Brauer algebras of Dynkin type

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research reported on is joint work
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Outline

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Motivation

Theorem

The Brauer algebra Br_n

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- *is a specialization of the Birman-Wenzl-Murakami algebra* BMW_n*;*
- *has a natural definition in terms of generators and relations.*

Motivation cont'd

The relations can be summarized by use of the Dynkin diagram of type A_{n-1}, whose Weyl group is Sym_n.

Similary for BMW_n.

To what extent are there similar algebras for other Dynkin types?

Outline

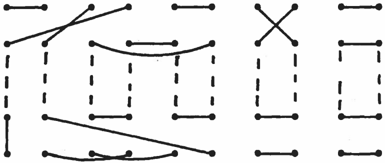
Br_n by diagrams, example for $n = 10$

As a $\mathbb{Z}[\delta]$-module, Br_n is spanned by Brauer diagrams on $2n$ nodes.

A Brauer diagram with $n = 10$

Br_n by diagrams, example cont'd

Multiplication of two Brauer diagrams



Br_n by diagrams cont'd

Known pictures for Sym_n.

Extended by cups and caps.

For a diagram T and a circle C in T

$$T = \delta \cdot (T \setminus C).$$

Main properties

Theorem (Brauer, 1937)

For $\delta \in \mathbb{N}$ and $V = \mathbb{C}^{\delta}$ there is a surjective homomorphism $\mathrm{Br}_n \to \mathrm{End}_{\mathrm{O}(V)}(\otimes^n(V))$.
For $\delta \in -2\mathbb{N}$ and $V = \mathbb{C}^{-\delta}$ there is a surjective homomorphism $\mathrm{Br}_n \to \mathrm{End}_{\mathrm{Sp}(V)}(\otimes^n(V))$.

This followed a result of Schur's for $\mathrm{End}_{\mathrm{GL}(V)}(\otimes^n(V))$.

Theorem (Wenzl, Hanlon & Wales, Doran, Rui & Si)

If $\delta \notin \mathbb{Z}$ or $|\delta| < n$, then Br_n is semisimple of dimension

$$\dim(\mathrm{Br}_n) = n!! = 1 \cdot 3 \cdots (2n-1).$$

TL_n by diagrams

Definition

TL_n is the subalgebra of Br_n spanned by the diagrams without crossings.

Lemma

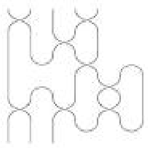
dim(TL_n) *is the n-th Catalan number.*

Theorem

TL_n *is the quotient of the Hecke algebra of type* A_{n-1} *by the central elements of the parabolic subalgebras of rank two.*

BMW_n by diagrams

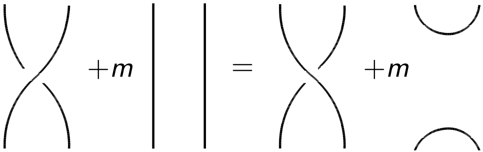
Same diagrams as for Brauer, but with distinction of over and under crossings.



Braid relations.

Skein relations.

The Kauffman skein relation



$$g_i \quad + m\, 1 \quad = \quad g_i^{-1} \quad + \quad m\, e_i$$

Properties of BMW_n

Theorem

The Birman-Wenzl-Murakami algebra BMW_n

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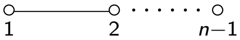
- *has a natural definition in terms of generators and relations;*
- *has a Markov trace leading to a knot theory invariant.*

BMW_n by presentation (Wenzl)

Diagram A_{n-1} = 

Coefficients δ, m, l such that $m(1-\delta) = l - l^{-1}$.

Single node i:

$$
\boxed{\begin{aligned}
g_i^2 &= 1 - m(g_i - l^{-1}e_i) \\
e_i g_i &= l^{-1} e_i \\
g_i e_i &= l^{-1} e_i \\
e_i^2 &= \delta e_i
\end{aligned}}
$$

BMW_n by presentation, cont'd

Two nodes i and j of A_{n-1} with $i \sim j$:

$$\boxed{\begin{aligned} g_i g_j &= g_j g_i \\ e_i g_j &= g_j e_i \\ e_i e_j &= e_j e_i \end{aligned}}$$

BMW_n by presentation, cont'd'

Two nodes i and j of A_{n-1} with $i \not\sim j$:

$$
\begin{aligned}
g_i g_j g_i &= g_j g_i g_j \\
g_j e_i g_j &= g_i e_j g_i + m(e_j g_i - e_i g_j + g_i e_j - g_j e_i) + m^2(e_j - e_i) \\
g_j g_i e_j &= e_i e_j \\
e_i g_j g_i &= e_i e_j \\
g_j e_i e_j &= g_i e_j + m(e_j - e_i e_j) \\
e_i g_j e_i &= l e_i \\
e_j e_i g_j &= e_j g_i + m(e_j - e_j e_i) \\
e_i e_j e_i &= e_i
\end{aligned}
$$

Presentations of the other algebras

- Br_n by specialization $m \mapsto 0, \qquad l \mapsto 1.$

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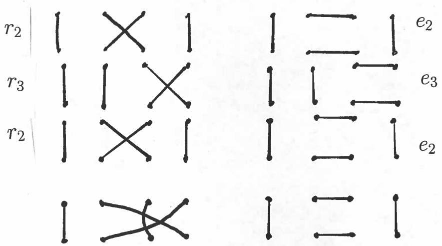
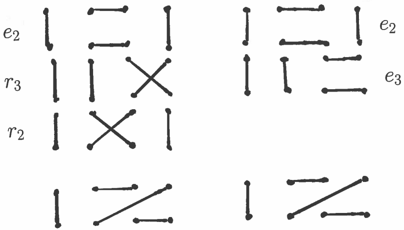
- TL_n generated by $e_1, \ldots, e_{n-1}$ subject to all relations given that involve only these generators.

Example for Brauer instead of BMW

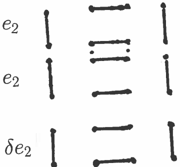


$r_2r_3r_2 = r_3r_2r_3$ $\qquad$ $e_2e_3e_2 = e_2$

More examples for Brauer instead of BMW



$e_2 r_3 r_2 = e_2 e_3$

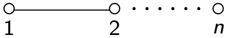


$e_2^2 = \delta e_2$

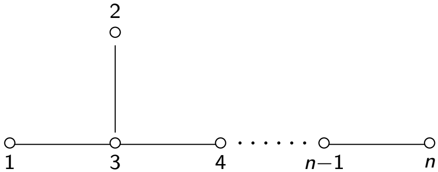
Outline

Simply laced diagrams

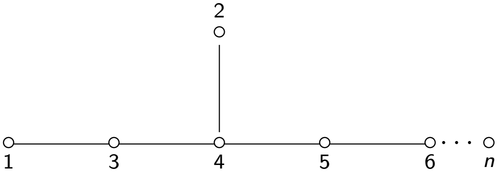
A_n



D_n



E_n $(n = 6, 7, 8)$



Presentations for simply laced types

Let M be a graph.

Definition

The BMW algebra $\mathrm{BMW}(M)$ of type M has presentation

- generators g_i, e_i (i node of M);
- relations as for $M = \mathrm{A}_{n-1}$.

Definition

- The *Brauer algebra* of type M is the specialization of $\mathrm{BMW}(M)$ with $m \mapsto 0, \qquad l \mapsto 1$;
- The *Temperley-Lieb algebra* $\mathrm{TL}(M)$ is the subalgebra of $\mathrm{BMW}(M)$ generated by $e_1, \ldots, e_{n-1}$ subject to all relations given that involve only these generators.

Results for simply laced Dynkin types

Theorem (C, Frenk, Gijsbers, Wales)

Let M be A_n $(n \geq 1)$, D_n $(n \geq 4)$, E_n $(n = 6, 7, 8)$.

- *The algebras* $\mathrm{BMW}(M)$ *and* $\mathrm{Br}(M)$ *are cellular with cells given by triples* (X, Y, w), *for* X, Y *certain (admissible) sets of commuting reflections of* $W(M)$ *in the same* $W(M)$*-orbit and elements* w *of a Coxeter group in* $C_W(X)$.

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- *The Temperley-Lieb algebras* $\mathrm{TL}(M)$ *coincide with those defined by Fan, Stembridge, and Graham.*
- *The subgroup of invertible elements generated by all* g_i *is the Artin group of type* M *(generalizing Krammer, Bigelow, Zinno).*

Reinterpretation of Brauer diagram as a triple

$n = 10$
diagram A_9
$X = \{\epsilon_1 - \epsilon_2, \epsilon_5 - \epsilon_6, \epsilon_9 - \epsilon_{10}\}$
$Y = \{\epsilon_3 - \epsilon_6, \epsilon_4 - \epsilon_5, \epsilon_9 - \epsilon_{10}\}$
$w =$ element $(1,2)(3,4)$ of $A = W(\mathrm{A}_3) = \mathrm{Sym}_4$

Dimensions of Brauer and BMW algebras of simply laced Dynkin type

M	$\dim(\mathrm{Br}(M))$
A_n	$(n+1)!!$
D_n	$(2^n+1)n!! - (2^{n-1}+1)n!$
E_6	$1,440,585$
E_7	$139,613,625$
E_8	$53,328,069,225$

Diagrams for D_n

- There is a diagram interpretation for $\mathrm{BMW}(\mathrm{D}_n)$ and $\mathrm{Br}(\mathrm{D}_n)$.

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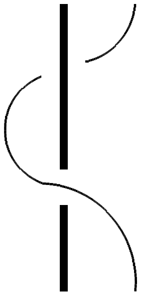
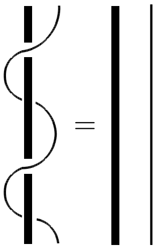
- There is a semisimplicity result for $\mathrm{BMW}(\mathrm{D}_n)$ and $\mathrm{Br}(\mathrm{D}_n)$ by Claire Levaillant.

Some relations for the tangle algebra of type D_n, $\mathrm{KT}(\mathrm{D}_n)$

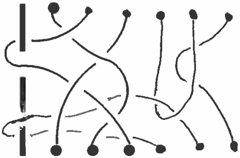
A pole twist and the relation of a pole of order two





Example D_n tangle

Tangle in $\mathbf{KT}(D_6)$.



Outline

Non-simply laced Dynkin types obtained from foldings

Theorem

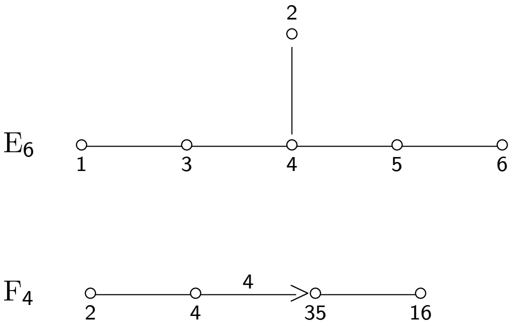
Let τ be a graph automorphism of the simply laced Dynkin diagram M. The τ-fixed subgroup of $W(M)$ is a Coxeter group of a well determined type M_τ.

Examples of non-simply laced Dynkin types from foldings

C_n	from	A_{2n-1}
B_n	from	D_{n+1}
F_4	from	E_6
G_2	from	D_4

Example of diagrams obtained from folding

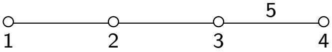
The one for F_4 uses E_6 and folds the diagram around its vertical axis.



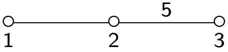
Further Dynkin diagrams

Obtained by folding of simply laced diagrams (need admissible partitions of the nodes):

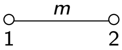
H_4



H_3



I_2^m



$\mathrm{I}_2^6 = \mathrm{G}_2, \qquad \mathrm{I}_2^4 = \mathrm{B}_2$

The simply laced Weyl group which works for

- H_3 is D_6,

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The simply laced Weyl group which works for

- H_3 is D_6,
- for H_4 is E_8,
- and for I_2^m is A_{m-1}, the symmetric group on m letters.

Extending to Brauer algebras of non-simply laced type

Let M be a simply laced Dynkin diagram with diagram automorphism τ.

Definition

The Brauer algebra $\mathrm{Br}(M_\tau)$ of type M_τ is the subalgebra of the Brauer algebra of type M generated by monomials in $\mathrm{Br}(M)$ fixed by τ.

Extending to Hecke algebras?

- In diagram of type A_3, with $\tau = (1, 3)$, for $M_\tau = \mathrm{C}_2$, the minimal polynomial for $T_1 T_3$ in $\mathrm{Br}(\mathrm{A}_3)$ contains terms involving $T_1 + T_3$. OUCH.

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- Hence no (obvious) extension to Hecke algebras.
- More promising approach: interpretation of the Artin group of type B_n as the fundamental group of the complement of the hyperplane arrangement in the complex reflection space and subsequent choice of cohomology space for a linear representation.
- What's left is the specialized case of the group algebra of $W(M)$: the Brauer algebra.

Result for C_n

Theorem (C, Shoumin Liu, Shona Yu)

If $M = \mathrm{A}_{2n-1}$ and $|\tau| = 2$, so $M_\tau \cong \mathrm{C}_n$, then $\mathrm{Br}(\mathrm{C}_n)$ is cellular of dimension

$$\sum_{i=0}^{n} \left(\sum_{p+2q=i} \frac{n!}{p!q!(n-i)!} \right)^2 2^{n-i}(n-i)!$$

with cells parameterized by triples (X, Y, w) such that X and Y are admissible sets of commuting reflections in $W(M_\tau)$ in the same $W(M_\tau)$-orbit and w is an element of a Coxeter subgroup of $C_{W(M_\tau)}(X)$.

Result for B_n

Let $n \geq 3$.

Theorem (C, Shoumin Liu)

If $M = \mathrm{D}_{n+1}$ and $|\tau| = 2$, so $M_\tau \cong \mathrm{B}_n$, then $\mathrm{Br}(\mathrm{B}_n)$ is cellular of dimension

$$2^{n+1}n!! - 2^n n! + (n+1)!! - (n+1)!$$

with cells parameterized by triples (X, Y, w) such that X and Y are **indexed** *admissible sets of commuting reflections in $W(M_\tau)$ in the same $W(M_\tau)$-orbit and w belongs to a Coxeter subgroup of $C_{W(M_\tau)}(X)$.*

Result for F_4

Theorem (Shoumin Liu)

If $M = \mathrm{E}_6$ and $|\tau| = 2$, so $M_\tau \cong \mathrm{F}_4$, then $\mathrm{Br}(\mathrm{F}_4)$ is cellular with cells parameterized by triples (X, Y, w) such that X and Y are admissible sets of commuting reflections in $W(\mathrm{F}_4)$ in the same $W(\mathrm{F}_4)$-orbit and w belongs to a Coxeter subgroup of $W(\mathrm{F}_4)$ centralizing X.

- Similar results by Zhi Chen using a flat connection.

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- Similar results by Zhi Chen using a flat connection.
- Similar results by Shoumin Liu for H_3, H_4, and I_2^m using admissible partitions.

Outline

Discrepancies with Zhi Chen and Häring Oldenburg

Both Zhi Chen and Häring Oldenburg define Brauer algebras of type B_2 but these have dimensions slightly smaller than this definition which is 25, so are not the same.

They don't appear to be subalgebras or homomorphic images, just different.

Diagram algebras for Temperley Lieb algebras

Theorem (tom Dieck, R.M. Green)

For $M = \mathrm{D}_n$ and $M = \mathrm{E}_6$, there is a diagram algebra presentation for $\mathrm{TL}(M)$ generalizing the one for $\mathrm{TL}_n = \mathrm{TL}(\mathrm{A}_{n-1})$.

Diagram algebras for Brauer and BMW algebras

Theorem (C, Gijsbers, Wales)

There is a diagram algebra presentation for $\mathrm{BMW}(\mathrm{D}_n)$ *and* $\mathrm{Br}(\mathrm{D}_n)$ *generalizing those for* $M = \mathrm{A}_{n-1}$.

How about E_n?

Theorem (Levaillant)

There is a diagram algebra presentation for $\mathrm{BMW}(\mathrm{E}_6)$ *generalizing those for* $M = \mathrm{A}_n$ *and* $M = \mathrm{D}_n$.

Other algebras with the label Brauer algebra

These are typically diagram algebras (many authors)

The Walled Brauer algebra, $B_{r,s}(\delta)$, where $r + s = n$

Brauer algebras of imprimitive complex reflection groups, labeled $G(m, p, d)$, where $pd = m$

q-Brauer algebras

Problems

- Try to find diagram algebras for the Brauer algebras of E_6, E_7, and E_8.

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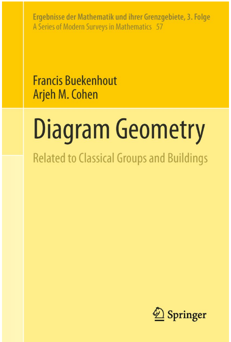
Problems

- Try to find diagram algebras for the Brauer algebras of E_6, E_7, and E_8.
- When semisimple? (Levaillant has results for D_n and E_6.)
- In the case in which the algebras are not semisimple find the blocks.
- Can BMW algebras of non-simply laced Dynkin type be defined from cohomological representations of the braid groups?

Thank you!



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