

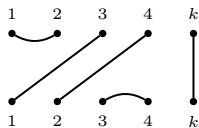
# Representation theory of the two-boundary Temperley-Lieb algebra

Zajj Daugherty  
(Joint work in progress with Arun Ram)

September 10, 2014

## Temperley-Lieb algebras

The *Temperley-Lieb algebra*  $TL_k(q)$  is the algebra of non-crossing pairings on  $2k$  vertices

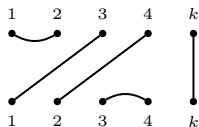


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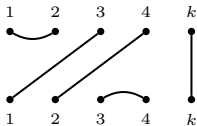
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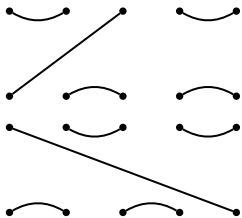
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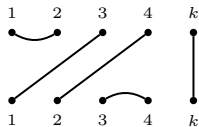
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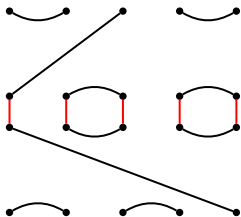
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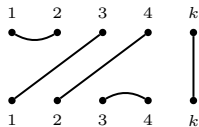
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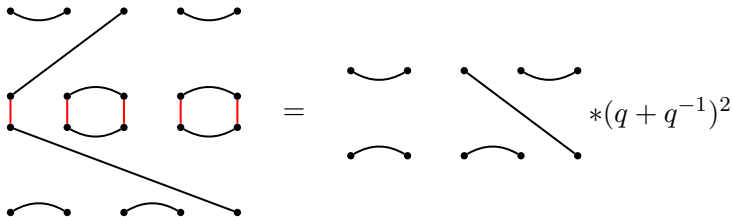
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## Odd/even relations

The algebra  $TL_k^{(1)}(q, z_0)$  is generated by

$$e_i = \left| \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right| \begin{array}{c} \overset{i}{\curvearrowright} \\ \vdots \\ \underset{i}{\curvearrowleft} \\ \vdots \\ \vdots \end{array} \left| \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right| \left| \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right| \quad \text{and} \quad e_0 = \left| \begin{array}{c} \overset{1}{\curvearrowright} \\ \vdots \\ \underset{1}{\curvearrowleft} \\ \vdots \\ \vdots \end{array} \right| \left| \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right| \left| \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right| \left| \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right| \left| \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right|$$

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for  $i = 1, \dots, k - 1$ , with relations

$$e_i e_{i \pm 1} e_i = e_i \text{ for } i \geq 1$$

$$\boxed{\begin{array}{c} \curvearrowright \\ \curvearrowleft \\ \curvearrowright \\ \curvearrowleft \\ \curvearrowright \end{array} = \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \left| \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right|}$$

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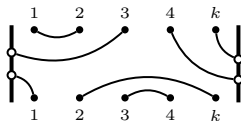
Side loops are resolved with a 1 or a  $z_0$  depending on whether there are an even or odd number of connections below their lowest point.



# Our main object: two-boundary Temperley-Lieb algebra

Nienhuis, De Gier, Batchelor (2004):

The *two-boundary Temperley-Lieb algebra*  $TL_k^{(2)}(q, z_0, z_k) = \mathcal{T}_k$  is the algebra of two-walled non-crossing pairings on  $2k$  vertices



so that each wall always has an even number of connections, with multiplication given by stacking diagrams, subject to the relations

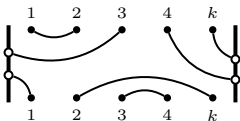
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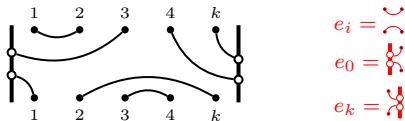
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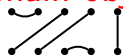


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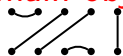


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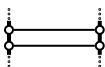
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**de Gier, Nichols (2008):** Explored representation theory of  $\mathcal{T}_k$ .

① Take quotients giving  =  $z$

to get finite-dimensional algebras.

② Establish connection to the affine Hecke algebras of type A and C to facilitate calculations.

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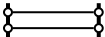
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## Quantum groups and braids

Fix  $q \in \mathbb{C}^*$ . Let  $U = U_q \mathfrak{g}$  be the Drinfel'd-Jimbo quantum group associated to a reductive Lie algebra  $\mathfrak{g}$ . Let  $V, M$  be  $U$ -modules. Then  $U \otimes U$  has invertible  $R = \sum_R R_1 \otimes R_2$  that yields a map

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that (1) satisfies braid relations, and  
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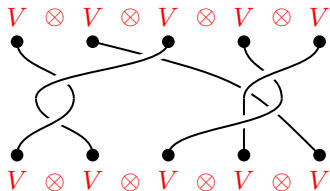
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The braid group shares a commuting action with  $U_q \mathfrak{g}$  on  $V^{\otimes k}$ :






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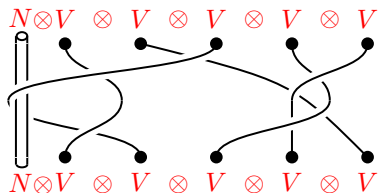
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The **one-boundary/affine** braid group shares a commuting action with  $U_q\mathfrak{g}$  on  $N \otimes V^{\otimes k}$ :



Around the pole:




$$= \check{R}_{NV} \check{R}_{VN}$$

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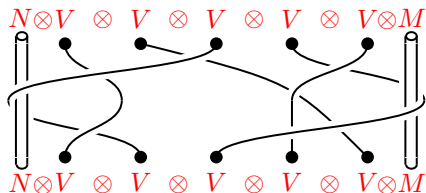
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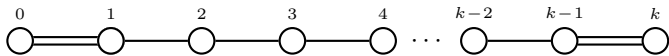


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# Affine type C Hecke algebra and two-boundary braids

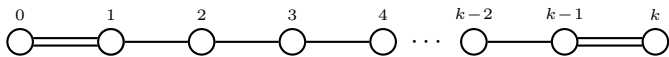


Fix constants  $t_0, t_k$ , and  $t = t_1 = \dots = t_{k-1}$ . The affine Hecke algebra of type C,  $\mathcal{H}_k$ , is generated by  $T_0, T_1, \dots, T_k$  with relations

$$\underbrace{T_i T_j \dots}_{m_{i,j} \text{ factors}} = \underbrace{T_j T_i \dots}_{m_{i,j} \text{ factors}} \quad \text{where} \quad m_{i,j} = \begin{array}{ll} 2 & \text{if } \begin{array}{c} i \quad j \\ \circ \quad \circ \end{array} \\ 3 & \text{if } \begin{array}{c} i \quad j \\ \circ \text{---} \circ \end{array} \\ 4 & \text{if } \begin{array}{c} i \quad j \\ \text{---} \circ \text{---} \circ \end{array} \end{array}$$

and  $T_i^2 = (t_i^{1/2} - t_i^{-1/2})T_i + 1$ .

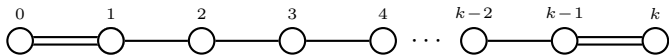
## Affine type C Hecke algebra and two-boundary braids



Fix constants  $t_0, t_k$ , and  $t = t_1 = \dots = t_{k-1}$ . The affine Hecke algebra of type C,  $\mathcal{H}_k$ , is generated by  $T_0, T_1, \dots, T_k$  with relations

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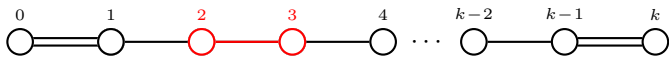
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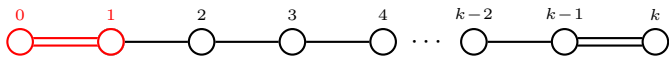
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Theorem (D.-Ram, degenerate versions of 1&2 in [D. 10])

- (1) Let  $U = U_q \mathfrak{g}$  for any complex reductive Lie algebras  $\mathfrak{g}$ .  
Let  $M$ ,  $N$ , and  $V$  be finite-dimensional modules.

The two-boundary braid group  $\mathcal{B}_k$  acts on  $N \otimes (V)^{\otimes k} \otimes M$  and this action commutes with the action of  $U$ .

- (2) If  $\mathfrak{g} = \mathfrak{gl}_n$ , then (for good simple choices of  $M$ ,  $N$ , and  $V$ ),  
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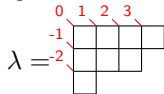
$$e_i e_{i \pm 1} e_i \quad \text{for } 1 \leq i \leq k-1 : \quad \begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{array}{c} \diagdown \\ \diagup \end{array} \quad | \quad \text{or} \quad \begin{array}{c} \diagdown \\ \diagup \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array} \quad \left( \begin{array}{c} \text{and} \\ \text{reverses} \end{array} \right)$$

- (3) When  $\mathfrak{g} = \mathfrak{gl}_2$ ,  $\mathcal{T}_k$  acts on  $N \otimes (V)^{\otimes k} \otimes M$  (for good choices).

Consider the fin-dim'l simple  $U_q\mathfrak{gl}_n$ -modules  $L(\lambda)$  indexed by **partitions**:

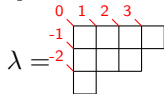
$$\lambda = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \\ \hline \square & & & \\ \hline \end{array}$$

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Fix  $V = L(\square)$ . The generators of  $\mathcal{H}_k$  acting on  $N \otimes V^{\otimes k} \otimes M$  look like

$$T_k = \begin{array}{c} V \otimes M \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ V \otimes M \end{array} \quad T_0 = \begin{array}{c} N \otimes V \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ N \otimes V \end{array} \quad \text{and} \quad T_i = \begin{array}{c} V \otimes V \\ \text{---} \text{---} \\ \text{---} \text{---} \\ V \otimes V \end{array}$$

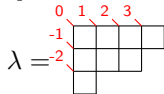
The eigenvalues of these operators (of which there should be two, since

$$(T_k - t_k^{1/2})(T_k + t_k^{-1/2}) = (T_0 - t_0^{1/2})(T_0 + t_0^{-1/2}) = (T_i - t^{1/2})(T_i + t^{-1/2}) = 0)$$

are controlled by contents of addable boxes.



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$$(a^c) = c \begin{array}{c} a \\ \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \begin{array}{c} \text{---} \uparrow \downarrow \text{---} \\ \text{---} \downarrow \uparrow \text{---} \end{array} \\ \text{---} \downarrow \uparrow \text{---} \\ \text{---} \uparrow \downarrow \text{---} \\ c \end{array}$$

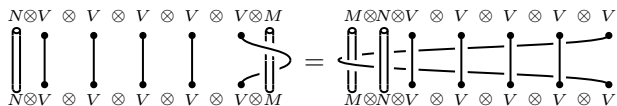
$\mathcal{H}_k$  has a commuting action with  $U_q \mathfrak{gl}_n$  on the space

$$L((b^d)) \otimes (L(\square))^{\otimes k} \otimes L((a^c)) \quad \text{with } c, d < n$$



## Exploring tensor space structure

Move the right pole to the left:





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The diagram shows an equality between two tensor network expressions. On the left, a pole from the space  $V \otimes M$  on the right passes through a series of tensor products  $N \otimes V \otimes V \otimes V \otimes V \otimes V \otimes V \otimes M$ . On the right, the pole from  $M \otimes N \otimes V$  on the left passes through the same series of tensor products. The equality indicates that the pole can be moved from right to left through the tensor products.

New favorite generators:

The diagram defines three generators:
 

- $T_0$ : A vertical line on the left with two poles on the right, one above and one below the line.
- $T_i$ : Two poles on the left, labeled  $i$  and  $i+1$ , crossing each other to become two poles on the right, also labeled  $i$  and  $i+1$ .
- $Y_j$ : A vertical line on the left with two poles on the right, labeled  $j$  and  $j$ . A horizontal line connects the two poles on the right, passing through a series of vertical lines representing tensor products.

Then

$$M \otimes N = L((a^c)) \otimes L((b^d)) = \bigoplus_{\lambda \in \Lambda} L(\lambda), \quad (\text{multiplicity one!})$$

where  $\Lambda$  is the following set of partitions:

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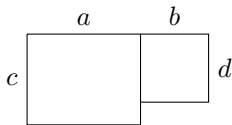
New favorite generators:

$$T_0 = \begin{array}{|c|} \hline \text{U} \\ \hline \end{array}, \quad T_i = \begin{array}{c} i \quad i+1 \\ \diagdown \quad \diagup \\ i \quad i+1 \end{array} \quad \text{and} \quad Y_j = \begin{array}{c} \text{U} \quad \text{U} \quad \text{U} \quad \text{U} \quad \text{U} \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \text{U} \quad \text{U} \quad \text{U} \quad \text{U} \quad \text{U} \end{array}$$

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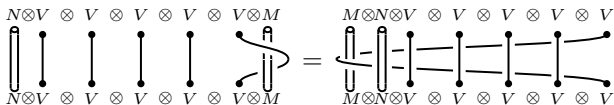
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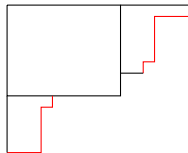
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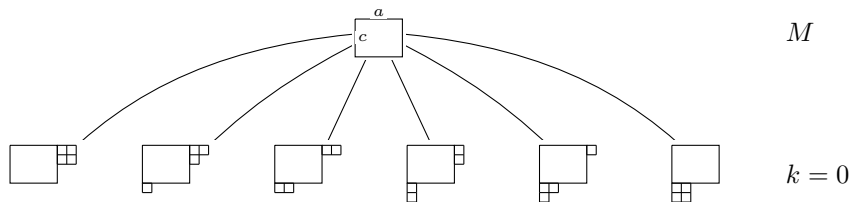
## Exploring tensor space structure



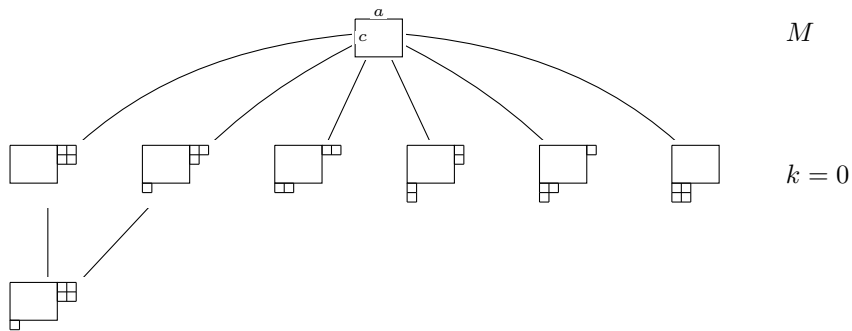
$M$



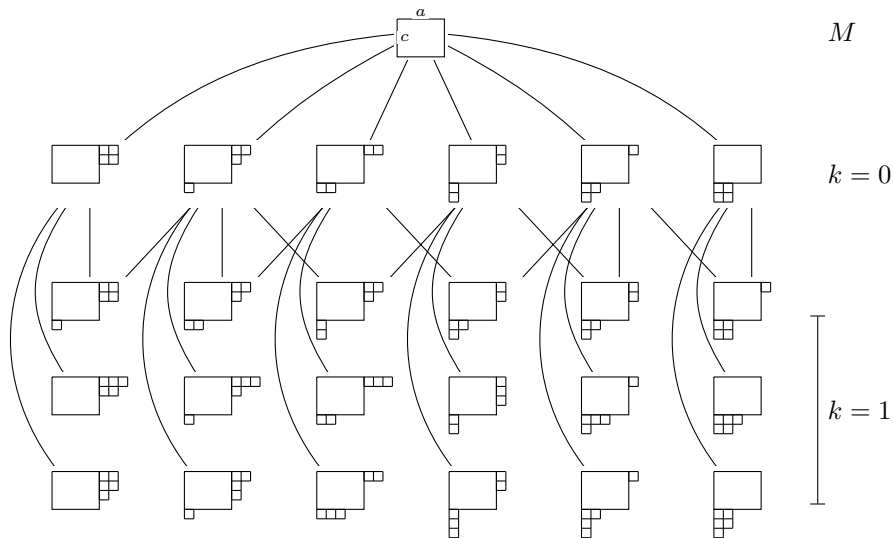
## Exploring tensor space structure



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## Central characters

The Hecke algebra  $\mathcal{H}_k$  features invertible, pairwise commuting elements  $Y_1, \dots, Y_k$  (weight lattice part).

The Weyl group  $\mathcal{W}$  of type C (the group of signed permutations) acts on  $\mathbb{C}[Y_1^{\pm 1}, \dots, Y_k^{\pm 1}]$  by permuting the subscripts, with  $Y_{-i} = Y_i^{-1}$ . Then the center of  $\mathcal{H}_k$  is symmetric Laurent polynomials

$$Z(\mathcal{H}_k) = \mathbb{C}[Y_1^{\pm 1}, \dots, Y_k^{\pm 1}]^{\mathcal{W}}.$$

We can encode central characters as maps

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$$\begin{aligned} \gamma &= (\gamma_1, \dots, \gamma_k) & \text{with} & \quad \gamma(Y_i^{\pm 1}) = (\gamma_i)^{\pm 1} \\ \mathbf{c} &= (c_1, \dots, c_k) & \text{with} & \quad \gamma(Y_i^{\pm 1}) = t^{\pm c_i} \end{aligned}$$

(when  $\mathbf{c}$  is real, favorite representatives satisfy  $0 \leq c_1 \leq \dots \leq c_k$ .)

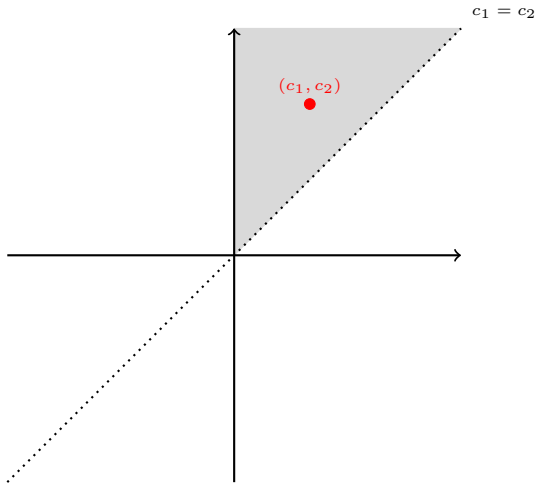


## Central characters as points

Restrict to real points.

Fav equivalence class reps:  $0 \leq c_1 \leq \cdots \leq c_k$ .

When  $k = 2$ :

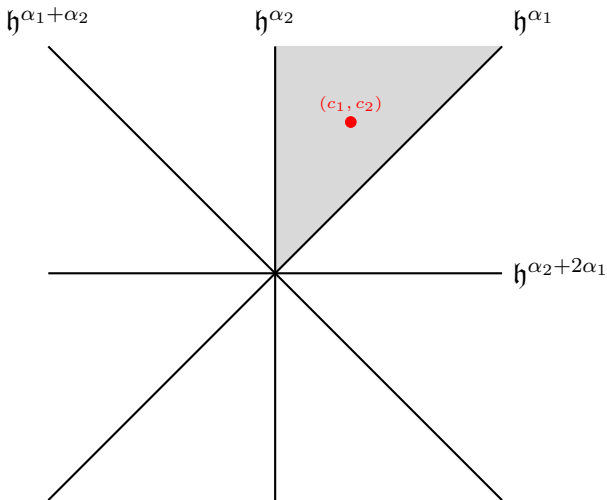


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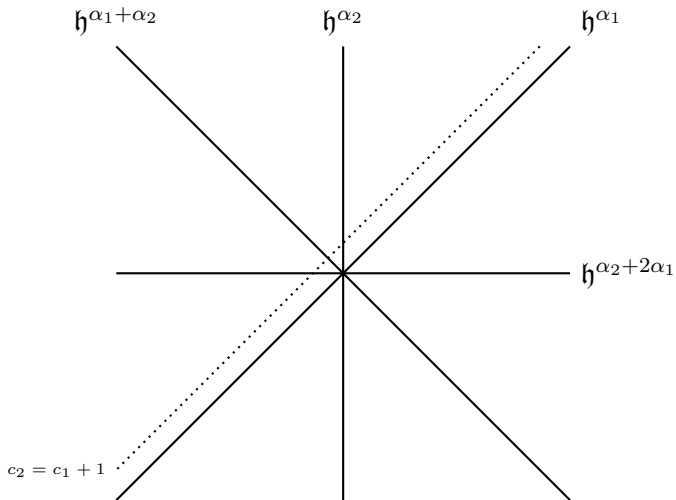


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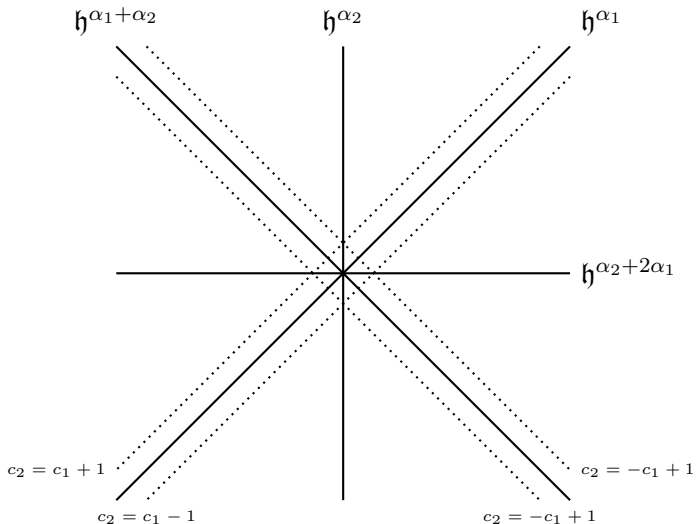


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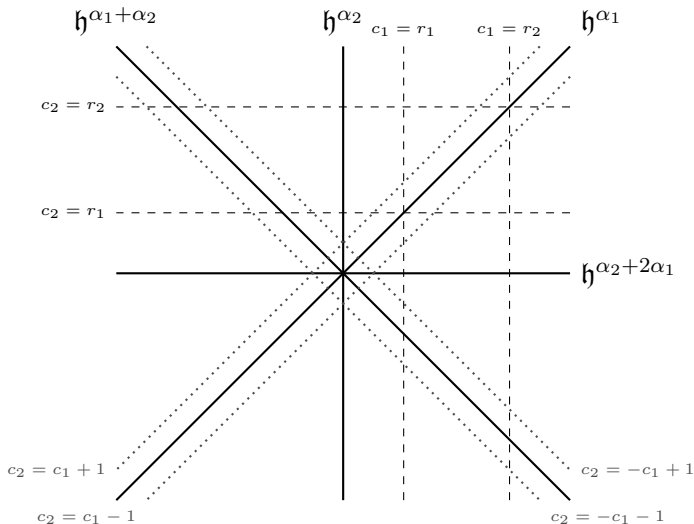


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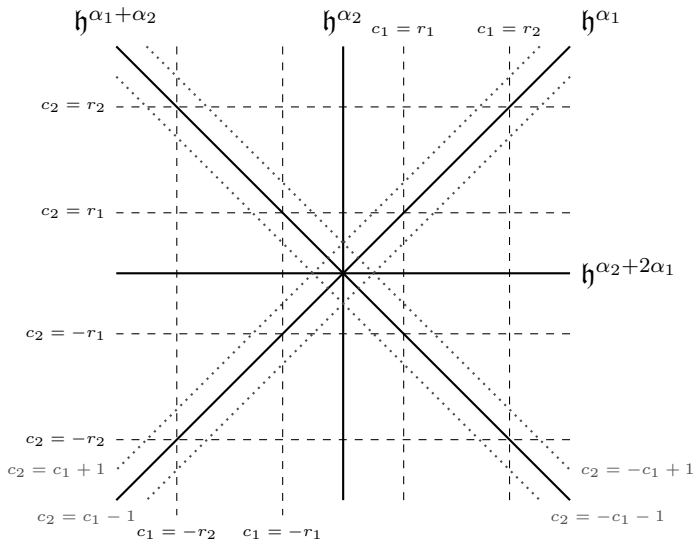
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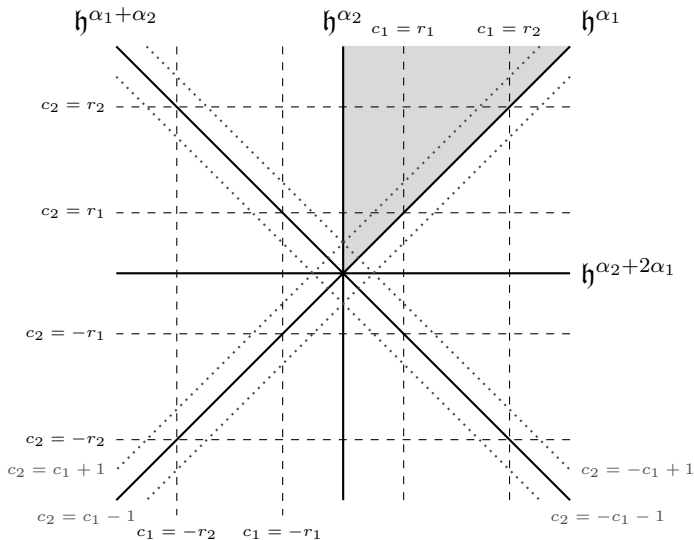
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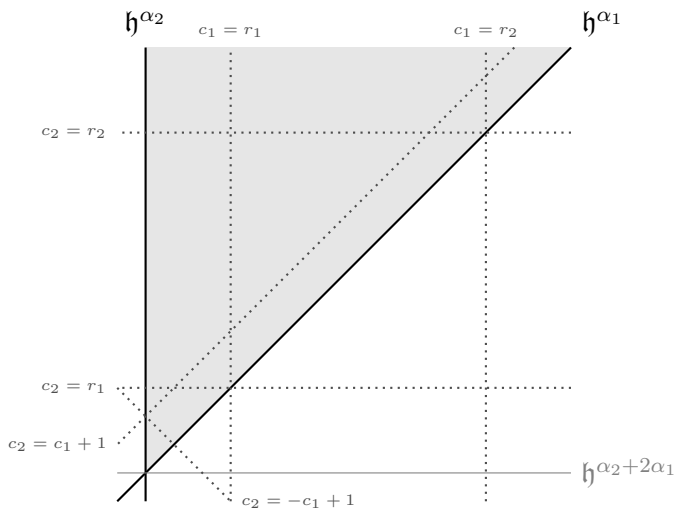
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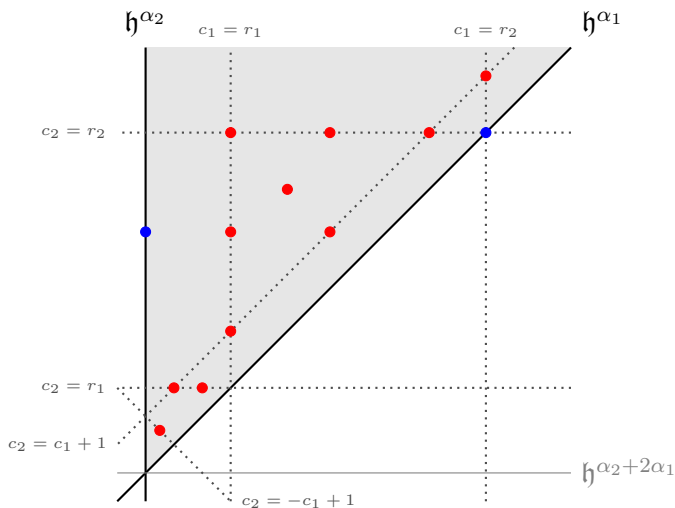
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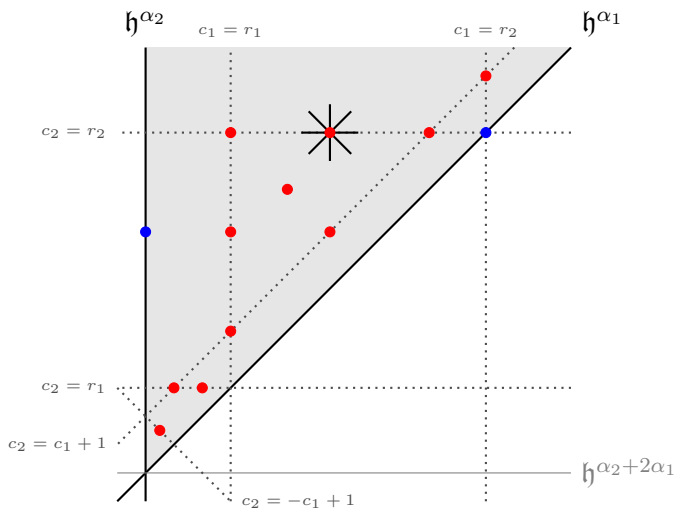


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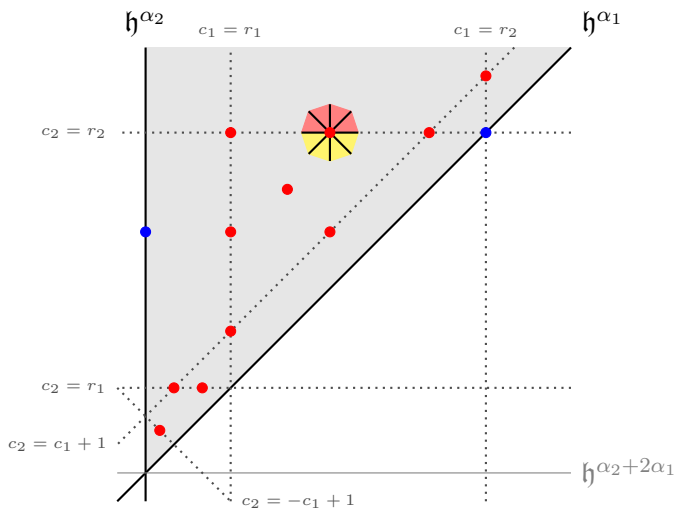
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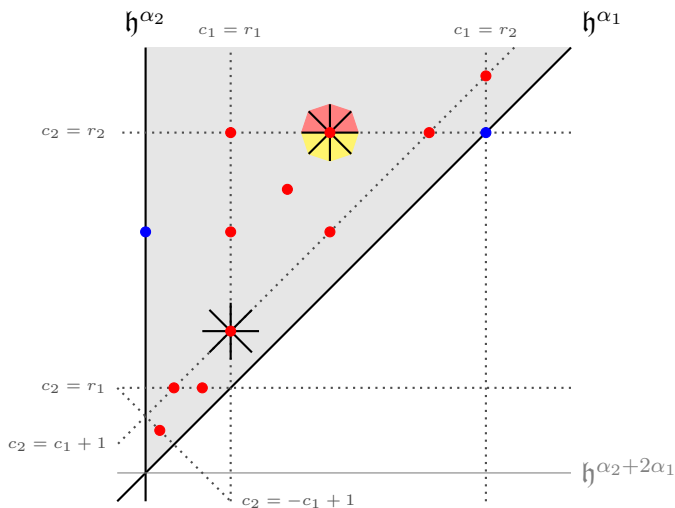
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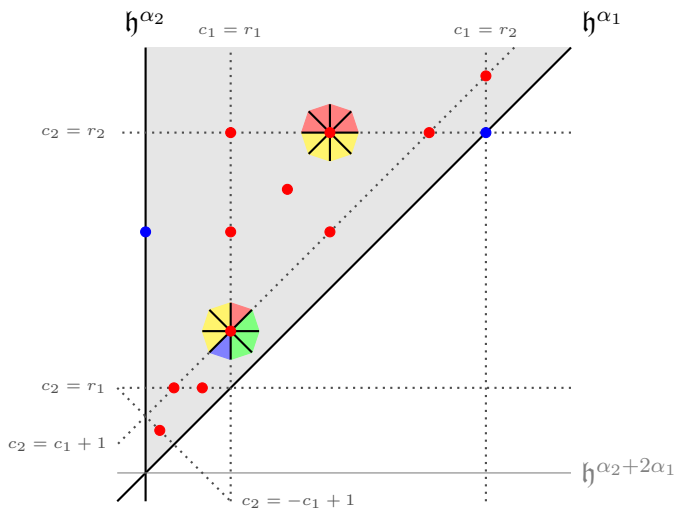
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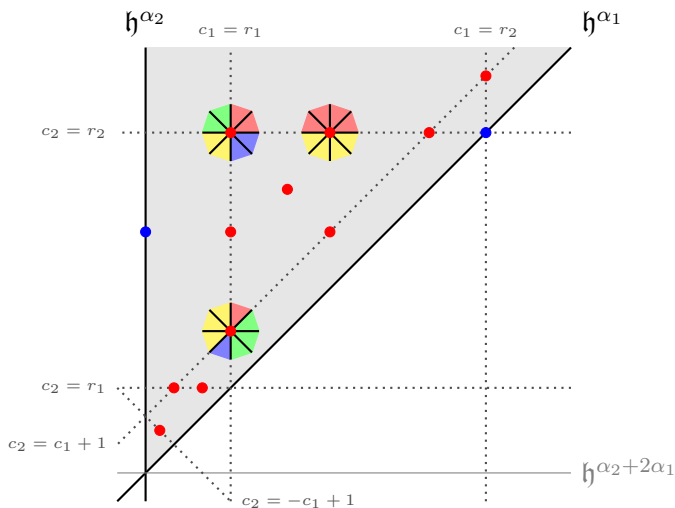
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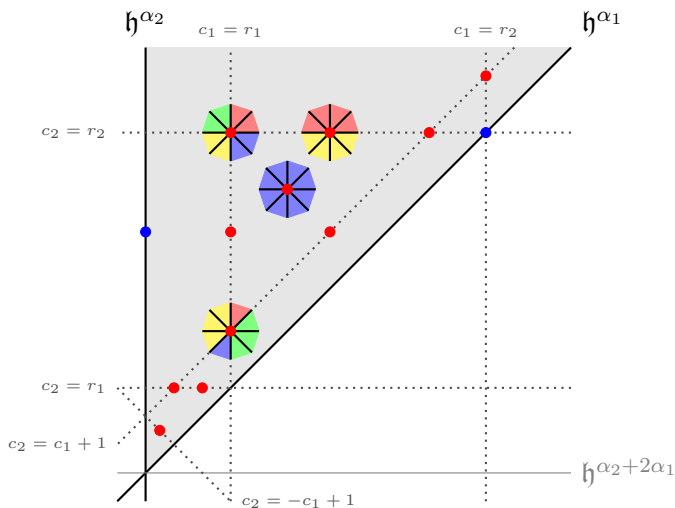
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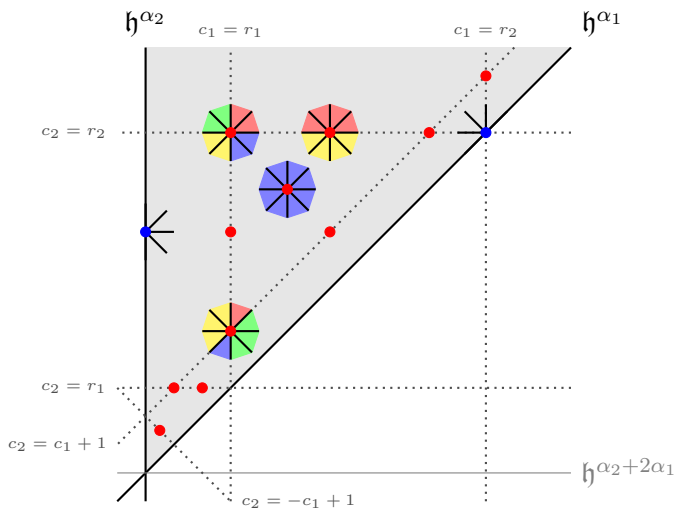
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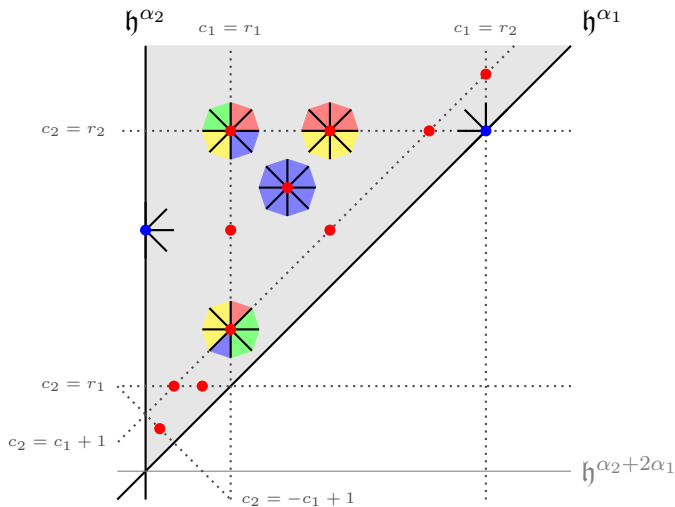
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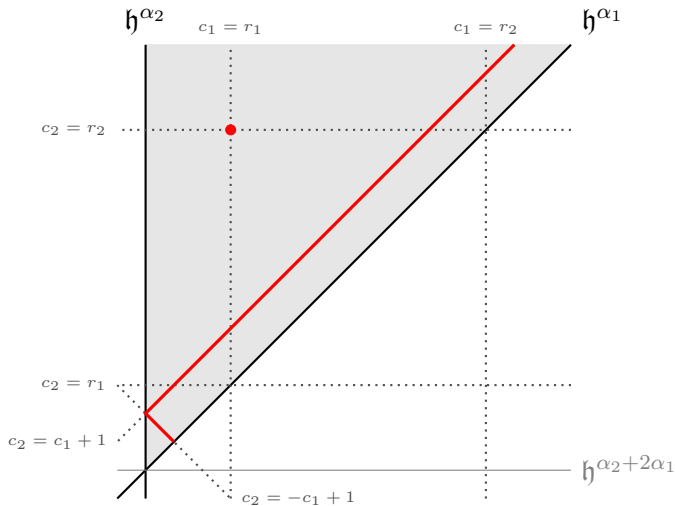
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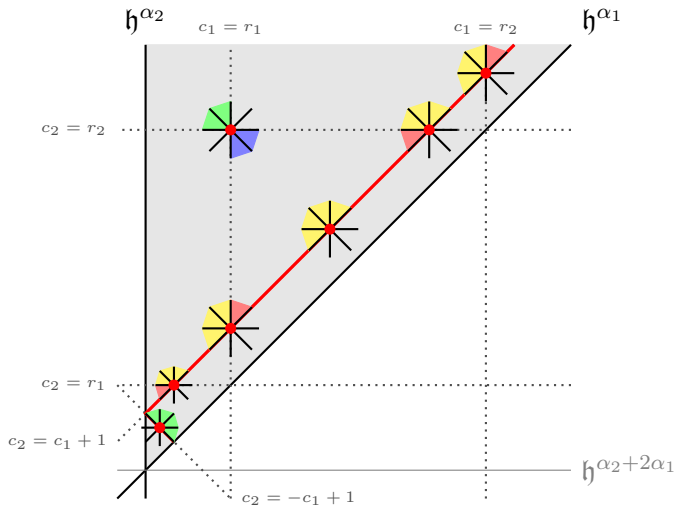
Thm. (D.-Ram)

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