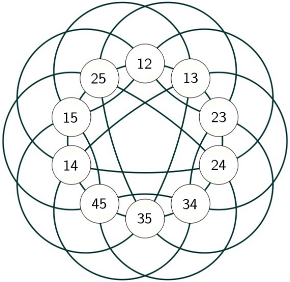
Idempotent generation in partition monoids

James East
University of Western Sydney



Workshop on Diagram Algebras
8–12 Sept 2014
Universität Stuttgart

University of Western Sydney

ALLY

Joint work with Bob Gray (and others)

Shona says Hi . . .

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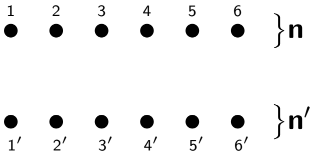

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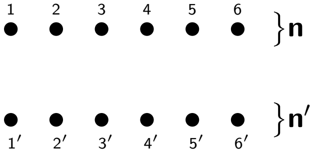

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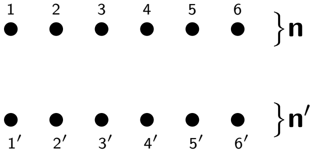

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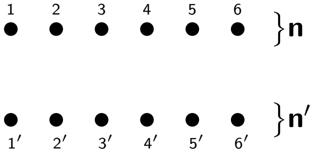

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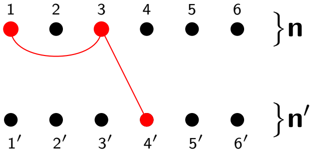

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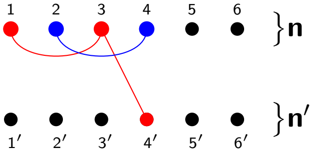

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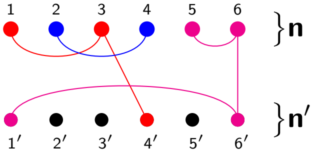

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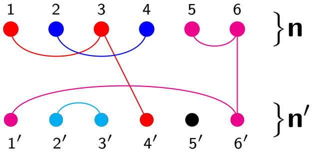

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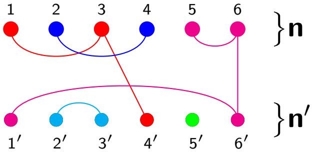

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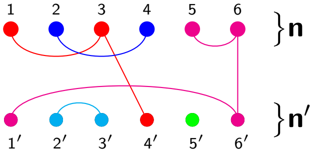

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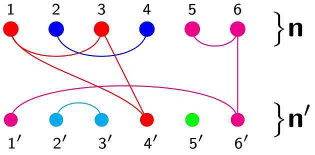

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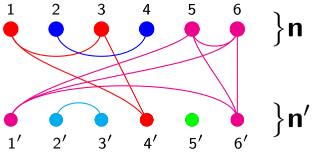

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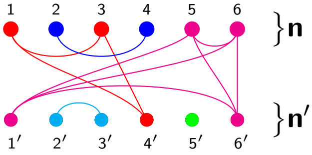

- Note: $\mathcal{P}_n$ is the basis of the partition algebra $\mathcal{P}_n^\delta$.

Product in $\mathcal{P}_n$

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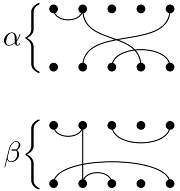
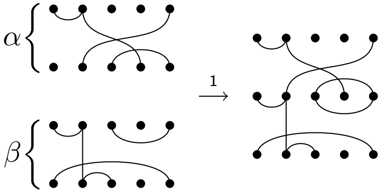
Let $\alpha, \beta \in \mathcal{P}_n$.



Let $\alpha, \beta \in \mathcal{P}_n$. To calculate $\alpha\beta$:

(1) connect bottom of α to top of β,



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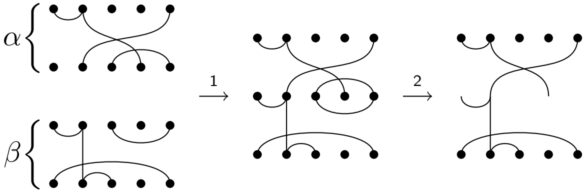

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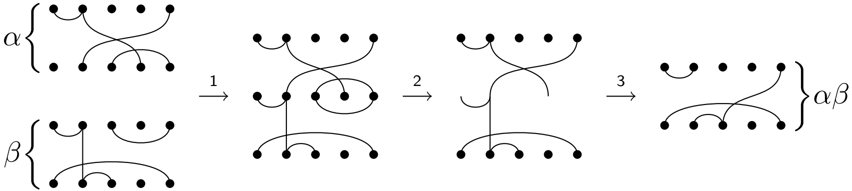

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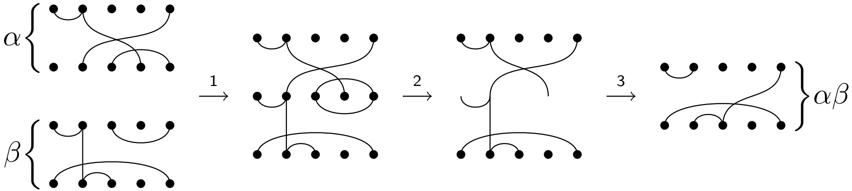

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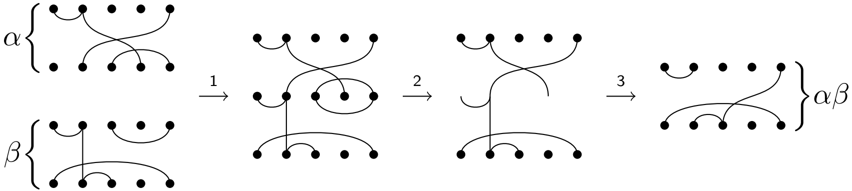

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- Note: usual multiplication in partition algebra $\mathcal{P}_n^\delta$ with $\delta = 1$.

Submonoids of $\mathcal{P}_n$

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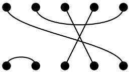
- $\mathcal{B}_n = \{\alpha \in \mathcal{P}_n : |A| = 2 \ (\forall A \in \alpha)\}$ — Brauer monoid

 $\in \mathcal{B}_5$

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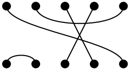

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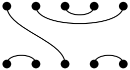

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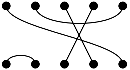 $\in \mathcal{B}_5$

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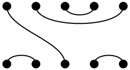
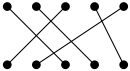
 $\in \mathcal{TL}_5$

- $\mathcal{S}_n = \{\alpha \in \mathcal{B}_n : |A \cap \mathbf{n}| = |A \cap \mathbf{n}'| = 1 \ (\forall A \in \alpha)\}$ — symmetric group

 $\in \mathcal{S}_5$

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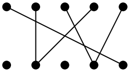
- $\mathcal{T}_n = \{\alpha \in \mathcal{P}_n : |A \cap \mathbf{n}'| = 1 \ (\forall A \in \alpha)\}$

— full transformation semigroup

 $\in \mathcal{T}_5$

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  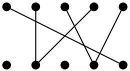 $\in \mathcal{T}_5$

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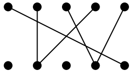 $\in \mathcal{T}_5$

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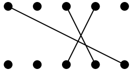
  — symmetric inverse monoid (aka rook monoid)

 $\in \mathcal{I}_5$

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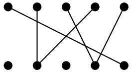
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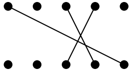
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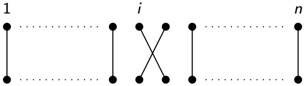
- In many ways, $\mathcal{P}_n$ is just like a transformation semigroup.

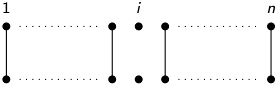
Generators

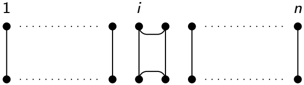
Proposition

There is a factorization $\mathcal{P}_n = \mathcal{T}_n \, \mathcal{I}_n \, \mathcal{T}_n^*$. Consequently,

$$\mathcal{P}_n = \langle s_1, \ldots, s_{n-1}, e_1, \ldots, e_n, t_1, \ldots, t_{n-1} \rangle.$$

$s_i =$ 

$e_i =$ 

$t_i =$ 

Presentations

Theorem (Halverson and Ram, 2005; E, 2011)

The partition monoid $\mathcal{P}_n$ has presentation

$$\mathcal{P}_n \cong \langle s_1, \ldots, s_{n-1}, e_1, \ldots, e_n, t_1, \ldots, t_{n-1} : \text{(R1—R16)} \rangle,$$

where

(R1) $s_i^2 = 1$

(R2) $s_i s_j = s_j s_i$

(R3) $s_i s_j s_i = s_j s_i s_j$

(R4) $e_i^2 = e_i$

(R5) $e_i e_j = e_j e_i$

(R6) $s_i e_j = e_j s_i$

(R7) $s_i e_i = e_{i+1} s_i$

(R8) $e_i e_{i+1} s_i = e_i e_{i+1}$

(R9) $t_i^2 = t_i$

(R10) $t_i t_j = t_j t_i$

(R11) $s_i t_j = t_j s_i$

(R12) $s_i s_j t_i = t_j s_i s_j$

(R13) $t_i s_i = s_i t_i = t_i$

(R14) $t_i e_j = e_j t_i$

(R15) $t_i e_j t_i = t_i$

(R16) $e_j t_i e_j = e_j$.

Theorem (Halverson and Ram, 2005; E, 2011)

The partition algebra $\mathcal{P}_n^\delta$ has presentation

$$\mathcal{P}_n^\delta \cong \langle s_1, \ldots, s_{n-1}, e_1, \ldots, e_n, t_1, \ldots, t_{n-1} : \text{(R1—R16)} \rangle,$$

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(R2) $s_i s_j = s_j s_i$

(R3) $s_i s_j s_i = s_j s_i s_j$

(R4) $e_i^2 = \delta e_i$

(R5) $e_i e_j = e_j e_i$

(R6) $s_i e_j = e_j s_i$

(R7) $s_i e_i = e_{i+1} s_i$

(R8) $e_i e_{i+1} s_i = e_i e_{i+1}$

(R9) $t_i^2 = t_i$

(R10) $t_i t_j = t_j t_i$

(R11) $s_i t_j = t_j s_i$

(R12) $s_i s_j t_i = t_j s_i s_j$

(R13) $t_i s_i = s_i t_i = t_i$

(R14) $t_i e_j = e_j t_i$

(R15) $t_i e_j t_i = t_i$

(R16) $e_j t_i e_j = e_j$.

Presentations

Theorem (E, 2011)

The singular partition monoid $\mathcal{P}_n \setminus \mathcal{S}_n$ has presentation

$$\mathcal{P}_n \setminus \mathcal{S}_n \cong \langle e_1, \ldots, e_n, t_{ij} \ (1 \leq i < j \leq n) : (\text{R1—R10}) \rangle,$$

where

(R1) $e_i^2 = e_i$

(R2) $e_i e_j = e_j e_i$

(R3) $t_{ij}^2 = t_{ij}$

(R4) $t_{ij} t_{kl} = t_{kl} t_{ij}$

(R5) $t_{ij} e_k t_{ij} = t_{ij}$

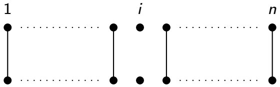
(R6) $e_k t_{ij} e_k = e_k$

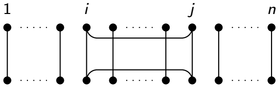
(R7) $t_{ij} e_k = e_k t_{ij}$

(R8) $t_{ij} t_{jk} = t_{jk} t_{ki} = t_{ki} t_{ij}$

(R9) $e_k t_{ki} e_i t_{ij} e_j t_{jk} e_k = e_k t_{kj} e_j t_{ji} e_i t_{ik} e_k$

(R10) $e_k t_{ki} e_i t_{ij} e_j t_{jl} e_l t_{lk} e_k = e_k t_{kl} e_l t_{li} e_i t_{ij} e_j t_{jk} e_k$.

$e_i =$ 

$t_{ij} =$ 

Presentations

Theorem (Kudryavtseva and Mazorchuk, 2006; see also Birman-Wenzl and Barcelo-Ram)

The Brauer monoid $\mathcal{B}_n$ has presentation

$$\mathcal{B}_n \cong \langle s_1, \ldots, s_{n-1}, u_1, \ldots, u_{n-1} : \text{(R1—R10)} \rangle,$$

where

(R1) $s_i^2 = 1$

(R2) $s_i s_j = s_j s_i$

(R3) $s_i s_j s_i = s_j s_i s_j$

(R4) $u_i^2 = u_i$

(R5) $u_i u_j = u_j u_i$

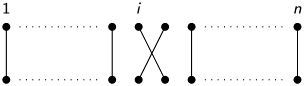
(R6) $s_i u_j = u_j s_i$

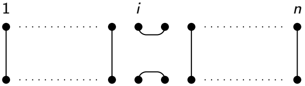
(R7) $s_i u_i = u_i s_i = u_i$

(R8) $u_i u_j u_i = u_i$

(R9) $s_i u_j u_i = s_j u_i$

(R10) $u_i u_j s_i = u_i s_j$.

$s_i =$ 

$u_i =$ 

Theorem (Maltcev and Mazorchuk, 2007)

The singular Brauer monoid $\mathcal{B}_n \setminus \mathcal{S}_n$ has presentation

$$\mathcal{B}_n \setminus \mathcal{S}_n \cong \langle u_{ij} \ (1 \leq i < j \leq n) : \text{(R1—R6)} \rangle,$$

where

(R1) $u_{ij}^2 = u_{ij}$

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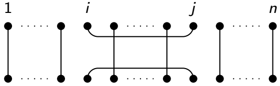
$u_{ij} =$ 

Presentations

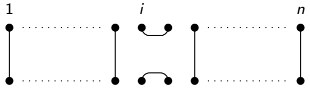
Theorem (Borisavljević, Došen, Petrić, 2002; see also Jones, Kauffman, etc)

The (singular) Temperley-Lieb monoid $\mathcal{TL}_n$ has presentation

$$\mathcal{TL}_n \cong \langle u_1, \ldots, u_{n-1} : \text{(R1—R3)} \rangle,$$

where

(R1) $u_i^2 = u_i$ (R2) $u_i u_j = u_j u_i$ (R3) $u_i u_j u_i = u_i$.

$u_i =$ 

Idempotent generation — questions

So the singular parts of $\mathcal{P}_n$, $\mathcal{B}_n$, $\mathcal{TL}_n$ are idempotent generated . . .

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- What about infinite partition monoids $\mathcal{P}_X$?
- Same questions for $\mathcal{B}_n$ and $\mathcal{TL}_n$ ...

Number of idempotents — $\mathcal{B}_n$

Theorem (Dolinka, E, Evangelou, FitzGerald, Ham, Hyde, Loughlin, 2014)

The number of idempotents in the Brauer monoid $\mathcal{B}_n$ is equal to

$$e_n = \sum_{\mu \vdash n} \frac{n!}{\mu_1! \cdots \mu_n! \cdot 2^{\mu_2} \cdots (2k)^{\mu_{2k}}}$$

where $k = \lfloor n/2 \rfloor$ — i.e., $n = 2k$ or $2k+1$.

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where $k = \lfloor n/2 \rfloor$ — i.e., $n = 2k$ or $2k+1$. The numbers e_n satisfy the recurrence

- $e_0 = 1$,
- $e_n = a_1 e_{n-1} + a_2 e_{n-2} + \cdots + a_n e_0$

where $a_{2i} = \binom{n-1}{2i-1}(2i-1)!$ and $a_{2i+1} = \binom{n-1}{2i}(2i+1)!$.

Number of idempotents — $\mathcal{B}_n^\delta$

Theorem (DEEFHHL, 2014)

The number of idempotent basis elements in the Brauer algebra $\mathcal{B}_n^\delta$ is equal to

$$\sum_\mu \frac{n!}{\mu_1!\mu_3!\cdots\mu_{2k+1}!},$$

where

- $k = \lfloor \frac{n-1}{2} \rfloor$,
- the sum is over all integer partitions $\mu \vdash n$ with only odd parts,
- δ is not a root of unity.

Number of idempotents — $\mathcal{P}_n$

Theorem (DEEFHHL, 2014)

The number of idempotents in the partition monoid $\mathcal{P}_n$ is equal to

$$n! \cdot \sum_{\mu \vdash n} \frac{c(1)^{\mu_1} \cdots c(n)^{\mu_n}}{\mu_1! \cdots \mu_n! \cdot (1!)^{\mu_1} \cdots (n!)^{\mu_n}},$$

where

- $c(k) = \sum_{r,s=1}^{k} (1+rs)c(k,r,s)$, and
- $$\begin{aligned} c(k,r,1) &= S(k,r) \\ c(k,1,s) &= S(k,s) \\ c(k,r,s) &= s \cdot c(k-1,r-1,s) + r \cdot c(k-1,r,s-1) + rs \cdot c(k-1,r,s) \\ &\quad + \sum_{m=1}^{k-2} \binom{k-2}{m} \sum_{a=1}^{r-1} \sum_{b=1}^{s-1} (a(s-b)+b(r-a))c(m,a,b)c(k-m-1,r-a,s-b) \\ &\qquad \text{if } r,s \geq 2. \end{aligned}$$

Number of idempotents — $\mathcal{P}_n^\delta$

Theorem (DEEFHHL, 2014)

The number of idempotent basis elements in the partition algebra $\mathcal{P}_n^\delta$ is equal to

$$n! \cdot \sum_{\mu \vdash n} \frac{c'(1)^{\mu_1} \cdots c'(n)^{\mu_n}}{\mu_1! \cdots \mu_n! \cdot (1!)^{\mu_1} \cdots (n!)^{\mu_n}},$$

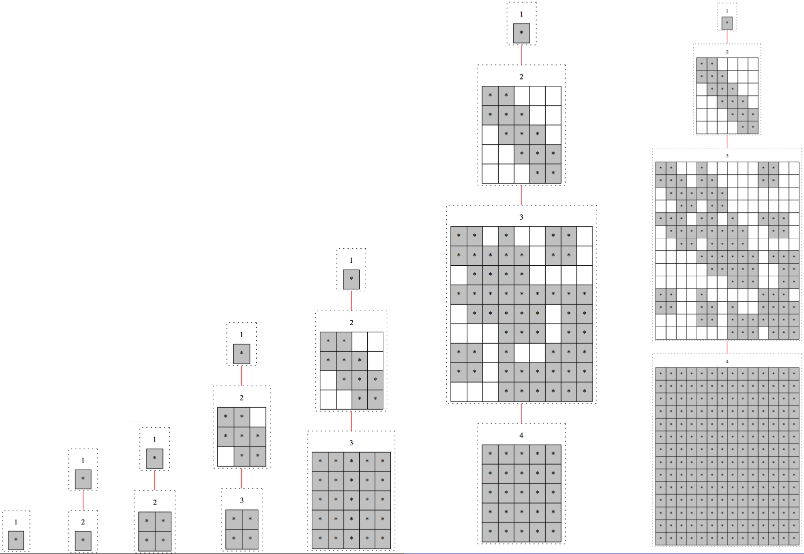
where

- $c'(k) = \displaystyle\sum_{r,s=1}^{k} rs \cdot c(k,r,s)$, and
- δ is not a root of unity.

Less algebra, more diagrams. . .

Number of idempotents — $\mathcal{TL}_1$–$\mathcal{TL}_7$ (GAP)

The number of idempotents in $\mathcal{TL}_n$ is currently unknown.



Number of idempotents — $\mathcal{TL}_8$–$\mathcal{TL}_{11}$ (GAP)

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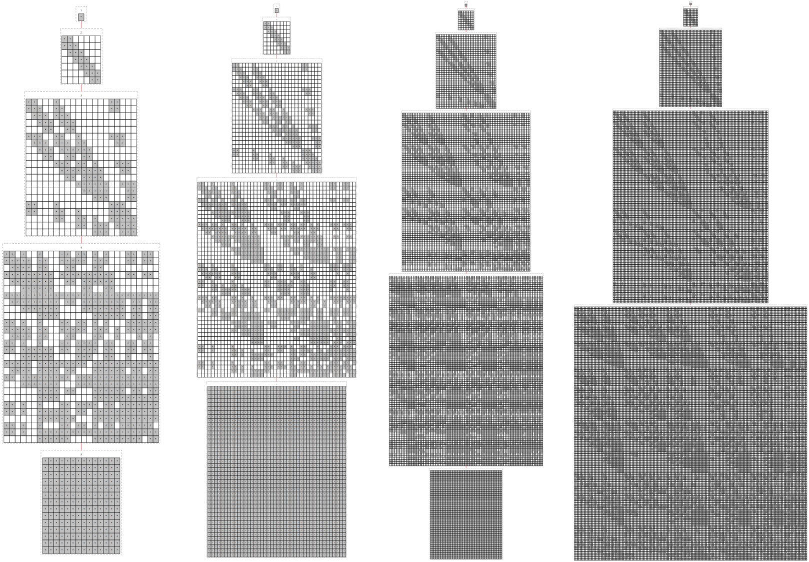
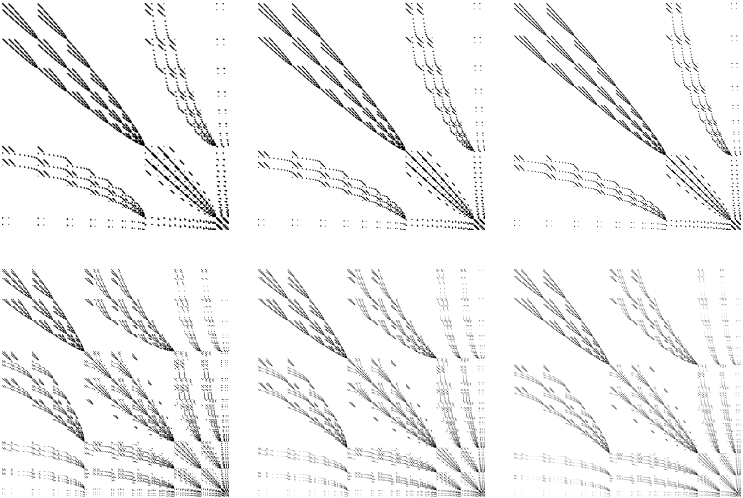


Number of idempotents — inside $\mathcal{TL}_{15}$–$\mathcal{TL}_{17}$ (GAP)

The number of idempotents in $\mathcal{TL}_n$ is currently unknown.



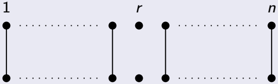
Thanks to Attila Egri-Nagy for these ...

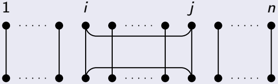
Rank and idempotent rank — $\mathcal{P}_n \setminus \mathcal{S}_n$

Rank and idempotent rank — $\mathcal{P}_n \setminus \mathcal{S}_n$

Theorem (E, 2011)

- $\mathcal{P}_n \setminus \mathcal{S}_n$ is idempotent generated.
- $\mathcal{P}_n \setminus \mathcal{S}_n = \langle e_1, \ldots, e_n, t_{ij} \ (1 \leq i < j \leq n) \rangle$.

$e_r =$ 

$t_{ij} =$ 

Rank and idempotent rank — $\mathcal{P}_n \setminus \mathcal{S}_n$

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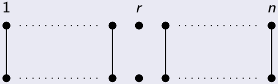
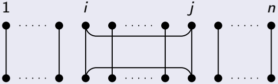
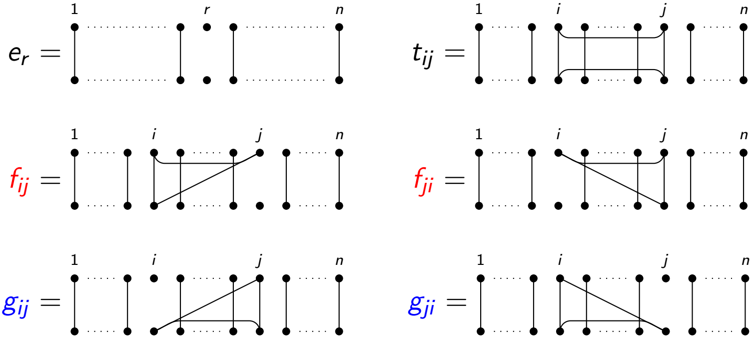
$e_r =$ 

$t_{ij} =$ 

- $\text{rank}(\mathcal{P}_n \setminus \mathcal{S}_n) = \text{idrank}(\mathcal{P}_n \setminus \mathcal{S}_n) = n + \binom{n}{2} = \binom{n+1}{2} = \frac{n(n+1)}{2}$.

Any minimal idempotent generating set for $\mathcal{P}_n \setminus \mathcal{S}_n$ is a subset of

$$\{e_r : 1 \leq r \leq n\} \cup \{t_{ij}, f_{ij}, f_{ji}, g_{ij}, g_{ji} : 1 \leq i < j \leq n\}.$$



Minimal idempotent generating sets — $\mathcal{P}_n \setminus \mathcal{S}_n$

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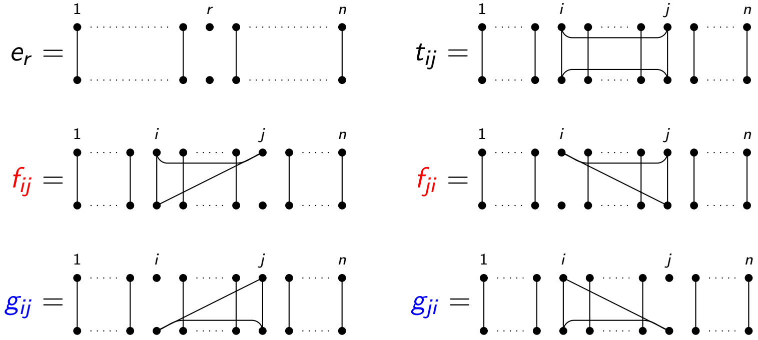


To see *which* subsets generate $\mathcal{P}_n \setminus \mathcal{S}_n$, we create a graph. . .

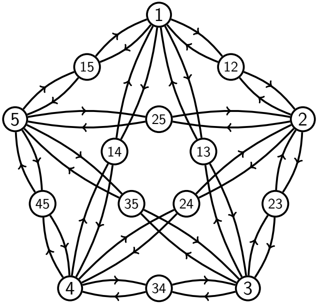
Minimal idempotent generating sets — $\mathcal{P}_n \setminus \mathcal{S}_n$

Let Γ_n be the di-graph with vertex set

$$V(\Gamma_n) = \{A \subseteq \mathbf{n} : |A| = 1 \text{ or } |A| = 2\}$$

and edge set

$$E(\Gamma_n) = \{(A, B) : A \subseteq B \text{ or } B \subseteq A\}.$$



Γ_5 (with loops omitted)

For only \$59.95. . .

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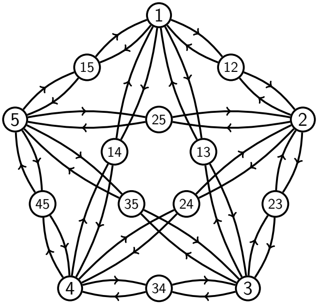

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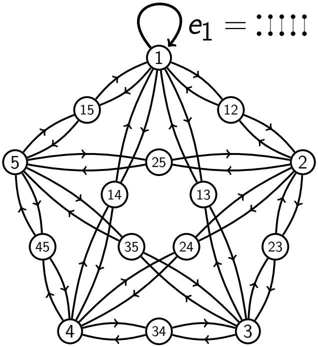

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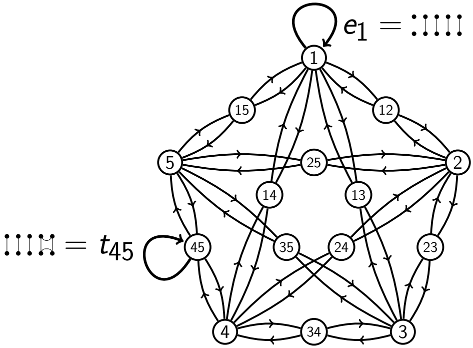

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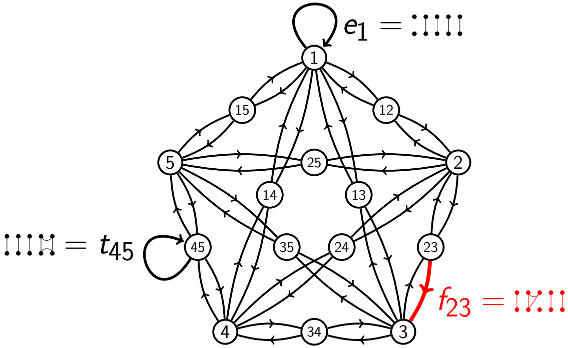

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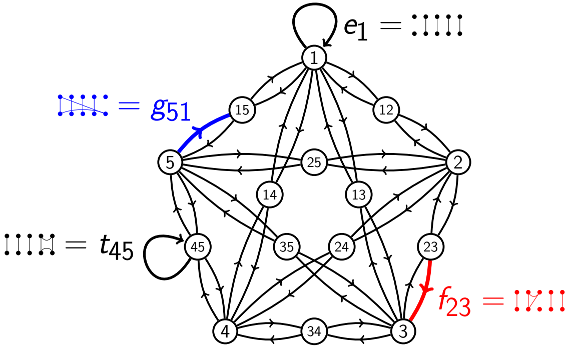

Γ_5 (with loops omitted)

Minimal idempotent generating sets — $\mathcal{P}_n \setminus \mathcal{S}_n$

A subgraph H of a di-graph G is a permutation subgraph if $V(H) = V(G)$ and the edges of H induce a permutation of $V(G)$.

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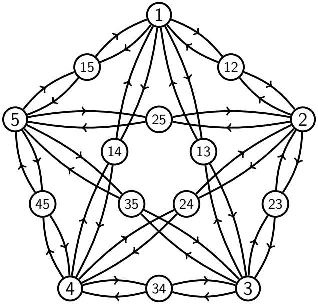

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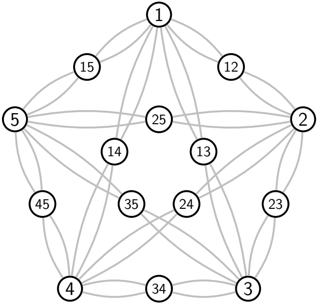

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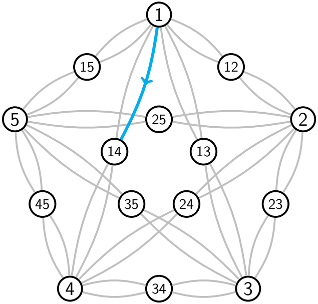

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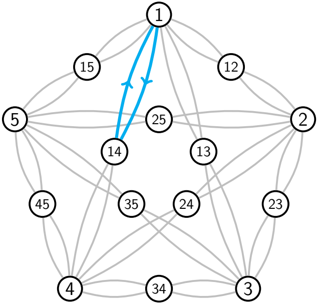

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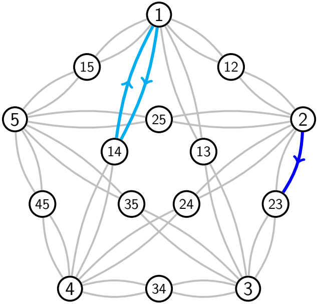

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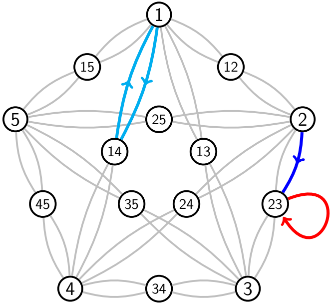

Minimal idempotent generating sets — $\mathcal{P}_n \setminus \mathcal{S}_n$

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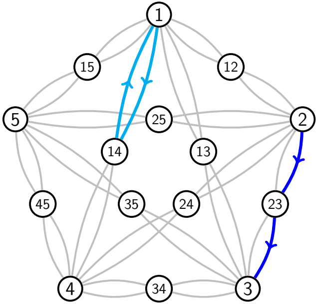

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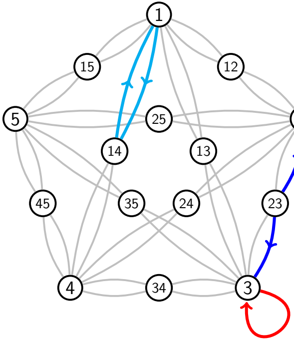

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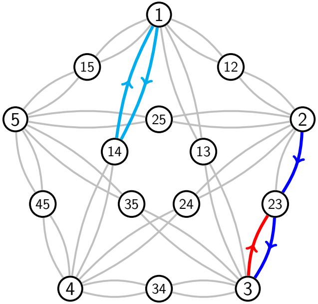

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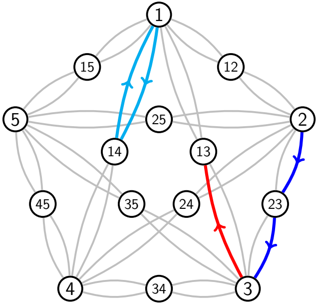

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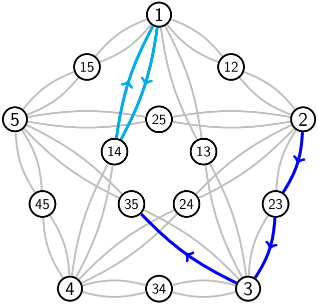

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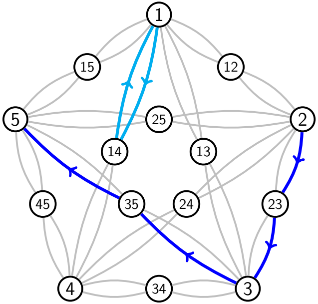

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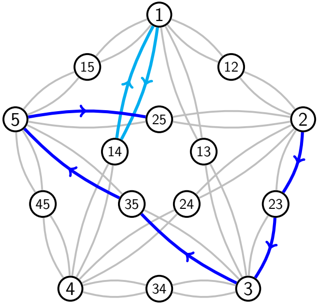

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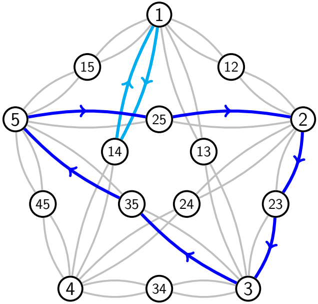

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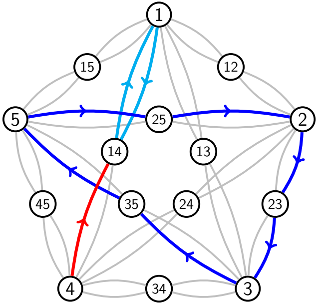

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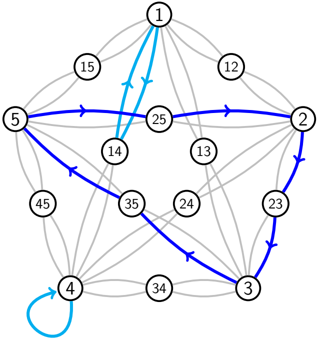

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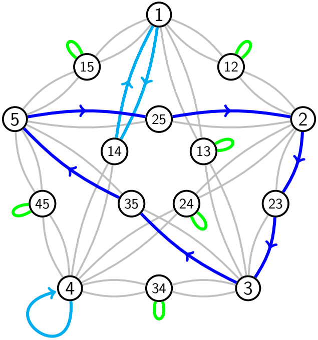

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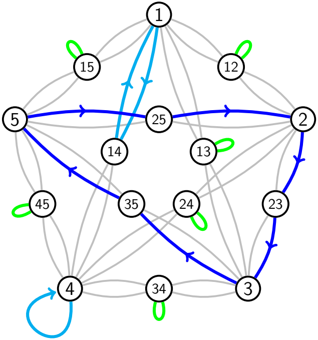

A permutation subgraph of Γ_n is determined by:

- a permutation of a subset A of $\mathbf{n}$ with no fixed points or 2-cycles ($A = \{2, 3, 5\}$, $2 \mapsto 3 \mapsto 5 \mapsto 2$), and
- a function $\mathbf{n} \setminus A \to \mathbf{n}$ with no 2-cycles ($1 \mapsto 4$, $4 \mapsto 4$).

Minimal idempotent generating sets — $\mathcal{P}_n \setminus \mathcal{S}_n$

Theorem (E+Gray, 2014)

The minimal idempotent generating sets of $\mathcal{P}_n \setminus \mathcal{S}_n$ are in one-one correspondence with the permutation subgraphs of Γ_n.

The number of minimal idempotent generating sets of $\mathcal{P}_n \setminus \mathcal{S}_n$ is equal to

$$\sum_{k=0}^{n} \binom{n}{k} a_k b_{n,n-k},$$

where $a_0 = 1$, $a_1 = a_2 = 0$, $a_{k+1} = ka_k + k(k-1)a_{k-2}$, and

$$b_{n,k} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^i \binom{k}{2i} (2i-1)!!n^{k-2i}.$$

n	0	1	2	3	4	5	6	7	$\cdots$
	1	1	3	20	201	2604	40915	754368	$\cdots$

Ideals — $\mathcal{P}_n \setminus \mathcal{S}_n$

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The ideals of $\mathcal{P}_n$ are

$$I_r = \{\alpha \in \mathcal{P}_n : \alpha \text{ has} \leq r \text{ transverse blocks}\}$$

for $0 \leq r \leq n$.

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Theorem (E+Gray, 2014)

If $0 \leq r \leq n-1$, then I_r is idempotent generated, and

$$\text{rank}(I_r) = \text{idrank}(I_r) = \sum_{j=r}^{n} S(n,j)\binom{j}{r}.$$

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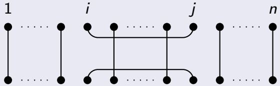
The idempotent generating sets of this size have not been classified/enumerated (for $1 \leq r \leq n-2$).

Rank and idempotent rank — $\mathcal{B}_n \setminus \mathcal{S}_n$

Rank and idempotent rank — $\mathcal{B}_n \setminus \mathcal{S}_n$

Theorem (Maltcev and Mazorchuk, 2007)

- $\mathcal{B}_n \setminus \mathcal{S}_n$ is idempotent generated.
- $\mathcal{B}_n \setminus \mathcal{S}_n = \langle u_{ij} \ (1 \leq i < j \leq n)\rangle$.

$u_{ij} =$ 

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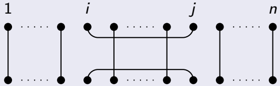
$u_{ij} =$ 

- $\text{rank}(\mathcal{B}_n \setminus \mathcal{S}_n) = \text{idrank}(\mathcal{B}_n \setminus \mathcal{S}_n) = \binom{n}{2} = \frac{n(n-1)}{2}$.

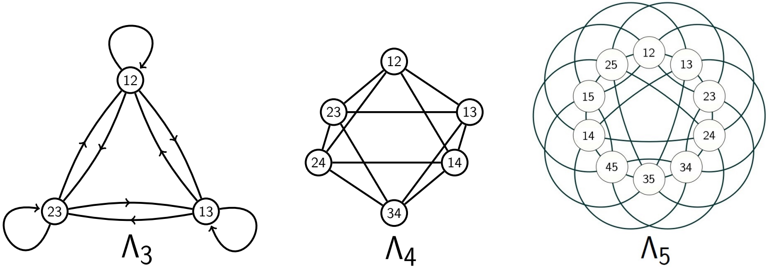
Minimal idempotent generating sets — $\mathcal{B}_n \setminus \mathcal{S}_n$

Let Λ_n be the di-graph with vertex set

$$V(\Lambda_n) = \{A \subseteq \mathbf{n} : |A| = 2\}$$

and edge set

$$E(\Lambda_n) = \{(A, B) : A \cap B \neq \emptyset\}.$$



Minimal idempotent generating sets — $\mathcal{B}_n \setminus \mathcal{S}_n$

Theorem (E+Gray, 2014)

The minimal idempotent generating sets of $\mathcal{B}_n \setminus \mathcal{S}_n$ are in one-one correspondence with the permutation subgraphs of Λ_n.

No formula is known for the number of minimal idempotent generating sets of $\mathcal{B}_n \setminus \mathcal{S}_n$ (yet). Very hard!

n	0	1	2	3	4	5	6	7
	1	1	1	6	265	126,140	855,966,441	????
	1	1	1	2	12	288	34,560	24,883,200

There are (way) more than $(n-1)! \cdot (n-2)! \cdots 3! \cdot 2! \cdot 1!$.

- Thanks to James Mitchell for $n = 5, 6$ (GAP).

Ideals — $\mathcal{B}_n \setminus \mathcal{S}_n$

The ideals of $\mathcal{B}_n$ are

$$I_r = \{\alpha \in \mathcal{B}_n : \alpha \text{ has} \leq r \text{ transverse blocks}\}$$

for $0 \leq r = n - 2k \leq n$.

Theorem (E+Gray, 2014)

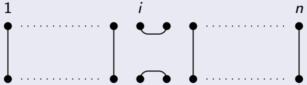
If $0 \leq r = n - 2k \leq n - 2$, then I_r is idempotent generated and

$$\text{rank}(I_r) = \text{idrank}(I_r) = \binom{n}{2k}(2k-1)!! = \frac{n!}{2^k k! r!}.$$

Rank and idempotent rank — $\mathcal{TL}_n$

Theorem (Borisavljević, Došen, Petrić, 2002, etc)

- $\mathcal{TL}_n$ is idempotent generated.
- $\mathcal{TL}_n = \langle u_1, \ldots, u_{n-1} \rangle$.

$u_i =$ 

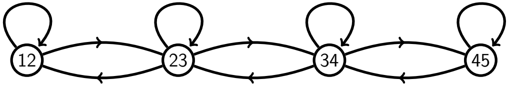
- $\text{rank}(\mathcal{TL}_n) = \text{idrank}(\mathcal{TL}_n) = n - 1$.

Let Ξ_n be the di-graph with vertex set

$$V(\Xi_n) = \big\{\{1,2\},\{2,3\},\ldots,\{n-1,n\}\big\}$$

and edge set

$$E(\Xi_n) = \{(A,B) : A \cap B \neq \emptyset\}.$$



Ξ_5

Minimal idempotent generating sets — $\mathcal{TL}_n$

Theorem (E+Gray, 2014)

The minimal idempotent generating sets of $\mathcal{TL}_n$ are in one-one correspondence with the permutation subgraphs of Ξ_n.

The number of minimal idempotent generating sets of $\mathcal{TL}_n$ is F_n, the nth Fibonacci number.

n	0	1	2	3	4	5	6	7	$\cdots$
	1	1	1	2	3	5	8	13	$\cdots$

Ideals — $\mathcal{TL}_n$

The ideals of $\mathcal{TL}_n$ are

$$I_r = \{\alpha \in \mathcal{TL}_n : \alpha \text{ has } \leq r \text{ transverse blocks}\}$$

for $0 \leq r = n - 2k \leq n$.

Theorem (E+Gray, 2014)

If $0 \leq r = n - 2k \leq n - 2$, then I_r is idempotent generated and

$$\text{rank}(I_r) = \text{idrank}(I_r) = \frac{r+1}{n+1}\binom{n+1}{k}.$$

Values of $\text{rank}(I_r) = \text{idrank}(I_r)$:

$n \setminus r$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1		1									
2	1		1								
3		2		1							
4	2		3		1						
5		5		4		1					
6	5		9		5		1				
7		14		14		6		1			
8	14		28		20		7		1		
9		42		48		27		8		1	
10	42		90		75		35		9		1

I have more problems but I should stop now. . .

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...unless I have a few minutes to spare...

Infinite partition monoids — $\mathcal{P}_X$

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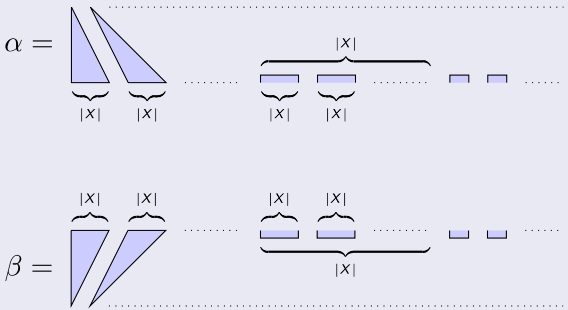
Theorem

$\mathcal{P}_X = \langle \mathcal{S}_X, \alpha, \beta \rangle$ where



Proof: Let $\gamma \in \mathcal{P}_X$.

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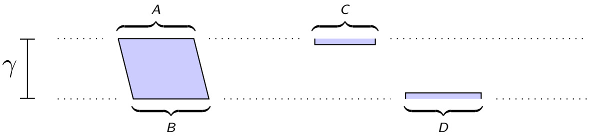

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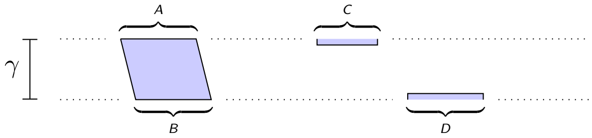
Proof: Let $\gamma \in \mathcal{P}_X$. We'll show that $\gamma = \alpha\pi\beta$ for some $\pi \in \mathcal{S}_X$.



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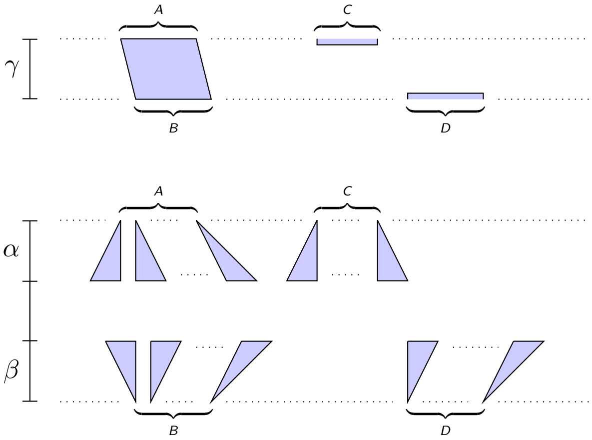

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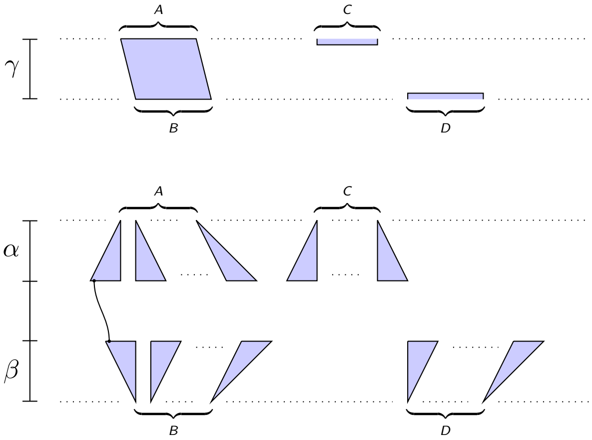

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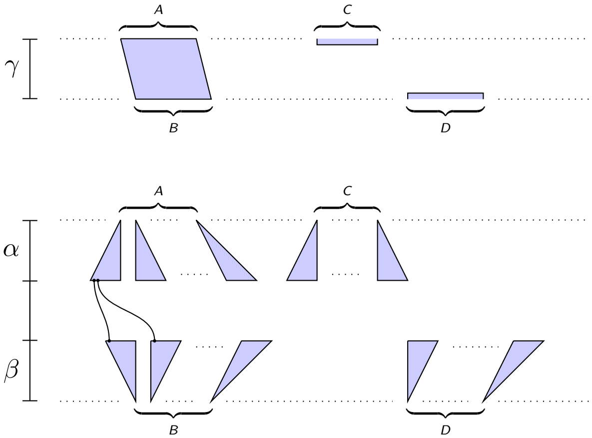

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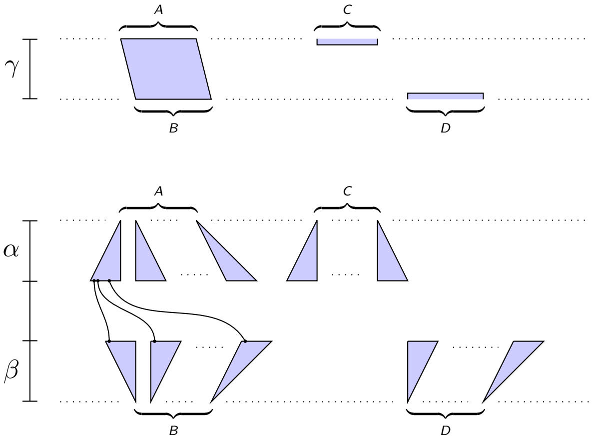

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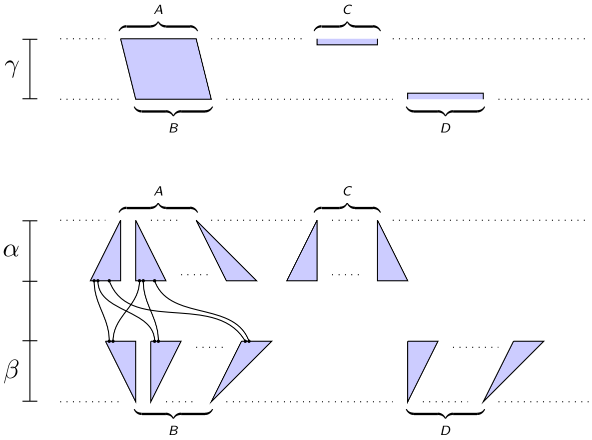

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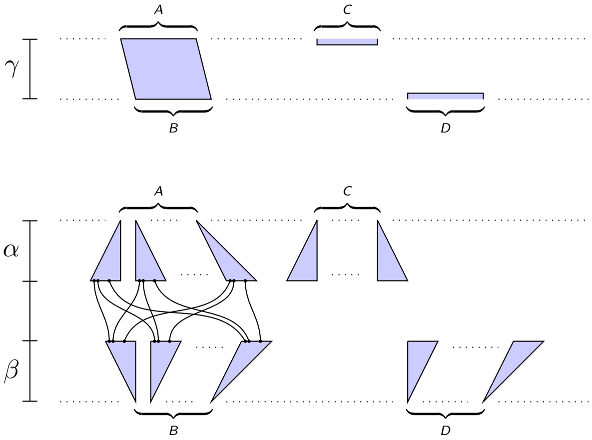

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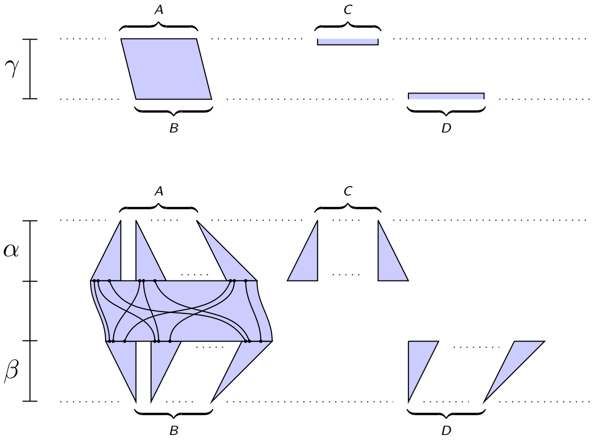

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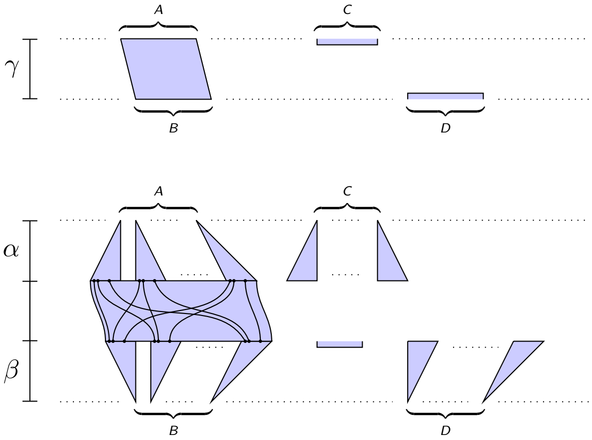

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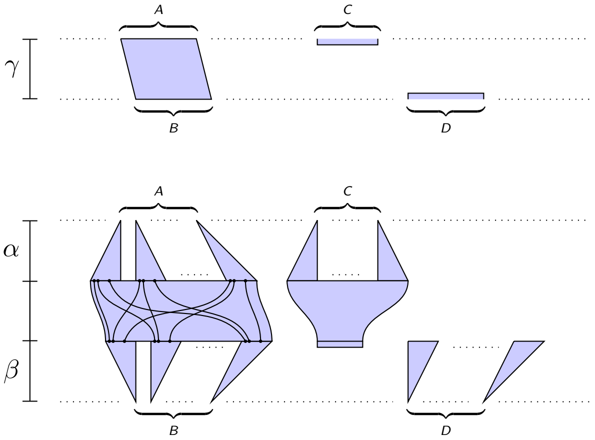

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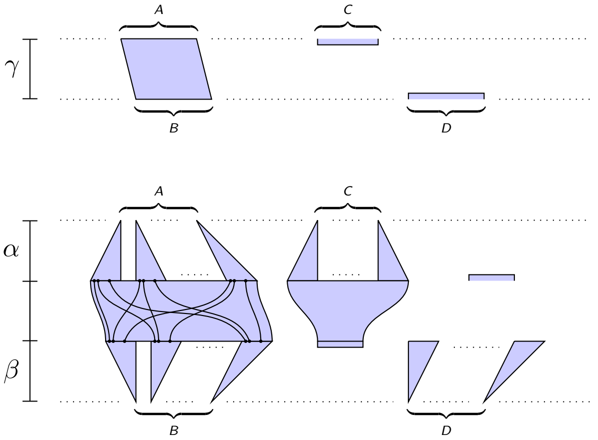

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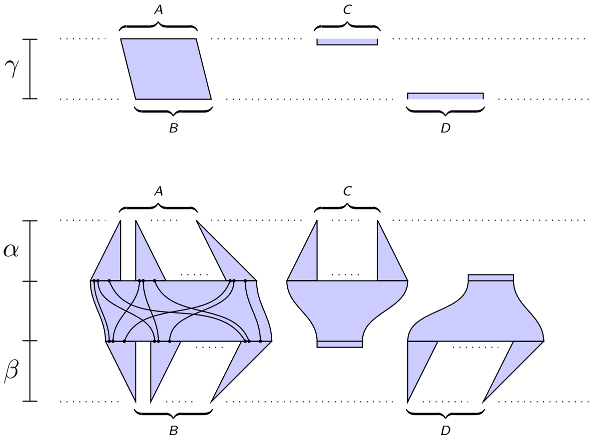

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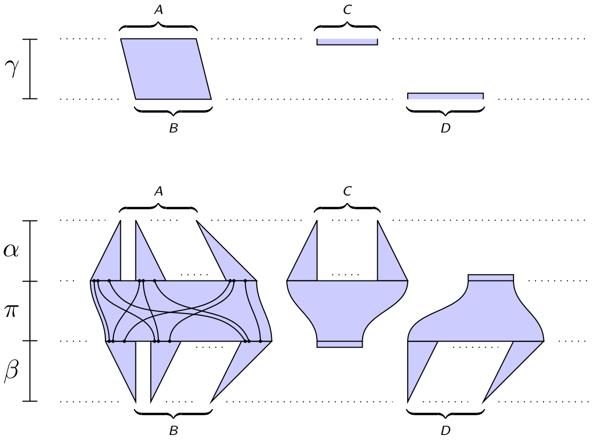

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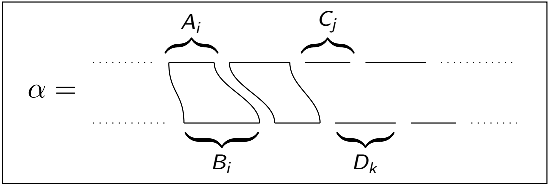

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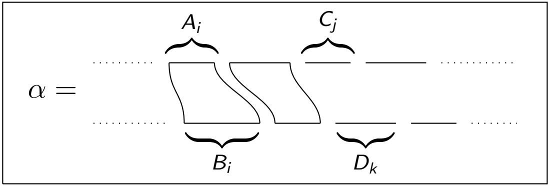

- Write $\alpha = \left(\begin{array}{c|c} A_i & C_j \\ \hline B_i & D_k \end{array} \right)_{i \in I,\ j \in J,\ k \in K}$.

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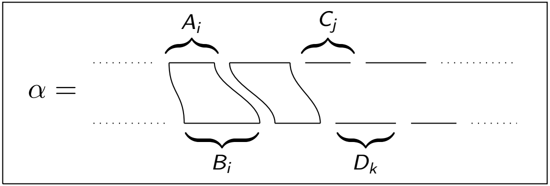

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- Define:
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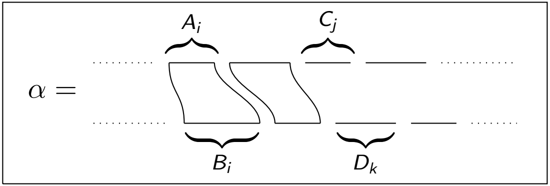

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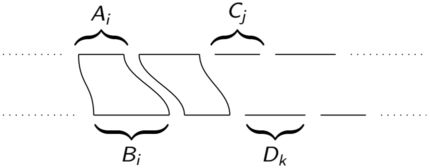
- Define:
  - $\text{def}(\alpha) = \sum_{j\in J} |C_j|$, $\text{codef}(\alpha) = \sum_{k\in K} |D_k|$,

Let $\alpha \in \mathcal{P}_X$.

$\alpha =$



- Write $\alpha = \begin{pmatrix} A_i & C_j \\ B_i & D_k \end{pmatrix}_{i\in I,\ j\in J,\ k\in K}$.
- Define:
  - $\mathrm{def}(\alpha) = \sum_{j\in J} |C_j|, \quad \mathrm{codef}(\alpha) = \sum_{k\in K} |D_k|,$
  - $\mathrm{col}(\alpha) = \sum_{i\in I} \big(|A_i| - 1\big)$

Infinite partition monoids — $\mathcal{P}_X$

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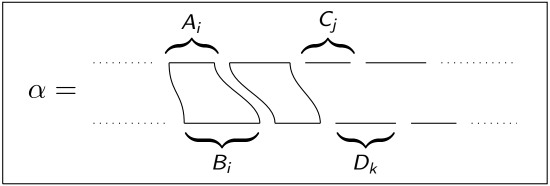

- Write $\alpha = \begin{pmatrix} A_i & C_j \\ B_i & D_k \end{pmatrix}_{i\in I,\ j\in J,\ k\in K}$.
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Infinite partition monoids — $\mathcal{P}_X$

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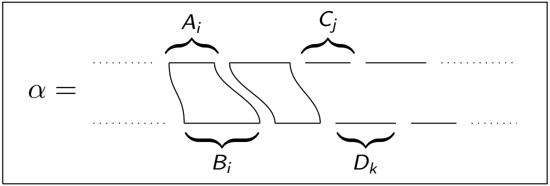

- Write $\alpha = \begin{pmatrix} A_i & C_j \\ B_i & D_k \end{pmatrix}_{i \in I,\ j \in J,\ k \in K}$.
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  - $\text{def}(\alpha) = \sum_{j \in J} |C_j|$, $\text{codef}(\alpha) = \sum_{k \in K} |D_k|$,
  - $\text{col}(\alpha) = \sum_{i \in I} \big(|A_i| - 1\big)$, $\text{cocol}(\alpha) = \sum_{i \in I} \big(|B_i| - 1\big)$,
  - $\text{sh}(\alpha) = \#\{i \in I : A_i \cap B_i = \emptyset\}$.

Infinite partition monoids — $\mathcal{P}_X$

Theorem (E+FitzGerald, 2012)

If X is infinite, then

$$\langle E(\mathcal{P}_X)\rangle = \{1\} \cup (\mathcal{P}_X^{\mathrm{fin}} \setminus \mathcal{S}_X^{\mathrm{fin}}) \cup \left\{ \alpha \in \mathcal{P}_X : \begin{array}{c} \mathrm{col}(\alpha) + \mathrm{def}(\alpha) = \mathrm{cocol}(\alpha) + \mathrm{codef}(\alpha) \\ \geq \max(\mathrm{sh}(\alpha), \aleph_0) \end{array} \right\}.$$

Infinite partition monoids — $\mathcal{P}_X$

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Theorem (E+FitzGerald, 2012)

For any X (finite or infinite),

$$\langle \mathcal{S}_X \cup E(\mathcal{P}_X)\rangle = \{\alpha \in \mathcal{P}_X : \text{col}(\alpha)+\text{def}(\alpha) = \text{cocol}(\alpha)+\text{codef}(\alpha)\}.$$

Thanks for having me in Stuttgart!