Calculated the limiting coefficients, $\overline{g}^{\nu}_{\lambda,\mu}$, in terms of the Littlewood–Richardson rule
- Shown how to pass from the limiting case to small n

Using the representation theory of the partition algebra.

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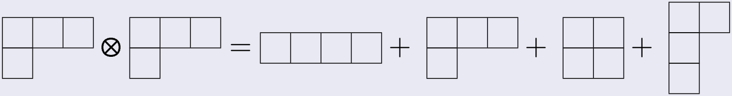
Example

Generic $P_2(n)$ has simple modules $\Delta(\emptyset), \Delta(1), \Delta(2), \Delta(1^2)$. We have for all λ that:

$$[\Delta(\lambda)\downarrow_{P_1 \times P_1} : \Delta(1) \otimes \Delta(1)] = 1$$

Therefore in the limit:



In the case $n = 1$ we have that $\Delta(1^2) = [L(1^2), L(1)]$ and so:

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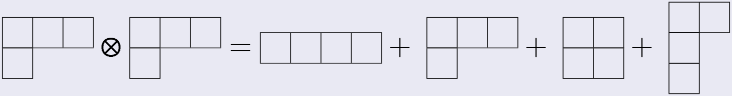

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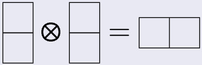

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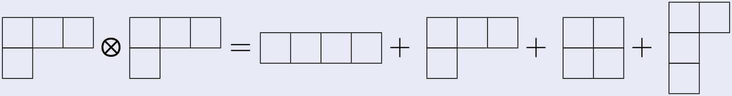

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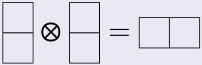

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Quasi-hereditary covers

- *The symmetric group is studied via the general linear group. The cyclotomic Hecke algebras are studied via the cyclotomic q-Schur algebras.*
- **When** *and* **how** *can we directly construct a quasi-hereditary cover of a (cellular) algebra?* **How much** *Lie theory is baked-in to this construction?*
- *In the case of diagram algebras, the existence question was answered in [HHKP].*
- *In my thesis, I gave a DJM-construction of these covers (in the case of the partition algebra and the walled and classical Brauer algebras). This used a characteristic-free definition of permutation modules, and the construction of 'cellular' homomorphisms between these modules.*

A semistandard basis theorem

Let A be the Brauer, walled Brauer, or partition algebra. Then the algebra S(A) has a basis:

$$\Phi = \{\varphi_{\mathrm{S}^\sigma \mathrm{T}^\tau} : \omega \vdash r - 2n, \mathrm{S}^\sigma, \mathrm{T}^\tau \in T_0^*(\omega)\},$$

where $\mathrm{S}^\sigma, \mathrm{T}^\tau$ *are* ω*-tableaux of type* (λ, i) *and* (μ, j)*-tableaux respectively, and we define* $\varphi_{\mathrm{S}^\sigma \mathrm{T}^\tau}$ *to be the extension of the element of* $\mathrm{Hom}_{B_r}(M(\lambda, l), M(\mu, m))$ *given by*

$$\varphi_{\mathrm{S}^\sigma \mathrm{T}^\tau}(\epsilon_\ell \otimes \epsilon_\ell \otimes x_\lambda) = [\sigma] \otimes [\tau] \otimes m_{\mathrm{S}\mathrm{T}}.$$

Moreover, S(A) is cellular with respect to this basis. We have that S(A) is a quasi-hereditary cover of A, and is 1-faithful for $p \neq 2, 3$ *and* $\delta \neq 0$.