A KLR grading of the Brauer algebras

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- Recently, Brundan and Kleshchev showed that cyclotomic Hecke algebras of type $G(\ell, 1, n)$ are isomorphic to the cyclotomic Khovanov-Lauda-Rouquier algebras $\mathscr{R}_n^\Lambda$ introduced by Khovanov and Lauda, and Rouquier, where a connection between the representation theory of Hecke algebras and Lusztig's canonical bases was established. In this way, cyclotomic Hecke algebras inherit a $\mathbb{Z}$-grading.

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- Hu and Mathas proved that $\mathscr{R}_n^\Lambda$ is graded cellular over a field, or an integral domain with certain properties, by constructing a graded cellular basis $\{\, \psi_{\mathsf{st}} \mid \mathsf{s}, \mathsf{t} \in \mathrm{Std}(\lambda), \lambda \vdash n \,\}$.

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Introduction

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- As a speical case of cyclotomic Hecke algebras, the symmetric group algebras $R\mathfrak{S}_n$ inherit the above properties.
- The goal of this talk is to study the $\mathbb{Z}$-grading of the Brauer algebra $\mathscr{B}_n(\delta)$ over a field R of characteristic $p = 0$, and as a byproduct, show the Brauer algebras $\mathscr{B}_n(\delta)$ are graded cellular algebras.

The Brauer algebras

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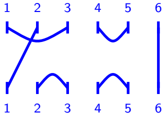
associated with relations

1. (Inverses) $s_k^2 = 1$.
2. (Essential idempotent relation) $e_k^2 = \delta e_k$.
3. (Braid relations) $s_k s_{k+1} s_k = s_{k+1} s_k s_{k+1}$ and $s_k s_r = s_r s_k$ if $|k - r| > 1$.
4. (Commutation relations) $s_k e_l = e_l s_k$ and $e_k e_r = e_r e_k$ if $|k - r| > 1$.
5. (Tangle relations) $e_k e_{k+1} e_k = e_k$, $e_{k+1} e_k e_{k+1} = e_{k+1}$, $s_k e_{k+1} e_k = s_{k+1} e_k$ and $e_k e_{k+1} s_k = e_k s_{k+1}$.
6. (Untwisting relations) $s_k e_k = e_k s_k = e_k$.

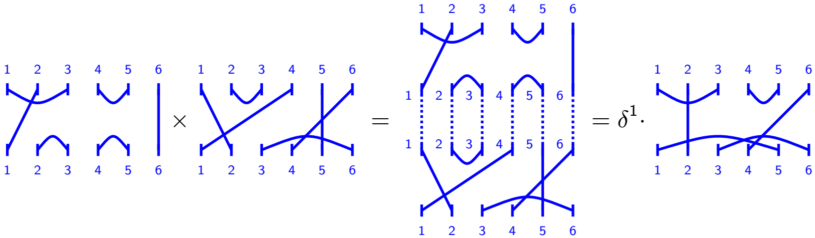
The Brauer algebra $\mathscr{B}_n(\delta)$ has R-basis consisting of Brauer diagrams D, which consist of two rows of n dots, labelled by $\{1, 2, \dots, n\}$, with each dot joined to one other dot. See the following diagram as an example:

$D =$  .

Two diagrams D_1 and D_2 can be composed to get $D_1 \circ D_2$ by placing D_1 above D_2 and joining corresponding points and deleting all the interior loops. The multiplication of $\mathscr{B}_n(\delta)$ is defined by

$$D_1 \cdot D_2 = \delta^{n(D_1, D_2)} D_1 \circ D_2,$$

where $n(D_1, D_2)$ is the number of deleted loops. For example:



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Let $P = \mathbb{Z} + \frac{\delta - 1}{2}$ and Γ_δ be the oriented quiver with vertex set P and directed edges $i \to i + 1$, for $i \in P$. Thus, Γ_δ is the quiver of type A_∞.

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Fix a *weight* $\Lambda = \Lambda_k$ for some $k \in P$. The *cyclotomic KLR algebras*, $\mathscr{R}_n^\Lambda$ of type Γ_δ is the unital associative R-algebra with generators

$$\{\, e(\mathbf{i}) \mid \mathbf{i} \in P^n \,\} \cup \{\, y_k \mid 1 \le k \le n \,\} \cup \{\, \psi_k \mid 1 \le k \le n-1 \,\},$$

and relations:

$$y_1^{\delta_{i_1,k}} e(\mathbf{i}) = 0, \qquad e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{ij}} e(\mathbf{i}), \qquad \textstyle\sum_{\mathbf{i}\in P^n} e(\mathbf{i}) = 1,$$

$$y_r e(\mathbf{i}) = e(\mathbf{i}) y_r, \qquad \psi_r e(\mathbf{i}) = e(s_r \cdot \mathbf{i}) \psi_r, \qquad y_r y_s = y_s y_r,$$

$$\psi_r y_s = y_s \psi_r, \qquad \text{if } s \neq r, r+1,$$

$$\psi_r \psi_s = \psi_s \psi_r, \qquad \text{if } |r-s| > 1,$$

$$\psi_r y_{r+1} e(\mathbf{i}) = \begin{cases} (y_r \psi_r + 1) e(\mathbf{i}), & \text{if } i_r = i_{r+1}, \\ y_r \psi_r e(\mathbf{i}), & \text{if } i_r \neq i_{r+1} \end{cases}$$

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$$\psi_r^2 e(\mathbf{i}) = \begin{cases} 0, & \text{if } i_r = i_{r+1}, \\ e(\mathbf{i}), & \text{if } i_r \neq i_{r+1} \pm 1, \\ (y_{r+1} - y_r)e(\mathbf{i}), & \text{if } i_{r+1} = i_r + 1, \\ (y_r - y_{r+1})e(\mathbf{i}), & \text{if } i_{r+1} = i_r - 1, \end{cases}$$

$$\psi_r\psi_{r+1}\psi_r e(\mathbf{i}) = \begin{cases} (\psi_{r+1}\psi_r\psi_{r+1} + 1)e(\mathbf{i}), & \text{if } i_{r+2} = i_r = i_{r+1} - 1, \\ (\psi_{r+1}\psi_r\psi_{r+1} - 1)e(\mathbf{i}), & \text{if } i_{r+2} = i_r = i_{r+1} + 1, \\ \psi_{r+1}\psi_r\psi_{r+1} e(\mathbf{i}), & \text{otherwise.} \end{cases}$$

for $\mathbf{i}, \mathbf{j} \in P^n$ and all admissible r and s. Moreover, $\mathscr{R}_n^\Lambda$ is naturally $\mathbb{Z}$-graded with degree function determined by

$$\deg e(\mathbf{i}) = 0, \qquad \deg y_r = 2 \qquad \text{and} \qquad \deg \psi_k e(\mathbf{i}) = \begin{cases} -2, & \text{if } i_k = i_{k+1}, \\ 0, & \text{if } i_k \neq i_{k+1} \pm 1, \\ 1, & \text{if } i_k = i_{k+1} \pm 1. \end{cases}$$

for $1 \le r \le n$, $1 \le k \le n$ and $\mathbf{i} \in P^n$.

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The cyclotomic KLR algebras

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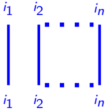
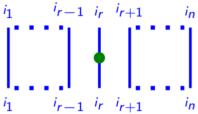
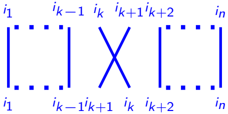
$e(\mathbf{i}) =$  ,

$e(\mathbf{i})y_r =$  ,

$e(\mathbf{i})\psi_k =$  ,

for $\mathbf{i} = (i_1, \ldots, i_n) \in P^n$, $1 \leq r \leq n$ and $1 \leq k \leq n-1$. The labels connected by a string have to be the same.

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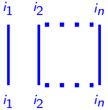
$e(\mathbf{i}) =$ 

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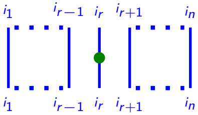
$e(\mathbf{i})y_r =$ 

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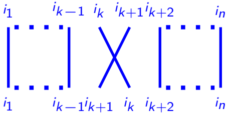
$e(\mathbf{i})\psi_k =$ 

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for $\mathbf{i} = (i_1, \ldots, i_n) \in P^n$, $1 \leq r \leq n$ and $1 \leq k \leq n-1$. The labels connected by a string have to be the same.

Diagrams are considered up to isotopy, and multiplication of diagrams is given by concatenation, subject to the relations of $\mathscr{R}_n^\Lambda$ listed before.

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The symmetric group algebras $R\mathfrak{S}_n$ are isomorphic to the cyclotomic KLR algebras $\mathscr{R}_n^\Lambda$.

Theorem (Hu-Mathas)

There exists a set of homogeneous elements of $\mathscr{R}_n^\Lambda$

$$\{\, \psi_{\mathsf{st}} \mid \mathsf{s}, \mathsf{t} \in \mathrm{Std}(\lambda), \lambda \vdash n \,\}$$

and these elements form a graded cellular basis of $\mathscr{R}_n^\Lambda$.

It is well-known that the symmetric group algebras $R\mathfrak{S}_n$ are subalgebras of the Brauer algebras $\mathscr{B}_n(\delta)$ by removing all Brauer diagrams with horizontal arcs. So we expect the $\mathbb{Z}$-grading of Brauer algebras has following properties:

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- The graded Brauer algebras are generated by homogeneous generators

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- The grading is compatible to the corresponding $\mathscr{R}_n^\Lambda$ if we restrict $\mathscr{B}_n(\delta)$ to $R\mathfrak{S}_n$.

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- There exists a diagrammatic representation of the graded Brauer algebras as P-labelled decorated planar diagram on $2n$ dots.

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For $\mathbf{i} \in P^n$ and $1 \leq k \leq n$, we define the function $h_k : P^n \longrightarrow \mathbb{Z}$ as

$$\begin{aligned} h_k(\mathbf{i}) \quad := \quad & \delta_{i_k, -\frac{\delta-1}{2}} + \#\{\, 1 \leq r \leq k-1 \mid i_r = -i_k \pm 1 \,\} \\ & +2\#\{\, 1 \leq r \leq k-1 \mid i_r = i_k \,\} \\ & -\delta_{i_k, \frac{\delta-1}{2}} - \#\{\, 1 \leq r \leq k-1 \mid i_r = i_k \pm 1 \,\} \\ & -2\#\{\, 1 \leq r \leq k-1 \mid i_r = -i_k \,\}. \end{aligned}$$

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We now categorize P^n using h_k. For $1 \leq k \leq n$, define $P^n_{k,+}$, $P^n_{k,-}$ and $P^n_{k,0}$ as subsets of P^n by

$$P^n_{k,+} := \{ \mathbf{i} \in P^n \mid i_k \neq 0, -\frac{1}{2} \text{ and } h_k(\mathbf{i}) = 0, \text{ or } i_k = -\frac{1}{2} \text{ and } h_k(\mathbf{i}) = -1 \} .$$

$$P^n_{k,-} := \{ \mathbf{i} \in P^n \mid i_k \neq 0, -\frac{1}{2} \text{ and } h_k(\mathbf{i}) = -2, \text{ or } i_k = -\frac{1}{2} \text{ and } h_k(\mathbf{i}) = -3 \} ,$$

$$P^n_{k,0} := P^n \backslash (P^n_{k,+} \cup P^n_{k,-}).$$

Clearly we have $P^n = P^n_{k,+} \sqcup P^n_{k,-} \sqcup P^n_{k,0}$.

For $\mathbf{i} \in P^n$ and $1 \leq k \leq n-1$, define $a_k(\mathbf{i}) \in \mathbb{Z}$ as

$$a_k(\mathbf{i}) = \begin{cases} \#\{\, 1 \leq r \leq k-1 \mid i_r \in \{-1, 1\} \,\} + 1 + \delta_{\frac{i_k - i_{k+1}}{2}, \frac{\delta-1}{2}}, & \text{if } \frac{i_k - i_{k+1}}{2} = 0, \\ \#\{\, 1 \leq r \leq k-1 \mid i_r \in \{-1, 1\} \,\} + \delta_{\frac{i_k - i_{k+1}}{2}, \frac{\delta-1}{2}}, & \text{if } \frac{i_k - i_{k+1}}{2} = 1, \\ \delta_{\frac{i_k - i_{k+1}}{2}, \frac{\delta-1}{2}}, & \text{if } \frac{i_k - i_{k+1}}{2} = 1/2, \\ \#\{\, 1 \leq r \leq k-1 \mid i_r \in \{\pm\frac{i_k - i_{k+1}}{2}, \pm\left(\frac{i_k - i_{k+1}}{2} - 1\right)\} \,\} & \\ \qquad + \delta_{\frac{i_k - i_{k+1}}{2}, \frac{\delta-1}{2}}, & \text{otherwise;} \end{cases}$$

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and $A^{\mathbf{i}}_{k,1}, A^{\mathbf{i}}_{k,2}, A^{\mathbf{i}}_{k,3}, A^{\mathbf{i}}_{k,4} \subset \{1, 2, \ldots, k-1\}$ as

$$A^{\mathbf{i}}_{k,1} := \{\,1 \leq r \leq k-1 \mid i_r = -i_k \pm 1\,\}, \quad A^{\mathbf{i}}_{k,2} := \{\,1 \leq r \leq k-1 \mid i_r = i_k\,\},$$
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and for $\mathbf{i} \in P^n_{k,0}$ and $1 \le k \le n-1$, define $z_k(\mathbf{i}) \in \mathbb{Z}$ by

$$z_k(\mathbf{i}) = \begin{cases} 0, & \text{if } h_k(\mathbf{i}) < -2, \text{ or } h_k(\mathbf{i}) \ge 0 \text{ and } i_k \ne 0, \\ (-1)^{a_k(\mathbf{i})}(1 + \delta_{i_k, -\frac{1}{2}}), & \text{if } -2 \le h_k(\mathbf{i}) < 0, \\ \frac{1 + (-1)^{a_k(\mathbf{i})}}{2}, & \text{if } i_k = 0. \end{cases}$$

Let $\mathscr{G}_n(\delta)$ be an unital associate R-algebra with generators

$$\{\, e(\mathbf{i}) \mid \mathbf{i} \in P^n \,\} \cup \{\, y_k \mid 1 \le k \le n \,\} \cup \{\, \psi_k \mid 1 \le k \le n-1 \,\} \cup \{\, \epsilon_k \mid 1 \le k \le n-1 \,\}$$

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associated with the following relations:

(1). Idempotent relations: Let $\mathbf{i}, \mathbf{j} \in P^n$ and $1 \le k \le n-1$. Then

$$y_1^{\delta_{i_1, \frac{\delta-1}{2}}} e(\mathbf{i}) = 0, \qquad \sum_{\mathbf{i} \in P^n} e(\mathbf{i}) = 1,$$

$$e(\mathbf{i}) e(\mathbf{j}) = \delta_{\mathbf{i},\mathbf{j}} e(\mathbf{i}), \qquad e(\mathbf{i}) \epsilon_k = \epsilon_k e(\mathbf{i}) = 0 \text{ if } i_k + i_{k+1} \neq 0;$$

(2). Commutation relations: Let $\mathbf{i} \in P^n$. Then

$$y_k e(\mathbf{i}) = e(\mathbf{i}) y_k, \qquad \psi_k e(\mathbf{i}) = e(\mathbf{i} \cdot s_k) \psi_k \qquad \text{and}$$

$$y_k y_r = y_r y_k, \qquad y_k \psi_r = \psi_r y_k, \qquad y_k \epsilon_r = \epsilon_r y_k,$$

$$\psi_k \psi_r = \psi_r \psi_k, \qquad \psi_k \epsilon_r = \epsilon_r \psi_k, \qquad \epsilon_k \epsilon_r = \epsilon_r \epsilon_k \qquad \text{if } |k - r| > 1;$$

(3). Essential commutation relations: Let $\mathbf{i} \in P^n$ and $1 \leq k \leq n-1$. Then

$$\begin{aligned} & e(\mathbf{i})y_k\psi_k = e(\mathbf{i})\psi_k y_{k+1} + e(\mathbf{i})\epsilon_k e(\mathbf{i}\cdot s_k) - \delta_{i_k,i_{k+1}} e(\mathbf{i}), \\ \text{and} \qquad & e(\mathbf{i})\psi_k y_k = e(\mathbf{i})y_{k+1}\psi_k + e(\mathbf{i})\epsilon_k e(\mathbf{i}\cdot s_k) - \delta_{i_k,i_{k+1}} e(\mathbf{i}). \end{aligned}$$

(4). Inverse relations: Let $\mathbf{i} \in P^n$ and $1 \leq k \leq n-1$. Then

$$e(\mathbf{i})\psi_k^2 = \begin{cases} 0, & \text{if } i_k = i_{k+1} \text{ or } i_k + i_{k+1} = 0 \text{ and } h_k(\mathbf{i}) \neq 0, \\ (y_k - y_{k+1})e(\mathbf{i}), & \text{if } i_k = i_{k+1} + 1 \text{ and } i_k + i_{k+1} \neq 0, \\ (y_{k+1} - y_k)e(\mathbf{i}), & \text{if } i_k = i_{k+1} - 1 \text{ and } i_k + i_{k+1} \neq 0, \\ e(\mathbf{i}), & \text{otherwise;} \end{cases}$$

(5). Essential idempotent relations: Let $\mathbf{i}, \mathbf{j}, \mathbf{k} \in P^n$ and $1 \leq k \leq n-1$. Then

$$e(\mathbf{i})\epsilon_k e(\mathbf{i}) = \begin{cases} (-1)^{a_k(\mathbf{i})} e(\mathbf{i}), & \text{if } \mathbf{i} \in P^n_{k,0} \text{ and } i_k = -i_{k+1} \neq \pm\frac{1}{2}, \\ (-1)^{a_k(\mathbf{i})+1}(y_{k+1} - y_k)e(\mathbf{i}), & \text{if } \mathbf{i} \in P^n_{k,+}; \end{cases}$$

$$\begin{aligned} y_{k+1}e(\mathbf{i}) &= y_k e(\mathbf{i}) - 2(-1)^{a_k(\mathbf{i})} y_k e(\mathbf{i})\epsilon_k e(\mathbf{i}) \\ &= y_k e(\mathbf{i}) - 2(-1)^{a_k(\mathbf{i})} e(\mathbf{i})\epsilon_k e(\mathbf{i}) y_k, \qquad \text{if } \mathbf{i} \in P^n_{k,0} \text{ and } i_k = -i_{k+1} = \frac{1}{2}, \end{aligned}$$

$$\begin{aligned} e(\mathbf{i}) = (-1)^{a_k(\mathbf{i})} & e(\mathbf{i})\epsilon_k e(\mathbf{i}) - 2(-1)^{a_{k-1}(\mathbf{i})} e(\mathbf{i})\epsilon_{k-1} e(\mathbf{i}) \\ &+ e(\mathbf{i})\epsilon_{k-1}\epsilon_k e(\mathbf{i}) + e(\mathbf{i})\epsilon_k\epsilon_{k-1} e(\mathbf{i}), \qquad \text{if } \mathbf{i} \in P^n_{k,0} \\ &\qquad\qquad \text{and } -i_{k-1} = i_k = -i_{k+1} = -\frac{1}{2}, \end{aligned}$$

$$e(\mathbf{i}) = (-1)^{a_k(\mathbf{i})} e(\mathbf{i})(\epsilon_k y_k + y_k \epsilon_k) e(\mathbf{i}), \qquad \text{if } \mathbf{i} \in P^n_{k,-} \text{ and } i_k = -i_{k+1},$$

$$e(\mathbf{j})\epsilon_k e(\mathbf{i})\epsilon_k e(\mathbf{k}) = \begin{cases} z_k(\mathbf{i}) e(\mathbf{j})\epsilon_k e(\mathbf{k}), & \text{if } \mathbf{i} \in P^n_{k,0}, \\ 0, & \text{if } \mathbf{i} \in P^n_{k,-}, \\ (-1)^{a_k(\mathbf{i})}(1 + \delta_{i_k,-\frac{1}{2}})(\sum_{r \in A^{\mathbf{i}}_{k,1}} y_r - 2\sum_{r \in A^{\mathbf{i}}_{k,2}} y_r \\ \qquad + \sum_{r \in A^{\mathbf{i}}_{k,3}} y_r - 2\sum_{r \in A^{\mathbf{i}}_{k,4}} y_r) e(\mathbf{j})\epsilon_k e(\mathbf{k}), & \text{if } \mathbf{i} \in P^n_{k,+}; \end{cases}$$

(6). Untwist relations: Let $\mathbf{i}, \mathbf{j} \in P^n$ and $1 \leq k \leq n-1$. Then

$$e(\mathbf{i})\psi_k\epsilon_k e(\mathbf{j}) = \begin{cases} (-1)^{a_k(\mathbf{i})} e(\mathbf{i})\epsilon_k e(\mathbf{j}), & \text{if } \mathbf{i} \in P^n_{k,+} \text{ and } i_k \neq 0, -\frac{1}{2}, \\ 0, & \text{otherwise;} \end{cases}$$

$$e(\mathbf{j})\epsilon_k\psi_k e(\mathbf{i}) = \begin{cases} (-1)^{a_k(\mathbf{i})} e(\mathbf{j})\epsilon_k e(\mathbf{i}), & \text{if } \mathbf{i} \in P^n_{k,+} \text{ and } i_k \neq 0, -\frac{1}{2}, \\ 0, & \text{otherwise;} \end{cases}$$

(7). Tangle relations: Let $\mathbf{i}, \mathbf{j} \in P^n$ and $1 < k < n$. Then

$$e(\mathbf{j})\epsilon_k\epsilon_{k-1}\psi_k e(\mathbf{i}) = e(\mathbf{j})\epsilon_k\psi_{k-1}e(\mathbf{i}), \quad e(\mathbf{i})\psi_k\epsilon_{k-1}\epsilon_k e(\mathbf{j}) = e(\mathbf{i})\psi_{k-1}\epsilon_k e(\mathbf{j}),$$

$$e(\mathbf{i})\epsilon_k\epsilon_{k-1}\epsilon_k e(\mathbf{j}) = e(\mathbf{i})\epsilon_k e(\mathbf{j}); \quad e(\mathbf{i})\epsilon_{k-1}\epsilon_k\epsilon_{k-1}e(\mathbf{j}) = e(\mathbf{i})\epsilon_{k-1}e(\mathbf{j});$$

$$e(\mathbf{i})\epsilon_k e(\mathbf{j})(y_k + y_{k+1}) = 0;$$

(8). Braid relations: Let $\mathcal{B}_k = \psi_k\psi_{k-1}\psi_k - \psi_{k-1}\psi_k\psi_{k-1}$, $\mathbf{i} \in P^n$ and $1 < k < n$. Then

$$e(\mathbf{i})\mathcal{B}_k = \begin{cases} e(\mathbf{i})\epsilon_k\epsilon_{k-1}e(\mathbf{i}{\cdot}s_ks_{k-1}s_k), & \text{if } i_k + i_{k+1} = 0 \text{ and } i_{k-1} = \pm(i_k - 1), \\ -e(\mathbf{i})\epsilon_k\epsilon_{k-1}e(\mathbf{i}{\cdot}s_ks_{k-1}s_k), & \text{if } i_k + i_{k+1} = 0 \text{ and } i_{k-1} = \pm(i_k + 1), \\ e(\mathbf{i})\epsilon_{k-1}\epsilon_k e(\mathbf{i}{\cdot}s_ks_{k-1}s_k), & \text{if } i_{k-1} + i_k = 0 \text{ and } i_{k+1} = \pm(i_k - 1), \\ -e(\mathbf{i})\epsilon_{k-1}\epsilon_k e(\mathbf{i}{\cdot}s_ks_{k-1}s_k), & \text{if } i_{k-1} + i_k = 0 \text{ and } i_{k+1} = \pm(i_k + 1), \\ -(-1)^{a_{k-1}(\mathbf{i})}e(\mathbf{i})\epsilon_{k-1}e(\mathbf{i}{\cdot}s_ks_{k-1}s_k), & \text{if } i_{k-1} = -i_k = i_{k+1} \neq 0, \pm\frac{1}{2} \\ & \qquad \text{and } h_k(\mathbf{i}) = 0, \\ (-1)^{a_k(\mathbf{i})}e(\mathbf{i})\epsilon_k e(\mathbf{i}{\cdot}s_ks_{k-1}s_k), & \text{if } i_{k-1} = -i_k = i_{k+1} \neq 0, \pm\frac{1}{2} \\ & \qquad \text{and } h_{k-1}(\mathbf{i}) = 0, \\ e(\mathbf{i}), & \text{if } i_{k-1} + i_k, i_{k-1} + i_{k+1}, i_k + i_{k+1} \neq 0 \\ & \qquad \text{and } i_{k-1} = i_{k+1} = i_k - 1, \\ -e(\mathbf{i}), & \text{if } i_{k-1} + i_k, i_{k-1} + i_{k+1}, i_k + i_{k+1} \neq 0 \\ & \qquad \text{and } i_{k-1} = i_{k+1} = i_k + 1, \\ 0, & \text{otherwise.} \end{cases}$$

The algebra is self-graded, where the degree of $e(\mathbf{i})$ is 0, y_k is 2 and

$$\deg e(\mathbf{i})\psi_k = \begin{cases} 1, & \text{if } i_k = i_{k+1} \pm 1, \\ -2, & \text{if } i_k = i_{k+1}, \\ 0, & \text{otherwise}; \end{cases}$$

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It is easy to verify that there exists an involution $*$ on $\mathscr{G}_n(\delta)$ such that $e(\mathbf{i})^* = e(\mathbf{i})$, $y_r^* = y_r$, $\psi_k^* = \psi_k$ and $\epsilon_k^* = \epsilon_k$ for $\mathbf{i} \in P^n$, $1 \leq r \leq n$ and $1 \leq k \leq n-1$.

We have an diagrammatic representation of $\mathscr{G}_n(\delta)$.

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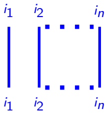
$e(\mathbf{i}) =$ 

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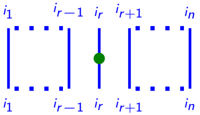
$e(\mathbf{i})y_r =$ 

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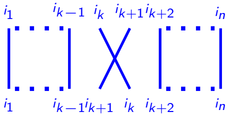
$e(\mathbf{i})\psi_k =$ 

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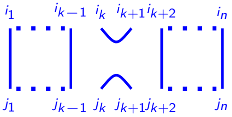
$e(\mathbf{i})\epsilon_k e(\mathbf{j}) =$ 

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for $\mathbf{i} = (i_1, \ldots, i_n) \in P^n$, $\mathbf{j} = (j_1, \ldots, j_n) \in P^n$, $1 \leq r \leq n$ and $1 \leq k \leq n-1$. The labels connected by a vertical string have to be the same, and the sum of labels connected by a horizontal string equals 0.

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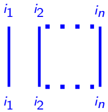
$e(\mathbf{i}) =$ 

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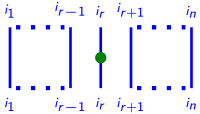
$e(\mathbf{i})y_r =$ 

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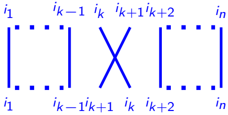
$e(\mathbf{i})\psi_k =$ 

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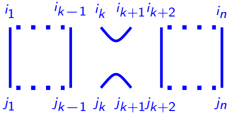
$e(\mathbf{i})\epsilon_k e(\mathbf{j}) =$ 

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Diagrams are considered up to isotopy, and multiplication of diagrams is given by concatenation, subject to the relations of $\mathscr{G}_n(\delta)$ listed before.

We will construct a set of homogeneous elements such that these elements span $\mathscr{G}_n(\delta)$.

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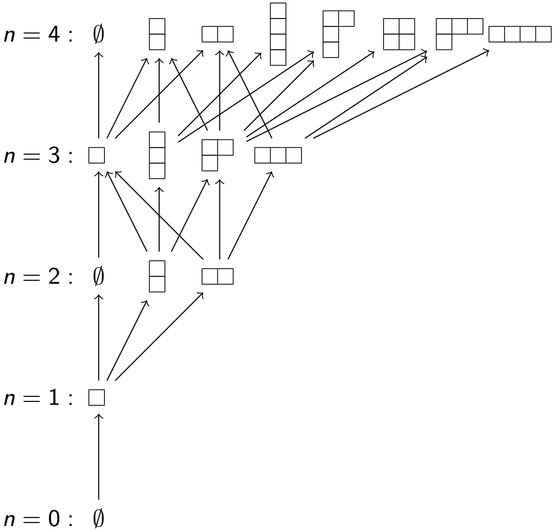
Define $\widehat{B}_n := \{\, (\lambda, f) \mid \lambda \vdash n - 2f,\ \text{and } 0 \leq f \leq \lfloor \frac{n}{2} \rfloor \,\}$ and $\widehat{B}$ to be the graph with

- vertices at level n: $\widehat{B}_n$, and
- an edge $(\lambda, f) \to (\mu, m)$, $(\lambda, f) \in \widehat{B}_{n-1}$ and $(\mu, m) \in \widehat{B}_n$, if either μ is obtained by adding a node to λ, or by deleting a node from λ.

Degree of up-down tableaux



Let $(\lambda, f) \in \widehat{B}_n$. An *up-down tableau* of shape (λ, f) is a sequence

$$\mathfrak{t} = ((\lambda^{(0)}, f_0), (\lambda^{(1)}, f_1), \dots, (\lambda^{(n)}, f_n)),$$

where $(\lambda^{(0)}, f_0) = (\emptyset, 0)$, $(\lambda^{(n)}, f_n) = (\lambda, f)$ and $(\lambda^{(k-1)}, f_{k-1}) \to (\lambda^{(k)}, f_k)$ is an edge in $\widehat{B}$, for $k = 1, \dots, n$.

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Suppose λ is a partition. A node $\alpha = (r, l) > 0$ is *addable* if $\lambda \cup \{\alpha\}$ is still a partition, and it is *removable* if $\lambda \backslash \{\alpha\}$ is still a partition. Let $\mathscr{A}(\lambda)$ and $\mathscr{R}(\lambda)$ be the sets of addable and removable nodes of λ, respectively.

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Recall $\delta \in R$. Suppose $\alpha = (r, l)$ is a node. The *residue* of α is defined to be $\operatorname{res}(\alpha) = \frac{\delta - 1}{2} + l - r$.

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Suppose we have $(\lambda, f) \to (\mu, m)$. Write $\lambda \ominus \mu = \alpha$ if $\lambda = \mu \cup \{\alpha\}$ or $\mu = \lambda \cup \{\alpha\}$.

For any up-down tableau $\mathfrak{t} = ((\lambda^{(0)}, f_0), (\lambda^{(1)}, f_1), \ldots, (\lambda^{(n)}, f_n))$ and an integer k with $1 \leq k \leq n$, let $\alpha = (r, l) = \lambda^{(k-1)} \ominus \lambda^{(k)}$. Define

$$\mathscr{A}_{\mathfrak{t}}(k) = \begin{cases} \{\, \beta = (k, c) \in \mathscr{A}(\lambda^{(k-1)}) \mid \operatorname{res}(\beta) = \operatorname{res}(\alpha) \text{ and } k > r \,\}, & \text{if } \lambda^{(k)} \supset \lambda^{(k-1)}, \\ \{\, \beta = (k, c) \in \mathscr{A}(\lambda^{(k)}) \mid \operatorname{res}(\beta) = -\operatorname{res}(\alpha) \text{ and } k \neq r \,\}, & \text{if } \lambda^{(k)} \subset \lambda^{(k-1)}; \end{cases}$$

$$\mathscr{R}_{\mathfrak{t}}(k) = \begin{cases} \{\, \beta = (k, c) \in \mathscr{R}(\lambda^{(k-1)}) \mid \operatorname{res}(\beta) = \operatorname{res}(\alpha) \text{ and } k > r \,\}, & \text{if } \lambda^{(k)} \supset \lambda^{(k-1)}, \\ \{\, \beta = (k, c) \in \mathscr{R}(\lambda^{(k)}) \mid \operatorname{res}(\beta) = -\operatorname{res}(\alpha) \,\}, & \text{if } \lambda^{(k)} \subset \lambda^{(k-1)}. \end{cases}$$

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Definition

For any up-down tableau $\mathfrak{t} = ((\lambda^{(0)}, f_0), (\lambda^{(1)}, f_1), \ldots, (\lambda^{(n)}, f_n))$ and an integer k with $1 \le k \le n$, let $\alpha = (r, l) = \lambda^{(k-1)} \ominus \lambda^{(k)}$. Define

$$\deg(\mathfrak{t}|_{k-1} \Rightarrow \mathfrak{t}|_k) := \begin{cases} |\mathscr{A}_{\mathfrak{t}}(k)| - |\mathscr{R}_{\mathfrak{t}}(k)|, & \text{if } \lambda^{(k)} \supset \lambda^{(k-1)}, \\ |\mathscr{A}_{\mathfrak{t}}(k)| - |\mathscr{R}_{\mathfrak{t}}(k)| + \delta_{\operatorname{res}(\alpha), -\frac{1}{2}}, & \text{if } \lambda^{(k)} \subset \lambda^{(k-1)}, \end{cases}$$

and the *degree* of $\mathfrak{t}$ is

$$\deg \mathfrak{t} := \sum_{k=1}^{n} \deg(\mathfrak{t}|_{k-1} \Rightarrow \mathfrak{t}|_k).$$

Theorem

There exist homogeneous elements $\{\, \psi_{\mathsf{st}} \mid (\lambda, f) \in \widehat{B}_n, \mathsf{s}, \mathsf{t} \in \mathscr{T}_n^{ud}(\lambda) \,\}$ *in* $\mathscr{G}_n(\delta)$ *with the following properties:*

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The homogeneous elements of $\mathscr{G}_n(\delta)$

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- *For any* $(\lambda, f) \in \widehat{B}_n$, $\mathsf{s}, \mathsf{t} \in \mathscr{T}_n^{ud}(\lambda)$ *and* $a \in \mathscr{G}_n(\delta)$, *we have*

$$\psi_{\mathsf{st}} a = \sum_{\mathsf{v} \in \mathscr{T}_n^{ud}(\lambda)} c_{\mathsf{v}} \psi_{\mathsf{sv}} + \sum_{\substack{(\mu,\ell) > (\lambda, f) \\ \mathsf{u},\mathsf{v} \in \mathscr{T}_n^{ud}(\mu)}} c_{\mathsf{uv}} \psi_{\mathsf{uv}},$$

with $c_{\mathsf{v}}, c_{\mathsf{uv}} \in R$ *and* $>$ *is the lexicographic ordering of* $\widehat{B}_n$.

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There exist homogeneous elements $\{\,\psi_{\mathsf{st}} \mid (\lambda, f) \in \widehat{B}_n, \mathsf{s}, \mathsf{t} \in \mathscr{T}_n^{ud}(\lambda)\,\}$ *in* $\mathscr{G}_n(\delta)$ *with the following properties:*

- $\deg \psi_{\mathsf{st}} = \deg \mathsf{s} + \deg \mathsf{t}$.
- *For any* $\mathbf{i} \in P^n$, $e(\mathbf{i}) = \sum_{\mathsf{s},\mathsf{t}} c_{\mathsf{st}} \psi_{\mathsf{st}}$ *with* $c_{\mathsf{st}} \in R$, *and* $c_{\mathsf{st}} \neq 0$ *only if* $\mathbf{i}$ *is the residue sequence of* s *and* t.
- *For any* $(\lambda, f) \in \widehat{B}_n$, $\mathsf{s}, \mathsf{t} \in \mathscr{T}_n^{ud}(\lambda)$ *and* $a \in \mathscr{G}_n(\delta)$, *we have*

$$\psi_{\mathsf{st}} a = \sum_{\mathsf{v} \in \mathscr{T}_n^{ud}(\lambda)} c_{\mathsf{v}} \psi_{\mathsf{sv}} + \sum_{\substack{(\mu,\ell) > (\lambda, f) \\ \mathsf{u},\mathsf{v} \in \mathscr{T}_n^{ud}(\mu)}} c_{\mathsf{uv}} \psi_{\mathsf{uv}},$$

  with $c_{\mathsf{v}}, c_{\mathsf{uv}} \in R$ *and* $>$ *is the lexicographic ordering of* $\widehat{B}_n$.

Moreover, $\{\,\psi_{\mathsf{st}} \mid (\lambda, f) \in \widehat{B}_n, \mathsf{s}, \mathsf{t} \in \mathscr{T}_n^{ud}(\lambda)\,\}$ *spans* $\mathscr{G}_n(\delta)$.

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- if $\dim \mathscr{G}_n(\delta) = (2n-1)!! = \dim \mathscr{B}_n(\delta)$, then

$$\{\, \psi_{\mathsf{st}} \mid (\lambda, f) \in \widehat{B}_n, \mathsf{s}, \mathsf{t} \in \mathscr{T}_n^{ud}(\lambda) \,\}$$

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  forms a graded cellular basis, which makes $\mathscr{G}_n(\delta)$ be a graded cellular algebra.
- y_k's are nilpotent, i.e. for $N \gg 0$, $y_k^N = 0$, because $\max_{\mathsf{s},\mathsf{t}} \deg \psi_{\mathsf{st}} < \infty$.

We focus on $\mathscr{B}_n(\delta)$ and construct a generating set of $\mathscr{B}_n(\delta)$

$$\{\, e(\mathbf{i}) \mid \mathbf{i} \in P^n \,\} \cup \{\, y_k \mid 1 \le k \le n \,\} \cup \{\, \psi_k, \epsilon_k \mid 1 \le k \le n-1 \,\},$$

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Let L_r, for $1 \le r \le n$, be Jucys-Murphy elements of $\mathscr{B}_n(\delta)$. For any finite dimensional $\mathscr{B}_n(\delta)$-module M, the eigenvalues of each L_r on M belongs to P. So M decomposes as the direct sum $M = \bigoplus_{\mathbf{i} \in P^n} M_{\mathbf{i}}$ of weight spaces

$$M_{\mathbf{i}} = \{\, v \in M \mid (L_r - i_r)^N v = 0 \text{ for all } r = 1, 2, \ldots, n \text{ and } N \gg 0 \,\}.$$

We deduce that there is a system $\{\, e(\mathbf{i}) \mid \mathbf{i} \in P^n \,\}$ of mutually orthogonal idempotents in $\mathscr{B}_n(\delta)$ such that $Me(\mathbf{i}) = M_{\mathbf{i}}$ for each finite dimensional module M, and $e(\mathbf{i}) \neq 0$ if and only if $\mathbf{i}$ is the residue sequence of some up-down tableaux.

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Define $I^n = \{\, \mathbf{i} \in P^n \mid \mathbf{i} \text{ is the residue sequence of some up-down tableaux} \,\}$.For an integer r with $1 \le r \le n$, define

$$y_r := \sum_{\mathbf{i} \in I^n} (L_r - i_r) e(\mathbf{i}) \in \mathscr{B}_n(\delta).$$

For any $\mathbf{i} \in I^n$, define $P_k(\mathbf{i})^{-1}$, $Q_k(\mathbf{i})^{-1}$ and $V_k(\mathbf{i})$ as elements generated by L_r's.

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$$e(\mathbf{i})\psi_k e(\mathbf{j}) := \begin{cases} 0, & \text{if } \mathbf{j} \neq \mathbf{i}{\cdot}s_k, \\ e(\mathbf{i})P_k(\mathbf{i})^{-1}(s_k - V_k(\mathbf{i}))Q_k(\mathbf{j})^{-1}e(\mathbf{j}), & \text{if } \mathbf{j} = \mathbf{i}{\cdot}s_k. \end{cases}$$

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$$\psi_k = \sum_{\mathbf{i}\in I^n}\sum_{\mathbf{j}\in I^n} e(\mathbf{i})\psi_k e(\mathbf{j}) \in \mathscr{B}_n(\delta), \qquad \epsilon_k = \sum_{\mathbf{i}\in I^n}\sum_{\mathbf{j}\in I^n} e(\mathbf{i})\epsilon_k e(\mathbf{j}) \in \mathscr{B}_n(\delta).$$

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Proposition

The elements

$$\{\, e(\mathbf{i}) \mid \mathbf{i} \in P^n \,\} \cup \{\, y_r \mid 1 \le k \le n \,\} \cup \{\, \psi_k, \epsilon_k \mid 1 \le k \le n-1 \,\}$$

generates $\mathscr{B}_n(\delta)$.

Theorem

The elements of $\mathscr{B}_n(\delta)$

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satisfy the relations of $\mathscr{G}_n(\delta)$.

The above Theorem tells us there exists a surjective homomorphism $\mathscr{G}_n(\delta) \longrightarrow \mathscr{B}_n(\delta)$ by sending

$$e(\mathbf{i}) \mapsto e(\mathbf{i}), \qquad y_r \mapsto y_r, \qquad \psi_k \mapsto \psi_k, \qquad \epsilon_k \mapsto \epsilon_k.$$

So the dimension of $\mathscr{G}_n(\delta)$ is bounded below by $(2n-1)!! = \dim \mathscr{B}_n(\delta)$, which forces $\dim \mathscr{G}_n(\delta) = (2n-1)!! = \dim \mathscr{B}_n(\delta)$ and the surjective homomorphism $\mathscr{G}_n(\delta) \longrightarrow \mathscr{B}_n(\delta)$ is actually an isomorphism.

Theorem

Suppose R is a field of characteristic 0 and $\delta \in R$. Then $\mathscr{B}_n(\delta) \cong \mathscr{G}_n(\delta)$. Moreover, $\mathscr{B}_n(\delta)$ is a graded cellular algebra with a graded cellular basis

$$\{ \psi_{\mathsf{st}} \mid (\lambda, f) \in \widehat{B}_n, \mathsf{s}, \mathsf{t} \in \mathscr{T}_n^{ud}(\lambda) \}.$$

Further research

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For $\mathscr{B}_n(\delta)$ over a field with positive characteristic, or more generally, for cyclotomic Nazarov-Wenzl algebras $\mathscr{W}_{r,n}(\mathbf{u})$ over arbitrary field, we should be able to construct a $\mathbb{Z}$-graded algebra similar to $\mathscr{G}_n(\delta)$ isomorphic to $\mathscr{W}_{r,n}(\mathbf{u})$.

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The algebras are generated with elements

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We are also able to construct a set of homogeneous elements

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Moreover, we are able to construct an affine version of the algebra and a weight such that the cyclotomic quotient of the affine algebra is isomorphic to $\mathscr{W}_{r,n}(\mathbf{u})$. The details are still in preparation.

Thank you!