A cellular basis of the q-Brauer algebra

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Wenzl (2012)

Version that contains $H_n(q)$

Fix $N \in \mathbb{Z} \setminus \{0\}$, let q and r be invertible elements. Moreover, assume that if $q = 1$ then $r = q^N$. The q-Brauer algebra $Br_n(r, q)$ is defined over the ring $\mathbb{Z}[q^{\pm 1}, r^{\pm 1}, ((r-1)/(q-1))^{\pm 1}]$ by generators g_1, g_2, g_3, ..., g_{n-1} and e and relations

(H) The elements g_1, g_2, g_3, ..., g_{n-1} satisfy the relations of the Hecke algebra H_n;

(E_1) $e^2 = \dfrac{r-1}{q-1} e$;

(E_2) $eg_i = g_i e$ for $i > 2$, $eg_1 = g_1 e = qe$, $eg_2 e = re$ and $eg_2^{-1} e = q^{-1} e$;

(E_3) $e_{(2)} = g_2 g_3 g_1^{-1} g_2^{-1} e_{(2)} = e_{(2)} g_2 g_3 g_1^{-1} g_2^{-1}$, where $e_{(2)} = e(g_2 g_3 g_1^{-1} g_2^{-1})e$.

The elements $e_{(k)}$ in $Br_n(r, q)$ are defined inductively by $e_{(1)} = e$ and by $e_{(k+1)} = eg^{+}_{2,2k+1} g^{-}_{1,2k} e_{(k)}$.

Dung (2014)

Version that contains $H_n(q^2)$

Let r and q be invertible elements over the ring $\mathbb{Z}[q^{\pm 1}, r^{\pm 1}, (\frac{r-r^{-1}}{q-q^{-1}})^{\pm 1}]$. Moreover, if $q = 1$ then assume that $r = q^N$ with $N \in \mathbb{Z} \setminus \{0\}$. The q-Brauer algebra $Br_n(r^2, q^2)$ over $\mathbb{Z}[q^{\pm 1}, r^{\pm 1}, (\frac{r-r^{-1}}{q-q^{-1}})^{\pm 1}]$ is the algebra defined via generators g_1, g_2, g_3, ..., g_{n-1} and e and relations

(H) The elements g_1, g_2, g_3, ..., g_{n-1} satisfy the relations of the Hecke algebra H_n;

(E_1) $e^2 = \frac{r-r^{-1}}{q-q^{-1}} e$;

(E_2) $eg_i = g_i e$ for $i > 2$, $eg_1 = g_1 e = q^2 e$, $eg_2 e = rqe$ and $eg_2^{-1} e = (rq)^{-1} e$;

(E_3) $g_2 g_3 g_1^{-1} g_2^{-1} e_{(2)} = e_{(2)} g_2 g_3 g_1^{-1} g_2^{-1}$.

Notations in Theorem 1

- k an integer, $0 \leq k \leq [n/2]$
- $B_{k,n} = \{u \in B_k \mid \ell(d) = \ell(u) \text{ with } d = e_{(k)}u \in \mathscr{D}_{k,n}\}$
- $S_{2k+1,n} = \mathbb{F}\{s_{2k+1}, s_{2k+2}, \cdots, s_{n-1}\}$ (the symmetric group)
- $H_{2k+1,n} = \mathbb{F}\{g_s,\ s \in S_{2k+1,n}\}$ (the Hecke algebra)
- S_λ : The Young subgroup of $S_{2k+1,n}$
- $Std(\lambda)$: The set of all standard λ- tableaux
- $\Lambda_n := \{(k, \lambda) \mid \lambda \text{ is a partition of } n - 2k\}$
- $\lambda \trianglerighteq \mu$: if $|\mu| > |\lambda|$ or $|\mu| = |\lambda|$ and $\sum_{i=1}^m \lambda_i \geq \sum_{i=1}^m \mu_i$
- $\mathcal{I}_n(k, \lambda) := \{(\mathfrak{s}, u) : \mathfrak{s} \in Std(\lambda) \text{ and } u \in B_{k,n}\}$
- $m_\mu = e_{(k)}c_\mu = c_\mu e_{(k)}$; $\qquad c_\mu = \sum_{\sigma \in S_\mu} g_\sigma$
- $\check{B}r_n^\lambda :=$
$$\left\{ x^\mu_{(\mathfrak{s},u)(\mathfrak{t},v)} := g_u^* g_{d(\mathfrak{s})}^* m_\mu g_{d(\mathfrak{t})} g_v \;\middle|\; \begin{array}{c} (\mathfrak{s}, u), (\mathfrak{t}, v) \in \mathcal{I}_n(l, \mu) \\ \mu \rhd \lambda \text{ for } (l, \mu), (k, \lambda) \in \Lambda_n \end{array} \right\}$$

Example

The Murphy basis of $H_{3,5}$: $\{c_{\mathfrak{s}\mathfrak{t}} = g^*_{d(\mathfrak{s})} c_\lambda g_{d(\mathfrak{t})}\}$

With $\mathfrak{t} = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}$, $\mathfrak{s} = \begin{array}{|c|c|}\hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}$, $\mathfrak{p} = \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline\end{array}$, $\mathfrak{q} = \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline 5 \\ \hline\end{array}$, we have $c^{(1^3)}_{\mathfrak{q}\mathfrak{q}} = 1$,
$c_{\mathfrak{t}\mathfrak{t}} = 1 + g_3$, $c_{\mathfrak{t}\mathfrak{s}} = (1 + g_3)g_4$, $c_{\mathfrak{s}\mathfrak{t}} = g_4(1 + g_3)$, $c_{\mathfrak{s}\mathfrak{s}} = g_4(1 + g_3)g_4$,
$c_{\mathfrak{p}\mathfrak{p}} = 1 + g_3 + g_4 + g_3g_4 + g_4g_3 + g_4g_3g_4$.

The presentation of $g_\pi = g_3g_4$ in The Murphy basis of $H_{3,5}$

$$g_\pi = g_3g_4 = \frac{q^2-1}{q^2} c_{\mathfrak{t}\mathfrak{s}} + \frac{1}{q^2} c_{\mathfrak{p}\mathfrak{p}} - \frac{1}{q^2} c_{\mathfrak{t}\mathfrak{t}} - \frac{1}{q^2} c_{\mathfrak{s}\mathfrak{t}} - \frac{1}{q^2} c_{\mathfrak{s}\mathfrak{s}} + c_{\mathfrak{q}\mathfrak{q}}$$

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The presentation of $g_\pi = g_3 g_4$ in The Murphy basis of $H_{3,5}$

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The presentation of $g_d = g_u^* e g_\pi g_v$ in the cell basis of $Br_5(r^2, q^2)$

$$\begin{aligned} g_d = g_u^* e g_\pi g_v = {} & \frac{q^2-1}{q^2} x^{(2,1)}_{(\mathfrak{t},u)(\mathfrak{s},v)} + \frac{1}{q^2} x^{(3)}_{(\mathfrak{p},u)(\mathfrak{p},v)} - \frac{1}{q^2} x^{(2,1)}_{(\mathfrak{t},u)(\mathfrak{t},v)} \\ & - \frac{1}{q^2} x^{(2,1)}_{(\mathfrak{s},u)(\mathfrak{t},v)} - \frac{1}{q^2} x^{(2,1)}_{(\mathfrak{s},u)(\mathfrak{s},v)} + x^{(1^3)}_{(\mathfrak{q},u)(\mathfrak{q},v)}, \end{aligned}$$

with $x^\lambda_{(\mathfrak{s},u)(\mathfrak{t},v)} = g_u^* e c_{\mathfrak{s}\mathfrak{t}} g_v = g_u^* g_{d(\mathfrak{s})}^* e c_\lambda g_{d(\mathfrak{t})} g_v = g_u^* g_{d(\mathfrak{s})}^* m_\lambda g_{d(\mathfrak{t})} g_v$

Notations in Theorem 2

- F: A field of characteristic p
- $rad(C(k,\lambda)) = \{x \in C(k,\lambda) |\ \langle x, y\rangle_\lambda = 0 \text{ for all } y \in C(k,\lambda)\}$
- $D(k,\lambda) = C(k,\lambda)/rad(C(k,\lambda))$.
- $d_{\lambda\mu} = [C(k,\lambda) \ : \ D(l,\mu)]$: the composition multiplicity of $D(l,\mu)$ in $C(k,\lambda)$

A semisimplicity criteria of the q-Brauer algebra for $n = 2, 3$

Let F be a field with $char(F) = p$. Then,

1. $Br_2(r^2, q^2)$ is semisimple $<=>$ $e(q^2) > 2$.
2. $Br_3(r^2, q^2)$ is semisimple $<=>$ $e(q^2) > 3$ and
$$\frac{3q^5(r^2 - q^2)^2(q^4 r^2 - 1)}{r^3(q^2 - 1)^3} \neq 0$$
3. $Br_2(r, q)$ is semisimple $<=>$ $e(q) > 2$.
4. $Br_3(r, q)$ is semisimple $<=>$ $e(q) > 3$ and
$$\frac{3q(r - q)^2(q^2 r - 1)}{(q - 1)^3} \neq 0$$
5. $Br_2(N)$ is semisimple $<=>$ $e(q) > 2$.
6. $Br_3(N)$ is semisimple $<=>$ $e(q) > 3$ and
$$3q^4(q^N - q[N])([N] + q^{N+1} + q^{N+3}) \neq 0$$

Ex1

Over field $\mathbb{C}$, the q-Brauer algebra and the BMW-algebra simultaneously depend on two parameters r and q. Calculation shows that

$\mathbb{C}$	BMW-algebra	q-Brauer algebra
$(r, q^2) = (q^{-1}, -i)$	$\mathscr{B}_3$ is not semisimple	$Br_3(r^2, q^2)$ is semisimple
$(r, q) = (q^{-1}, i\sqrt{i})$	$\mathscr{B}_3$ is not semisimple	$Br_3(r, q)$ is semisimple

Ex2

Over field $\mathbb{F}$ with $char(\mathbb{F}) = 5$. The total parameter values, such that the algebras are not semisimple, are summarized in the following table.

The non-semisimple case	$\mathbb{F}_5 \times \mathbb{F}_5$
The BMW-algebra $\mathscr{B}_2$	$(r, q) \in (\{\bar{1}, \bar{2}, \bar{3}, \bar{4}\} \times \{\bar{2}, \bar{3}\}) \cup (\{\bar{2},$
The q-Brauer algebra $Br_2(r^2, q^2)$	$(r, q) \in \{\bar{2},\ \bar{3}\} \times \{\bar{2}, \bar{3}\}$
The q-Brauer algebra $Br_2(r, q)$	$(r, q) \in \{\bar{2}, \bar{3}, \bar{4}\} \times \{\bar{4}\}$

Thank for your attention