Quantalic fields

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Dedicated to B. V. M.

The whole thing is of course pure magic.

S. Lang

The lecture starts with the observation that every field extension $K|k$ is given by a quantale, a complete lattice with a compatible multiplication. The multiplicative group of the field K coincides with the unit group of the quantale, while the additive group has no counterpart in the new structure. In particular, there is no additive group for the "base field", like in the case of number fields where the roots of unity together with zero take the rôle of a base field. Nevertheless, if $(K : k) \geqslant 3$, the base field k of $K|k$ with its internal field structure, though k is just an *atom* (a minimal element > 0) in the quantale, can be recovered from this quantale, the *q-field* associated to $K|k$.

In the first part, the theory of function fields is developed in the quantalic framework up to the Riemann-Roch theorem which is basic for all further developments (ζ-function, abelian integrals, etc.). Instead of the ring of adèles, the *double* of a q-field is introduced, a quantale containing the q-field and its dual, together with the group of divisors. In contrast to adèles, no completion is involved. Local duality (Tate's thesis) naturally appears in the quantale without assuming local compactness.

In the second part, the double is characterized as a quantale. It is shown that in such a quantale, the Riemann-Roch theorem admits a simple formulation, a visualization, and a simple proof. Our general approach yields a Riemann-Roch theorem for algebraic curves where the underlying projective geometry need not be connected. A worked-out example exhibits a totally disconnected curve with only two points, which shows that existence of infinitely many valuations is not essential for the Riemann-Roch theorem.

The third part deals with the category of modular q-fields. With respect to stable morphisms, the non-exceptional objects form a category which is fibred over abelian groups and co-fibred over classical field extensions. It is shown that this category has enough projectives, and every objects admits an injective envelope.

1 Quantalic groups

Recall that a complete lattice Q with a semigroup structure is said to be a *quantale* [25] if it satisfies the equations

$$a \cdot \Big(\bigvee_{i \in I} a_i\Big) = \bigvee_{i \in I} (a \cdot a_i), \qquad \Big(\bigvee_{i \in I} a_i\Big) \cdot a = \bigvee_{i \in I} (a_i \cdot a)$$

for all $a, a_i \in Q$. We write $1 := \bigvee Q$ for the greatest and $0 := \bigvee \varnothing$ for the smallest element of Q. By Q^c we denote the set of *compact* $c \in Q$, that is, $c \leqslant \bigvee S$ with $S \subset Q$ implies that $c \leqslant \bigvee S'$ for some finite subset $S' \subset S$. A quantale Q is said to be *unital* if as a semigroup, Q admits a unit element. We use Greek letters $\alpha, \beta, \ldots$ to denote the invertible elements of Q. They form a group $Q^\times$, the *unit group* of Q. The unit element will always be denoted by ε. A *morphism* of unital quantales is a monoid homomorphism $f \colon Q \to Q'$ which satisfies $f(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} f(a_i)$ for arbitrary sets I and $a_i \in Q$.

A *closure operator* of Q, that is, a monotone map $j \colon Q \to Q$ with $a \leqslant j(a) = j^2(a)$ for all $a \in Q$, is said to be a *quantic nucleus* [23, 25] if it satisfies $j(a)j(b) \leqslant j(ab)$ for $a, b \in Q$. There is a bijection between quantic nuclei $j \colon Q \to Q$ and congruence relations on Q, where the congruence classes correspond to the closed elements with respect to j. The closed elements form a quantale jQ with multiplication $a \cdot b := j(ab)$, and there is a surjective quantale morphism $Q \twoheadrightarrow jQ$. A finitary version of a quantale is given by the following

Definition 1. We define a *quantalic monoid* (or *q-monoid* for brevity) to be a monoid M which is a $\vee$-semilattice with smallest element 0 such that the equations $x(y \vee z) = xy \vee xz$ and $(x \vee y)z = xz \vee yz$, and $x0 = 0x = 0$, hold in M.

The q-monoids form a category $\mathbf{Mon}_q$ with monoid homomorphisms $f \colon M \to N$ satisfying $f(0) = 0$ and $f(x \vee y) = f(x) \vee f(y)$ for all $x, y \in M$ as morphisms. For any q-monoid M, the power set $\mathfrak{P}(M)$ is a unital quantale with $AB := \{xy \mid x \in A,\, y \in B\}$ for subsets A and B of M. We define the *closure* $\overline{A}$ of $A \in \mathfrak{P}(M)$ to be the set of all $x \in M$ with $x \leqslant x_1 \vee \cdots \vee x_n$ for some $x_1, \ldots, x_n \in A$. Since $A \mapsto \overline{A}$ is a quantic nucleus of $\mathfrak{P}(M)$, the set $\widehat{M} := \{A \subset M \mid \overline{A} = A\}$ of *closed* subsets of M is a unital quantale with unit element $\downarrow\!\varepsilon := \{x \in M \mid x \leqslant \varepsilon\}$. We call $\widehat{M}$ the *completion* of M. By $\mathbf{Mon}^q$ we denote the category of *pre-coherent* quantales, that is, unital quantales Q for which Q^c is a sub-monoid and every $a \in Q$ is a join of compact elements. Morphisms in $\mathbf{Mon}^q$ are unital quantale morphisms $f \colon Q_1 \to Q_2$ which are *proper*, that is, $f(Q_1^c) \subset Q_2^c$.

Proposition 1. *Let M be a q-monoid. Then $x \mapsto \downarrow\! x := \{y \in M \mid y \leqslant x\}$ gives a an isomorphism $M \xrightarrow{\sim} \widehat{M}^c$ of q-monoids. The functor $M \mapsto \widehat{M}$ is an equivalence $\mathbf{Mon}_q \xrightarrow{\sim} \mathbf{Mon}^q$.*

Proof. For any $x \in M$, the element $\downarrow x \in \widehat{M}$ is compact. Indeed, $\downarrow x \leqslant \bigvee_{i \in I} A_i$ with $A_i \in \widehat{M}$ implies that $x \leqslant x_1 \vee \cdots \vee x_n$ with $x_i \in \bigcup_{i \in I} A_i$. Moreover, $\downarrow(xy) = \overline{(\downarrow x)(\downarrow y)} = \downarrow x \cdot \downarrow y$ and $\downarrow(x \vee y) = \downarrow x \vee \downarrow y$ holds for $x, y \in M$. Every $A \in \widehat{M}$ is a join $A = \bigvee_{x \in A} \downarrow x$. If A is compact, then $A = \downarrow x_1 \vee \cdots \vee \downarrow x_n = \downarrow(x_1 \vee \cdots \vee x_n)$ for some $x_i \in A$. Thus $\widehat{M} \in \mathbf{Mon}^q$, and $x \mapsto \downarrow x$ is an isomorphism $M \xrightarrow{\sim} \widehat{M}^c$ of q-monoids.

Any morphism $f\colon M \to N$ of q-monoids extends uniquely to a quantale morphism $\widehat{f}\colon \widehat{M} \to \widehat{N}$ with $\widehat{f}(A) := \overline{f(A)}$. Indeed, since $f(\overline{A}) \subset \overline{f(A)}$ holds for all $A \subset M$, we have $\widehat{f}(\bigvee A_i) = \widehat{f}(\overline{\bigcup A_i}) = \overline{f(\overline{\bigcup A_i})} = \overline{f(\bigcup A_i)} = \overline{\bigcup f(A_i)} = \overline{\bigcup \overline{f(A_i)}} = \bigvee \widehat{f}(A_i)$ and $\widehat{f}(A \cdot B) = \overline{f(\overline{AB})} = \overline{f(AB)} = \overline{f(A)f(B)} = \overline{\overline{f(A)}\ \overline{f(B)}} = \widehat{f}(A) \cdot \widehat{f}(B)$ for $A_i, A, B \in \widehat{M}$.

Conversely, let Q be a quantale in $\mathbf{Mon}^q$. Then $M := Q^c$ is a q-monoid. For any $a \in Q$, the set $A := \{x \in M \mid x \leqslant a\}$ belongs to $\widehat{M}$, and $a = \bigvee A$. On the other hand, every $A \in \widehat{M}$ gives an element $a := \bigvee A \in Q$. Since A is closed, A coincides with the set $\{x \in M \mid x \leqslant a\}$. Thus $Q \cong \widehat{M}$. Any morphism $g\colon \widehat{M} \to \widehat{N}$ in $\mathbf{Mon}^q$ induces a morphism $f\colon M \to N$ with $g = \widehat{f}$. So $M \mapsto \widehat{M}$ is an equivalence $\mathbf{Mon}_q \xrightarrow{\sim} \mathbf{Mon}^q$. □

Recall that a lattice is said to be *modular* [13] if it satisfies the implication

$$a \leqslant c \implies (a \vee b) \wedge c = a \vee (b \wedge c).$$

Proposition 2. *Let M be a q-monoid which is a lattice. Then M is a sublattice of $\widehat{M}$. Furthermore, M is modular if and only if $\widehat{M}$ is modular.*

Proof. The first assertion follows since $\downarrow x \cap \downarrow y = \downarrow(x \wedge y)$ holds for $x, y \in M$. Assume that M is modular. For $A, B, C \in \widehat{M}$ with $A \subset C$, we have to verify that $\overline{A \cup B} \cap C \subset \overline{A \cup (B \cap C)}$. Assume that $z \in \overline{A \cup B} \cap C$. Then $z \leqslant x \vee y$ for some $x \in A$ and $y \in B$. Hence $z \leqslant x \vee z = (x \vee y) \wedge (x \vee z) = x \vee \big(y \wedge (x \vee z)\big) \in \overline{A \cup (B \cap C)}$. Thus $\widehat{M}$ is modular. The converse follows since M is a sublattice of $\widehat{M}$. □

Definition 2. We define a *quantalic structure* of a group G to be a q-monoid $E(G)$ with $G = E(G)^\times$ such that $E(G) = \{\alpha_1 \vee \cdots \vee \alpha_n \,|\, \alpha_i \in G\}$. A group G with a quantalic structure will be called a *quantalic group* or simply a *q-group*. The elements of $E(G)$ will be called *flats* of G.

Examples. 1. The unit group of any q-monoid M is a q-group with $E(M^\times) = \{\alpha_1 \vee \cdots \vee \alpha_n \mid \alpha_i \in M^\times\}$.

2. Let G be a lattice-ordered group (*ℓ-group* for short) with unit element ε. Then G is a quantalic group with $E(G) = G \sqcup \{0\}$, where $0 < \alpha$ for all $\alpha \in G$.

3. Let G be a partially ordered group, and let $E(G)$ be the set of subsets $\downarrow S := \{\alpha \in G \mid \exists\, \sigma \in S\colon \alpha \leqslant \sigma\}$ of G with $S \subset G$ finite. With the multiplication

$$ST := \{\alpha \in G \mid \exists\, \sigma \in S,\, \tau \in T\colon \alpha \leqslant \sigma\tau\}$$

and inclusion as partial order, $E(G)$ defines a quantalic structure on G. Note that if G is lattice-ordered, this structure is finer than that of Example 2.

For q-groups G and H, every morphism $g\colon E(G) \to E(H)$ of q-monoids restricts to a group homomorphism $f\colon G \to H$, and g is uniquely determined by f. Therefore, we write $Ef := g$ and define a *morphism* f of q-groups to be the restriction of a morphism Ef of q-monoids. In particular, q-groups form a full subcategory $\mathbf{Gr}_q$ of $\mathbf{Mon}_q$. We define the *completion* of G to be $\widehat{G} := \widehat{E(G)}$. Accordingly, every morphism $f\colon G \to H$ in $\mathbf{Gr}_q$ gives rise to a commutative diagram

$$\begin{array}{ccccc} G & \hookrightarrow & E(G) & \hookrightarrow & \widehat{G} \\ \downarrow f & & \downarrow Ef & & \downarrow \widehat{f} \\ H & \hookrightarrow & E(H) & \hookrightarrow & \widehat{H} \end{array}$$

with $\widehat{f} := \widehat{E(f)}$.

Definition 3. By $\mathbf{Gr}^q$ we denote the category of unital quantales Q with compact unit element ε such that every $a \in Q$ is of the form $a = \bigvee A$ with $A \subset Q^\times$. For simplicity, we call them *complete q-groups*. Morphisms in $\mathbf{Gr}^q$ are unital quantale morphisms.

The compactness of ε implies that every unit is compact. So the quantales in $\mathbf{Gr}^q$ are pre-coherent. Furthermore, morphisms in $\mathbf{Gr}^q$ are proper, which shows that $\mathbf{Gr}^q$ is a full subcategory of $\mathbf{Mon}^q$. So we obtain

Proposition 3. *Let G be a quantalic group. Then $E(G) = \widehat{G}^c$ and $G = \widehat{G}^\times$. The functor $G \mapsto \widehat{G}$ gives an equivalence $\mathbf{Gr}_q \xrightarrow{\sim} \mathbf{Gr}^q$.*

Remarks. 1. In the construction of $\widehat{G}$ we may skip the passage through $E(G)$. For a subset $A \subset G$, define the *closure* $\overline{A}$ to be the set of all $\alpha \in G$ with $\alpha \leqslant \alpha_1 \vee \cdots \vee \alpha_n$ for some $\alpha_1, \ldots, \alpha_n \in A$. Then $A \mapsto \overline{A}$ is a nucleus in the quantale $\mathfrak{P}(G)$, and $\widehat{G}$ can be identified with the quantale of closed subsets of G.

2. A q-group G can thus be defined by the closure operation in $\mathfrak{P}(G)$. Similarly, a topological group G is given by a closure operation.

3. For a partially ordered group G (Example 3), the completion $\widehat{G}$ consists of the downsets $A = \downarrow A \subset G$. For example, consider the multiplicative group $\mathbb{Q}_{>0}$ of positive rationals. Then the non-units of the completion $\widehat{\mathbb{Q}_{>0}}$ are $0 := \varnothing$, $\infty := \mathbb{Q}_{>0}$, and $\alpha_- := \{\beta \in \mathbb{Q}_{>0} \mid \beta < \alpha\}$, an element with $\beta < \alpha_- < \alpha$ for each $\beta \in \mathbb{Q}_{>0}$ with $\beta < \alpha$.

Definition 4. Let Q be a unital quantale. We define an *order* in Q to be an idempotent $e \in Q$ with $\varepsilon \leqslant e$. If $\alpha e \alpha^{-1} = e$ for all $\alpha \in Q^\times$, we call e *central*. We say that e is *solid* if $e = \bigvee H$ for a subgroup H of $Q^\times$. A subgroup H of $Q^\times$ will be called *solid* if $\alpha \vee \alpha^{-1} \leqslant \alpha_1 \vee \cdots \vee \alpha_n$ with $\alpha \in Q^\times$ and $\alpha_i \in H$ implies that $\alpha \in H$.

Note that for a q-group G, a central order $e \in \widehat{G}$ satisfies $ea = ae$ for all $a \in \widehat{G}$. For a solid order e, the set $e^\times$ of all $\alpha \in G$ with $\alpha \vee \alpha^{-1} \leqslant e$ is a solid subgroup with $\bigvee e^\times = e$. So the map $e \mapsto e^\times$ is a one-to-one correspondence between solid orders e and solid subgroups of G. Under this correspondence, e is central if and only if $e^\times$ is a normal subgroup of G. Note that every solid subgroup of a quantalic group G carries an induced quantalic structure.

Proposition 4. *The kernel* $\operatorname{Ker} f$ *of a morphism* $f\colon G \to H$ *of q-groups is a normal solid subgroup of G. Conversely, any normal solid subgroup H of G gives rise to a canonical surjective morphism* $\widehat{G} \twoheadrightarrow j\widehat{G}$ *of complete q-groups.*

Proof. Assume that $\alpha \vee \alpha^{-1} \leqslant \alpha_1 \vee \cdots \vee \alpha_n$ for some $\alpha_i \in \operatorname{Ker} f$. Then $f(\alpha) \leqslant f(\alpha_1) \vee \cdots \vee f(\alpha_n) = \varepsilon$, and similarly, $f(\alpha)^{-1} = f(\alpha^{-1}) \leqslant \varepsilon$. Hence $f(\alpha) = \varepsilon$, which shows that $\operatorname{Ker} f$ is solid and normal.

Conversely, let H be a normal solid subgroup of G. Then $e := \bigvee H$ is a central solid order in $\widehat{G}$. The map $j\colon \widehat{G} \to \widehat{G}$ with $j(a) := ea$ is a quantic nucleus of $\widehat{G}$. So there is a unital quantale morphism $\widehat{G} \twoheadrightarrow j\widehat{G}$. The unit group of $j\widehat{G}$ contains $j(G)$, and e is compact in $j\widehat{G}$. So $j\widehat{G}$ belongs to $\mathbf{Gr}^q$. ☐

In what follows, we write $\widehat{G}/e$ with $e := \bigvee H$ instead of $j\widehat{G}$.

2 Quantalic fields

In Section 1, it was shown that a quantalic group G, its q-monoid $E(G)$ of flats, and its completion $\widehat{G}$ determine each other. Now we introduce a quantalic version of fields.

Definition 5. We call a (complete) q-group Q *discrete* if its partial order is trivial on $Q^\times$. A commutative discrete complete q-group F will be called a *quantalic field* or simply a *q-field*. We say that F is *modular* if F is modular as a lattice.

By Definition 3, the units of a q-field F coincide with the *atoms*, that is, minimal elements > 0. Furthermore, every $a \in F$ is of the form $a = \bigvee [a]$ with

$$[a] := \{\alpha \in F^\times \mid \alpha \leqslant a\}.$$

If F is modular, the flats $a \in E(F) := E(F^\times)$ coincide with the elements $a \in F$ of finite *height* $h(a)$. So they form a sublattice of F. The following two are our guiding examples.

Examples. 4. Let $K|k$ be a field extension. The set $F(K|k)$ of all k-subspaces of K is a modular q-field. By the Veblen-Young theorem [30], the field k is an invariant of $F(K|k)$ if the degree $(K : k)$ is at least three.

5. Let $\mathfrak{P}(G)$ be the power set of a commutative group G. With the multiplication

$$AB := \{\alpha\beta \mid \alpha \in A,\ \beta \in B\}, \tag{1}$$

$\mathfrak{P}(G)$ is a modular q-field with unit element $\varepsilon := \{1\}$ and unit group $\mathfrak{P}(G)^\times \cong G$.

6. Let F be the set of all non-empty subsets A of $\mathbb{Q}$ which satisfy

$$|\alpha - \beta| \leqslant |\gamma| \implies \alpha - \beta \in A$$

for all $\alpha, \beta, \gamma \in A$. Since F is closed under arbitrary intersection, it consists of the closed sets of a closure operation $A \mapsto \overline{A}$ in $\mathfrak{P}(\mathbb{Q})$. For $A \subset \mathbb{Q}$, let $\langle A \rangle$ be the additive subgroup of $\mathbb{Q}$ generated by A, and let $s(A) \in \widehat{\mathbb{Q}_{>0}}$ be the supremum of all $|\alpha|$ with $\alpha \in A$. Then

$$\overline{A} = \{\alpha \in \langle A \rangle \mid |\alpha| \leqslant s(A)\}.$$

Indeed, any $\alpha \in \langle A \rangle$ belongs to a finitely generated subgroup $\langle \alpha_1, \ldots, \alpha_n \rangle$ of $\langle A \rangle$ with $0 < \alpha_i \in A \cup -A$. Then α can be obtained by taking differences $|\alpha_i - \alpha_j| \leqslant \alpha_i$, then differences of such differences, and so on, until a generator of the cyclic group $\langle \alpha_1, \ldots, \alpha_n \rangle$ of $\mathbb{Q}$ has been found. Note that $|\alpha| \leqslant s(A)$ is equivalent to $|\alpha| \leqslant |\gamma|$ for some $\gamma \in A$. Hence $A \mapsto \overline{A}$ is a nucleus of $\mathfrak{P}(\mathbb{Q})$, and thus F is a quantale. The units are of the form $\overline{\alpha} = \{-\alpha, 0, \alpha\}$ with $\alpha \in \mathbb{Q}_{>0}$. In particular, $\varepsilon = \{-1, 0, 1\}$ is the unit element of F. If $\varepsilon \leqslant \bigvee_{i \in I} A_i$ with $A_i \in F$, then $1 \in \overline{\bigcup_{i \in I} A_i}$. So there is an element $\alpha \in A_i$ for some $i \in I$ such that $1 \leqslant \alpha$, and $1 \in \langle \bigcup_{i \in I} A_i \rangle$. Hence $\varepsilon \leqslant \bigvee_{j \in J} A_j$ for a finite set $J \subset I$. Thus F is a q-field. (In a similar fashion, any algebraic number field can be made into a q-field.)

However, F is not modular. For example, $\overline{1} \leqslant \overline{\{1,2\}}$ and $(\overline{1} \vee \overline{3}) \wedge \overline{\{1,2\}} = \overline{\{1,2\}}$, but $\overline{1} \vee (\overline{3} \wedge \overline{\{1,2\}}) = \overline{1} \vee 0 = \overline{1}$.

7. A subset C of an ℓ-group G is said to be *convex* [9] if $\alpha \leqslant \beta \leqslant \gamma$ with $\alpha, \gamma \in C$ and $\beta \in G$ implies that $\beta \in C$. Let $C(G)$ be the set of all convex sublattices of G. For $A \subset G$, let jA be the convex sublattice generated by A. Then $j\colon \mathfrak{P}(G) \to \mathfrak{P}(G)$ is a quantic nucleus. So we get a quantale epimorphism $\mathfrak{P}(G) \twoheadrightarrow C(G)$. For $A, B \in C(G)$, the product $AB := \{ab \mid a \in A,\ b \in B\}$ is again in $C(G)$. Indeed, if $a_1 b_1 \leqslant c \leqslant a_2 b_2$ with $a_1 \leqslant a_2$ in A and $b_1 \leqslant b_2$ in B, the basic inequality $1 \leqslant a \vee a^{-1}$ in ℓ-groups implies that $c = (cb_1^{-1} \wedge a_2)(a_2^{-1}c \vee b_1)$ with $a_1 \leqslant cb_1^{-1} \wedge a_2 \leqslant a_2$ and $b_1 \leqslant a_2^{-1}c \vee b_1 \leqslant b_2$. The unit group $C(G)^\times$ consists of the singletons, which are compact in $C(G)$. Thus G is a discrete q-group with completion $C(G)$. If G is commutative, $C(G)$ is a q-field.

If $G \neq \{1\}$, the lattice $C(G)$ is not modular. For arbitrary $\alpha < \beta < \gamma$ in G, consider the elements $A := \{\gamma\}, B := \{\alpha\}$, and $C := [\beta, \gamma]$ of $C(G)$. Then $(A \vee B) \wedge C = [\alpha, \gamma] \cap [\beta, \gamma] = [\beta, \gamma]$ and $A \vee (B \wedge C) = \{\gamma\} \vee \varnothing = \{\gamma\}$. So $C(G)$ is not modular.

Thus any non-trivial abelian ℓ-group G yields a non-modular q-field $C(G)$. An important consequence of modularity is the following exchange lemma:

Lemma 1. *Let L be a modular lattice with $c \in L$ and atoms $\alpha, \beta \in L$. Assume that $\alpha \leqslant \beta \vee c$ and $\alpha \nleqslant c$. Then $\beta \leqslant \alpha \vee c$.*

Proof. Since $c\vee\big(\beta\wedge(\alpha\vee c)\big) = (c\vee\beta)\wedge(\alpha\vee c) = \alpha\vee c > c$, we have $0 < \beta\wedge(\alpha\vee c) \leqslant \beta$. Whence $\beta \leqslant \alpha \vee c$. □

Definition 6. Let F be a q-field. For an order $e \in F$ we define:

$$\begin{aligned} e \text{ is } \textit{rational} \quad &:\Longleftrightarrow \ \forall\, \alpha \in F^\times:\ \alpha \leqslant e \Rightarrow \alpha^{-1} \leqslant e \\ e \text{ is } \textit{prime} \quad &:\Longleftrightarrow \ \forall\, \alpha \in F^\times:\ \alpha \leqslant e \text{ or } \alpha^{-1} \leqslant e. \end{aligned}$$

A unit $\alpha \in F^\times$ will be called *integral over* e if $\alpha^{n+1} \leqslant e(\varepsilon \vee \alpha \vee \cdots \vee \alpha^n)$ for some $n \in \mathbb{N}$, otherwise *transcendental*. For orders $e \leqslant f$ we call f an *overorder* of e. We say that an overorder f is *finite* over e if there is a non-zero flat $a \in E(F)$ with $ea = fa$. If $f = \bigvee e_i$ with $e_i \geqslant e$ finite, we call f *algebraic* over e, otherwise *transcendental*.

Thus e is rational if and only if $[e]$ is a group. In particular, this implies that e is solid, with corresponding solid subgroup $[e]$ of $F^\times$. If e is an order in F and $\alpha_1, \dots, \alpha_r \in F^\times$, we write $e[\alpha_1, \dots, \alpha_r] := \bigvee\{e\alpha_1^{n_1} \cdots \alpha_r^{n_r} \mid n_1, \dots, n_r \in \mathbb{N}\}$ for the smallest overorder of e which contains $\alpha_1, \dots, \alpha_r$. Thus $\alpha \in F^\times$ is integral over e if and only if $\alpha \leqslant e[\alpha^{-1}]$.

Proposition 5. *Let F be a q-field. A solid order $e \in F$ is rational if and only if the complete q-group F/e (see Proposition 4) is a q-field.*

Proof. The defining condition for e to be rational says that $e\alpha \leqslant e$ implies that $e\alpha = e$. Thus e is rational if and only if F/e is discrete. □

Proposition 6. *Let $e \leqslant f$ be orders in a modular q-field F. If $f = ea$ with $a \in E(F)$, then f is finite over e. The converse holds if e is rational.*

Proof. If $f = ea$, then $a > 0$ and $fa \leqslant fea = f^2 = f = ea \leqslant fa$. Conversely, assume that $ea = fa$ with $a = \alpha_1 \vee \cdots \vee \alpha_n$ and $n > 0$ for some $\alpha_i \in F^\times$. Then $f\alpha_1 \leqslant fa = ea$. Hence $f \leqslant ea\alpha_1^{-1}$. If e is rational, then Proposition 5 implies that the interval $[e, f]$ is of finite length in F/e. □

Proposition 7. *Let e be a rational order in a modular q-field F. A unit $\alpha \in F^\times$ is integral over e if and only if there is an order f with $f \geqslant e$ finite and $\alpha \leqslant f$.*

Proof. Assume that $\alpha \leqslant f$ with f finite over e. By Proposition 6, there is a flat $a \in E(F)$ with $f = ea$. Hence $e[\alpha] \leqslant ea \in E(F/e)$, and thus $e\alpha^{n+1} \leqslant e(\varepsilon \vee \alpha \vee \cdots \vee \alpha^n)$ for some $n \in \mathbb{N}$. So α is integral over e. Conversely, $\alpha^{n+1} \leqslant e(\varepsilon \vee \alpha \vee \cdots \vee \alpha^n)$ implies that the order $e[\alpha] = e(\varepsilon \vee \alpha \vee \cdots \vee \alpha^n)$ is finite over e. □

Corollary 1. *Let $e \leqslant f$ be orders in a modular q-field F with e rational. Then f is algebraic over e if and only if each $\alpha \in [f]$ is integral over e.*

Proof. If f is algebraic over e, any $\alpha \in [f]$ satisfies $\alpha \leqslant f_1 \cdots f_n$ with orders f_i, finite over e. By Proposition 6, $f_i = ea_i$ with $a_i \in E(F)$. Hence $f_1 \cdots f_n = ea_1 \cdots a_n$. So $f_1 \cdots f_n$ is finite over e, and α is integral over e. The converse is trivial. □

Corollary 2. *Let $e \leqslant f$ be orders in a modular q-field F with f algebraic over e. Then e is rational if and only if f is rational.*

Proof. Assume that e is rational. Then every $\alpha \in [f]$ is integral over e. So $e[\alpha] \in E(F/e)$. The $a_n := \bigvee_{i \geqslant n} e\alpha^i$ form a descending sequence $e[\alpha] = a_0 \geqslant a_1 \geqslant a_2 \geqslant \cdots$ in F/e. So there is an $n \in \mathbb{N}$ with $a_{n+1} = a_n$, that is, $\alpha^n \leqslant e(\alpha^{n+1} \vee \alpha^{n+2} \vee \cdots)$. Hence $\alpha^{-1} \leqslant e[\alpha] \leqslant f$. Thus f is rational. Conversely, let f be rational. For a unit $\alpha \leqslant e$ this implies that $\alpha^{-1} \leqslant f$. By Corollary 1 of Proposition 7, α^{-1} is integral over e. Thus $\alpha^{-1} \leqslant e[\alpha] \leqslant e$. □

Now we turn our attention to prime orders. Note that the greatest element 1 is the only rational prime order. For any order e in a q-field F, consider the set $\mathrm{Mod}(e) := \{a \in F \mid ea = a\}$ of *e-modules*. We call $a \in \mathrm{Mod}(e)$ *finitely generated* if $a = eb$ for some $b \in E(F)$. The set of finitely generated e-modules will be denoted by $\mathrm{mod}(e)$. If e is rational, $\mathrm{mod}(e) = E(F/e)$.

Lemma 2. *Let e be an order in a q-field F. For any e-module $a < e$, there is a maximal e-module $m < e$ with $a \leqslant m$.*

Proof. This follows immediately by the compactness of ε and Zorn's lemma. □

Proposition 8. *Let F be a q-field. An order e in F is prime if and only if* $\mathrm{Mod}(e)$ *– or equivalently:* $\mathrm{mod}(e)$ *– is a chain. For a prime order e, there is a greatest e-module $m < e$. Either $m^2 = m$ or $m = e\pi$ for some $\pi \in F^\times$.*

Proof. Assume that e is prime, and that $a, b \in \mathrm{Mod}(e)$ satisfy $a \not\leqslant b$. Then there is a unit $\alpha \in F^\times$ with $\alpha \leqslant a$ and $\alpha \not\leqslant b$. For all units $\beta \leqslant b$, this implies that $\alpha \not\leqslant e\beta$. Hence, $\alpha\beta^{-1} \not\leqslant e$, and thus $\alpha^{-1}\beta \leqslant e$. So we obtain $\beta \leqslant e\alpha \leqslant a$, which yields $b \leqslant a$. Conversely, let $\mathrm{mod}(e)$ be a chain. Then $\alpha \not\leqslant e$ implies that $e \leqslant e\alpha$, which gives $\alpha^{-1} \leqslant e$. So e is a prime order. By Lemma 2, there is a greatest e-module $m < e$.

Assume that $m^2 < m$. So there is a unit $\pi \leqslant m$ with $\pi \not\leqslant m^2$. Hence $m^2 < e\pi \leqslant m$, and thus $m^2 \leqslant m\pi \leqslant m^2$. So we obtain $m^2 = m\pi < e\pi$, which shows that $e\pi$ covers m^2 and is unique since $\mathrm{Mod}(e)$ is a chain. Whence $e\pi = \bigvee e\big([m] \smallsetminus [m^2]\big) = m$. □

Definition 7. Let F be a q-field. We call an order e *local* if there is a greatest $m < e$ in $\mathrm{Mod}(e)$, the *radical* of e.

Thus, rational orders and prime orders e are local. By Proposition 8, there are two exclusive cases. If the radical m is idempotent, we call e *continuous*, otherwise *discrete*.

Example 8. Consider the subgroup $\mathbb{Z}^2$ in the additive group of $\mathbb{R}^2$. By Example 5, $F := \mathfrak{P}(\mathbb{Z}^2)$ is a modular q-field. Any 1-dimensional subspace $\mathbb{R}v$ of $\mathbb{R}^2$ splits $\mathbb{Z}^2$ into two halves which are prime orders of F. If $\mathbb{R}v \cap \mathbb{Z}^2 = \{0\}$, these prime orders are continuous, otherwise discrete.

Our next aim is to show that there are enough prime orders in any q-field. Like in ring theory, orders can be localized. Let us call $p < e$ in $\mathrm{Mod}(e)$ a *prime ideal* of e if $[e] \setminus [p]$ is a submonoid of $F^\times$.

Proposition 9. *Let e be an order in a q-field F and $p \in \mathrm{Mod}(e)$ a prime ideal. Then $e_p := \bigvee\{\alpha^{-1}e \mid \alpha \in [e] \setminus [p]\}$ is a local overorder with radical pe_p, and $e \wedge pe_p = p$.*

Proof. Since $[e] \setminus [p]$ is a monoid, e_p is an overorder of e. Let $\alpha \in F^\times$ satisfy $\alpha \leqslant e \wedge pe_p$. Then $\alpha \leqslant \beta^{-1}p$ for some $\beta \in [e] \setminus [p]$. Hence $\alpha\beta \leqslant p$. Since p is prime, this yields $\alpha \leqslant p$. Thus $e \wedge pe_p = p$. Finally, assume that $\alpha \in F^\times$ and $e_p\alpha < e_p$. Then $\alpha \leqslant \beta^{-1}e$ for some $\beta \in [e] \setminus [p]$, and $\gamma^{-1}e \nleqslant e_p\alpha$ for some $\gamma \in [e] \setminus [p]$. Hence $\alpha\beta \leqslant e$ and $e_p\alpha\beta \leqslant e_p\alpha$. Thus $\gamma^{-1}e \nleqslant e_p\alpha\beta$. So we have $(\alpha\beta\gamma)^{-1}e \nleqslant e_p$, which gives $\alpha\beta\gamma \in [p]$. Since p is prime, we infer that $\alpha\beta \leqslant p$. Thus, $\alpha \leqslant p\beta^{-1} \leqslant pe_p$. This proves that e_p is local with radical pe_p. □

In particular, the prime ideal 0 of an order e gives the *quotient order* e_0 of e, the smallest rational overorder of e. For $\alpha_1, \ldots, \alpha_n \in F^\times$ we write $e(\alpha_1, \ldots, \alpha_n)$ for the quotient order of $e[\alpha_1, \ldots, \alpha_n]$. The following is an analogue to Nakayama's lemma.

Lemma 3. *Let F be a modular q-field, and let e be a local order with radical m. Let $a \in \mathrm{mod}(e)$ satisfy $a = ma$. Then $a = 0$.*

Proof. Suppose that $a \neq 0$. Then $a = e\alpha \vee b$ with $\alpha \in F^\times$ such that $b \in \mathrm{mod}(e)$ needs less generators than a. Hence $e\alpha \leqslant ma \leqslant m\alpha \vee b$, and thus $e\alpha = (m\alpha \vee b) \wedge e\alpha = m\alpha \vee (b \wedge e\alpha)$. So $b \wedge e\alpha \nleqslant m\alpha$, which yields $e\alpha \leqslant b$, a contradiction. □

Proposition 10. *Let F be a modular q-field, and let e be a local order in F with radical m. Then e is prime if and only if every overorder $f > e$ satisfies $mf = f$.*

Proof. Assume that every overorder $f > e$ satisfies $mf = f$. Let $\alpha \in F^\times$ be a unit with $\alpha \nleqslant e$. Then $e < e[\alpha] = e \vee e\alpha \vee e\alpha^2 \vee \cdots$. Hence $\varepsilon \leqslant e[\alpha] = me[\alpha]$, and thus $\varepsilon \leqslant m \vee m\alpha \vee \cdots \vee m\alpha^n$ for some $n \in \mathbb{N}$. With $a := e\alpha \vee \cdots \vee e\alpha^n$, this gives $e = (m \vee a) \wedge e = m \vee (a \wedge e)$. Thus $a \wedge e \nleqslant m$, which yields $e \leqslant a$. So we obtain $e\alpha^{-n} \leqslant e\alpha^{-(n-1)} \vee \cdots \vee e$, hence $e[\alpha^{-1}] \in \mathrm{mod}(e)$. By Lemma 3, $me[\alpha^{-1}] < e[\alpha^{-1}]$. Thus $e[\alpha^{-1}] \ngtr e$, which yields $\alpha^{-1} \leqslant e$. So e is a prime order.

Conversely, let e be prime, and let $f > e$ be an overorder. Then $m \leqslant mf$. Suppose that $mf = m$. Choose $\alpha \in [f] \setminus [e]$. Then $m\alpha \leqslant mf = m < e$, which gives $e\alpha \leqslant e$, contrary to $\alpha \nleqslant e$. Hence $m < mf$, and thus $e \leqslant mf$. So we get $f = ef \leqslant mf^2 = mf \leqslant ef = f$. □

Lemma 4. *Let e be an order in a q-field F. Every maximal e-module $m < e$ is a prime ideal.*

Proof. This follows since $\alpha \in [e] \smallsetminus [m]$ is equivalent to $e\alpha \vee m = e$. $\square$

Proposition 11. *Let F be a modular q-field, and let e be a local order in F with radical m. There exists a prime overorder e' with radical m' and $e \wedge m' = m$.*

Proof. Let $(e_i)_{i\in I}$ be a chain of local overorders of e with radical $m_i \geqslant m$, indexed by a linearly ordered set I such that $i \leqslant j$ in I implies that $e_i \leqslant e_j$ and $m_i \leqslant m_j$. Define $e' := \bigvee_{i\in I} e_i$ and $m' := \bigvee_{i\in I} m_i$. Then e' is an order with $m' \in \mathrm{Mod}(e')$. Since $\varepsilon \nleqslant m'$, we have $m' < e'$. Now let $\alpha \in F^\times$ be a unit with $e'\alpha < e'$. Then $\alpha \leqslant e_i$ for some $i \in I$. If $e_i\alpha = e_i$, then $\varepsilon \leqslant e'\alpha$, which is impossible. Thus $\alpha \leqslant m_i \leqslant m'$, which proves that e' is local. By Zorn's lemma, we obtain a local overorder e' of e with radical $m' \geqslant m$ such that e' is maximal with this property. Let $f > e'$ be any overorder with $m'f < f$. By Lemma 2, there is a maximal f-module $p < f$ with $m'f \leqslant p$. By Lemma 4, p is a prime ideal. So Proposition 9 yields a local order f_p with radical $pf_p \geqslant m' \geqslant m$. By the maximality of e', we get $f_p = e'$, hence $f = e'$. Now Proposition 10 implies that e' is a prime order. Since $m \leqslant m'$, it follows that $e \wedge m' = m$. $\square$

3 Valuations of a q-field

In this section, we associate prime orders to morphisms in the category of q-groups. By Example 2, every (linearly) ordered group G is a q-group with $E(G) = G \sqcup \{0\}$.

Definition 8. Let F be a q-field, and let G be an ordered abelian group. We define a *valuation* of F to be a surjective morphism $\varphi\colon F^\times \twoheadrightarrow G$ of q-groups. We say that two valuations $\varphi\colon F^\times \twoheadrightarrow G$ and $\psi\colon F^\times \twoheadrightarrow H$ are *equivalent* if there exists an isomorphism $\omega\colon G \xrightarrow{\sim} H$ with $\psi = \omega\varphi$.

We show first that the classical concept of valuation is a special case. As usual, we extend the concept of valuation to maps $\varphi\colon F^\times \to G$ which need not be surjective, so that $\varphi\colon F^\times \twoheadrightarrow \varphi(F^\times)$ is a valuation in the strict sense.

Proposition 12. *Let $K|k$ be a field extension. A valuation of the q-field $F(K|k)$ is the same as a valuation of K which is trivial on k.*

Proof. Let $\varphi\colon F(K|k)^\times \twoheadrightarrow G$ be a q-field valuation. Define $v\colon K \to E(G)$ by $v(\alpha) := \varphi(k\alpha)$. Then $v(0) = 0$. For $\alpha, \beta \in K$ we have $\varphi(k(\alpha+\beta)) \leqslant \varphi(k\alpha + k\beta) = \varphi(k\alpha) \vee \varphi(k\beta)$ and $\varphi(k\alpha\beta) = \varphi(k\alpha k\beta) = \varphi(k\alpha)\varphi(k\beta)$. Hence v is a classical valuation. For $\alpha \in k^\times$, we have $\varphi(k\alpha) = 1$. Thus, v is trivial on k.

Conversely, let $v\colon K \twoheadrightarrow E(G)$ be a classical valuation which is trivial on k. For $a \in F(K|k)$ with $\dim_k a < \infty$, we define

$$\varphi(a) := \bigvee\{v(\alpha) \mid \alpha \in a\}.$$

Then $\varphi(a)\varphi(b) = \bigvee\{v(\alpha\beta) \mid \alpha \in a,\ \beta \in b\} = \bigvee\{v\big(\sum_{i=1}^n \alpha_i\beta_i\big) \mid \alpha_i \in a,\ \beta_i \in b\} = \bigvee\{v(\gamma) \mid \gamma \in ab\} = \varphi(ab)$. Furthermore, $\varphi(a) \vee \varphi(b) \leqslant \varphi(a \vee b)$ holds for $a, b \in E(F(K|k))$. Since any $\gamma \in a \vee b$ is of the form $\gamma = \alpha + \beta$ with $\alpha \in a$ and $\beta \in b$, we have $v(\gamma) \leqslant v(\alpha) \vee v(\beta) \leqslant \varphi(a) \vee \varphi(b)$. Thus $\varphi(a) \vee \varphi(b) = \varphi(a \vee b)$. This proves that $\varphi\colon F(K|k)^\times \twoheadrightarrow G$ is a morphism of q-groups. Since v is trivial on k, we get $\varphi(k\alpha) = v(\alpha)$ for all $\alpha \in K^\times$. Finally, any valuation $\varphi\colon F(K|k)^\times \twoheadrightarrow G$ with corresponding valuation $v\colon K \twoheadrightarrow E(G)$ satisfies $\bigvee\{v(\alpha) \mid \alpha \in a\} = \bigvee\{\varphi(k\alpha) \mid \alpha \in a\} = \varphi(a)$ for all $\alpha \in E(F(K|k))$. □

Proposition 13. *Let F be a q-field. There is a one-to-one correspondence between equivalence classes of valuations and prime orders of F.*

Proof. Let $\varphi\colon F^\times \twoheadrightarrow G$ be a valuation of F. Then $e := \bigvee\{\alpha \in F^\times \mid \varphi(\alpha) \leqslant 1\}$ is an order in F with $\alpha \leqslant e \Leftrightarrow \varphi(\alpha) \leqslant 1$ for all $\alpha \in F^\times$. Hence e is prime. Equivalent valuations φ give the same order e. Conversely, let e be a prime order. By Proposition 8, $G_e := \{e\alpha \mid \alpha \in F^\times\}$ is an ordered group, and $\varphi(\alpha) := e\alpha$ is a valuation $\varphi\colon F^\times \twoheadrightarrow G$. The corresponding prime order is $\bigvee\{\alpha \in F^\times \mid e\alpha \leqslant e\} = e$. On the other hand, let $e := \bigvee\{\alpha \in F^\times \mid \varphi(\alpha) \leqslant 1\}$ be the order of a valuation $\varphi\colon F^\times \twoheadrightarrow G$. Then $e\alpha = \widehat{\varphi}^{-1}(\varphi(\alpha))$ and $\varphi(\alpha) = \widehat{\varphi}(e\alpha)$. So the map $e\alpha \mapsto \varphi(\alpha)$ gives an equivalence between the valuations $\alpha \mapsto e\alpha$ and φ. □

In particular, the prime order 1 corresponds to the *trivial* valuation $\varphi\colon F^\times \to 1$. For a prime order e we call $G_e := \{e\alpha \,|\, \alpha \in F^\times\}$ the *value group* of e. So $\varphi_e(\alpha) := e\alpha$ is the corresponding valuation $\varphi_e\colon F^\times \twoheadrightarrow G_e$. Note that for an ordered q-group G, solid subgroups coincide with the convex subgroups of G.

Proposition 14. *For a prime order e of a q-field F, the map $H \mapsto \bigvee H$ is a bijection between the solid subgroups H of G_e and the overorders of e.*

Proof. Let H be a solid subgroup of G_e. Then $\bigvee H$ is an overorder of e. For an overorder $f \geqslant e$, the subgroup $H := \{e\alpha \mid \alpha \in f^\times\}$ of G_e is solid. Indeed, $e\alpha \vee e\alpha^{-1} \leqslant e\alpha_1 \vee \cdots \vee e\alpha_n$ with $e\alpha_i \in H$ gives $\alpha \vee \alpha^{-1} \leqslant f$. Thus $e\alpha \in H$. Furthermore, $\bigvee H \leqslant f$. If $\alpha \in [f] \smallsetminus [e]$, then $\alpha^{-1} \leqslant e \leqslant f$, which yields $\alpha \in f^\times$. Hence $e\alpha \in H$, and thus $\bigvee H = f$. Conversely, a solid subgroup H of G_e satisfies $\{e\alpha \mid \alpha \in (\bigvee H)^\times\} = H$. □

Corollary 1. *A prime order $e < 1$ of a q-field F is maximal if and only if G_e is isomorphic to an ordered subgroup of $\mathbb{R}_{>0}$.*

Proof. By Proposition 14, e is maximal if and only if G_e has no non-trivial solid subgroup $H \neq G_e$, that is, G_e is archimedean. So the corollary follows by Hölder's theorem [9]. □

By Corollary 1, we have

Corollary 2. *Let F be a q-field. There is a bijection between the equivalence classes of non-trivial valuations $\varphi\colon F^\times \to \mathbb{R}_{>0}$ and maximal prime orders $e < 1$ of F.*

For elements $a \geqslant b$ in a lattice, we write $\ell(a/b)$ for the length of the interval $[b, a]$. Now we turn our attention to the set $V(F)$ of all prime orders in a q-field F. The length of a maximal chain in $V(F)$ will be called the *dimension* $\dim F$ of F. Thus, $\dim F = 0$ if and only if $V(F) = \{1\}$. Let e be a local order in F with radical m. Then $e^\times = [e] \smallsetminus [m]$. If F is modular, Lemma 1 implies that the interval $\partial e := [m, e]$ is a q-field with $(\partial e)^\times = \{\alpha \vee m \mid \alpha \in e^\times\}$. The map $\alpha \mapsto \alpha \vee m$ is a group homomorphism $e^\times \twoheadrightarrow (\partial e)^\times$. We call ∂e the *residue q-field* of e. The length $d_e := \ell(e/m)$ will be called the *residue degree* of e.

Valuations exhibit a close interaction between q-fields and ordered groups, or in other words, between rational orders and prime orders. Indeed, any rational order e of a q-field F corresponds to a short exact sequence $\partial e \hookrightarrow F \twoheadrightarrow F/e$ of q-fields, with $F/e = \operatorname{Mod}(e)$, while a prime order e gives an ordered group G_e, with $\widehat{G_e} = \operatorname{Mod}(e)$.

Proposition 15. *Let e be a prime order in a q-field F. If ∂e is a q-field (e. g. if F is modular), then $V(\partial e) = \{f \in V(F) \mid f \leqslant e\}$.*

Proof. Let m be the radical of e. Any prime order $f \leqslant e$ satisfies $f \nleqslant m \in \operatorname{Mod}(f)$. Hence $m \leqslant f$, and thus $f \in \partial e$. For any $\alpha \in F^\times$ we have $\alpha \leqslant f \Leftrightarrow \alpha \vee m \leqslant f$. Hence $f \in V(F) \Leftrightarrow f \in V(\partial e)$. □

As an immediate consequence, we obtain

Corollary. *A prime order e in a modular q-field F is minimal if and only if* $\dim \partial e = 0$.

Proposition 16. *Let F be a q-field. For every prime order e in F there is a minimal prime order $f \leqslant e$ in F.*

Proof. Let $\mathscr{C}$ be a chain of prime orders $f \leqslant e$ in F. then $f_0 := \bigwedge \mathscr{C}$ is an order in F. Assume that $\alpha \in F^\times$ satisfies $\alpha \nleqslant f_0$. Then $\alpha \nleqslant f$ for some $f \in \mathscr{C}$. So $\alpha^{-1} \leqslant f'$ for all $f' \leqslant f$ in $\mathscr{C}$. Hence $\alpha^{-1} \leqslant f_0$, which shows that f_0 is a prime order. Now Zorn's lemma completes the proof. □

Proposition 17. *Let e be an order in a modular q-field F. A unit $\alpha \in F^\times$ is integral over e if and only if $\alpha \leqslant e'$ for every prime overorder e' of e.*

Proof. Assume first that α is integral over e, and let e' be a prime overorder of e. Suppose that $\alpha \nleqslant e'$. Then $\alpha^{-1} \leqslant e'$, which yields $\alpha \in e[\alpha^{-1}] \leqslant e'$, a contradiction. Whence $\alpha \leqslant e'$.

Conversely, assume that α is not integral over e. Then $e[\alpha^{-1}]$ is an overorder of e with $\alpha \nleqslant e[\alpha^{-1}]$. Hence $\varepsilon \nleqslant \alpha^{-1}e[\alpha^{-1}]$. By Lemma 2, there is a maximal $e[\alpha^{-1}]$-module $m < e[\alpha^{-1}]$ with $\alpha^{-1} \leqslant m$. Proposition 9 implies that $e[\alpha^{-1}]_m$ is a local order, and Proposition 11 yields a prime overorder e' of $e[\alpha^{-1}]_m$ with radical m' such that $m \leqslant m'$. Hence $e'\alpha^{-1} \leqslant m' < e'$, which yields $\alpha \nleqslant e'$. □

Corollary 1. *Let e be an order in a modular q-field F. The units $\alpha \in F^\times$ which are integral over e form an overorder $\overline{e}$ of e.*

Proof. Indeed, $\overline{e} = \bigwedge\{e' \in V(F) \mid e' \geqslant e\}$. □

We call $\overline{\varepsilon} = \bigwedge V(F)$ the *integral closure* of e. In particular, Corollary 1 yields

Corollary 2. *Let F be a modular q-field. Then* $\dim F = 0$ *if and only if* $\overline{\varepsilon} = 1$.

Corollary 3. *Let e be a rational order in a modular q-field F. Then $\overline{e}$ is the greatest overorder of e which is algebraic over e.*

For a q-field F, let $X(F)$ denote the set of minimal prime orders in F. The valuations $\varphi_e \colon F^\times \twoheadrightarrow G_e$ with $\varphi_e(\alpha) := e\alpha$ combine to a morphism of q-groups

$$\varphi\colon F^\times \longrightarrow \prod_{e\in X(F)} G_e. \tag{2}$$

Corollary 4. *Let F be a modular q-field. The map* (2) *is injective if and only if* $\overline{\varepsilon} = \varepsilon$. *If* (2) *is injective, the image $\varphi(F^\times)$ is an antichain in $\prod_{e\in X(F)} G_e$.*

Proof. Let $\alpha \in F^\times$ be a unit with $\alpha \leqslant \overline{\varepsilon}$. By Corollary 2 of Proposition 7, $\overline{\varepsilon}$ is rational. Hence $\alpha^{-1} \leqslant \overline{\varepsilon}$, and thus $\overline{\varepsilon}\alpha = \overline{\varepsilon}$. So the kernel of φ consists of the units $\alpha \in F^\times$ with $\alpha \leqslant \overline{\varepsilon}$. □

Lemma 5. *Let F be a finite dimensional modular q-field with a rational order e. Then*

$$\dim F \geqslant \dim \partial e + \dim F/e.$$

Proof. Let $e_1 < \cdots < e_n < e$ be a chain of prime orders in ∂e, and let $f_1 < \cdots < f_m < 1$ be a chain of prime orders in F with $e \leqslant f_1$. If m_1 denotes the radical of f_1, then $e \wedge m_1 = 0$. Indeed, $\alpha \leqslant e \wedge m_1$ implies that $\alpha^{-1} \leqslant e \leqslant f_1$ and $\alpha^{-1} \nleqslant f_1$, a contradiction. Thus ∂e embeds into ∂f_1. So $e \vee m_1$ is a rational order in ∂f_1, and the $e_i \vee m_1$ form a chain of prime orders in $e \vee m_1$. By Proposition 11, there is a prime overorder e_1' of $e_1 \vee m_1$ in ∂f_1 with radical m_1' such that $m_1' \wedge e_1$ is the radical of e_1. So the $e_i e_1'$ form a chain of prime orders in ∂f_1. By Proposition 15, $e_1' \leqslant e_2 e_1' \leqslant \cdots \leqslant e_n e_1' \leqslant f_1 < \cdots < f_m < 1$ is a chain of prime orders in F. If $\alpha \in [e \wedge e_1'] \smallsetminus [e_1]$, then $\alpha^{-1} \leqslant m_1' \wedge e_1 \leqslant m_1'$, which gives $\alpha \nleqslant e_1'$, a contradiction. Thus $e \wedge e_1' = e_1$, which implies that $e_1' < e_2 e_1' < \cdots < e_n e_1' < f_1$. □

Lemma 6. *Let F be a modular q-field with prime orders $e < f$. Then $\varepsilon(\alpha) \leqslant f$ holds for every $\alpha \in [f] \smallsetminus [e]$.*

Proof. Let m denote the radical of e. Then $\alpha \notin [e]$ implies that $\alpha^{-1} \leqslant m$. Every element of $[\varepsilon(\alpha)]$ is of the form $\beta\gamma^{-1}$ with $\beta \leqslant \varepsilon[\alpha]$ and $\gamma \leqslant \alpha^r \vee \cdots \vee \alpha^s$ for some $r, s \in \mathbb{N}$ with $s - r \in \mathbb{N}$ minimal. By Lemma 1, this implies that $\alpha^s \leqslant \alpha^r \vee \cdots \vee \alpha^{s-1} \vee \gamma$. Hence $\varepsilon \leqslant \alpha^{r-s} \vee \cdots \vee \alpha^{-1} \vee \gamma\alpha^{-s} \leqslant m \vee \gamma\alpha^{-s}$, and thus $\gamma\alpha^{-s} \nleqslant m$, which yields $\gamma^{-1}\alpha^s \leqslant e$. So we obtain $\gamma^{-1} = \gamma^{-1}\alpha^s \cdot \alpha^{-s} \leqslant e$, which proves that $\varepsilon(\alpha) \leqslant f$. □

Definition 9. We define the *transcendence degree* $\operatorname{trd} F$ of a finite dimensional q-field F to be the maximum number of units $\alpha_1, \ldots, \alpha_n \in F^\times$ with α_i transcendental over $\varepsilon(\alpha_1, \ldots, \alpha_{i-1})$, for $i \in \{1, \ldots, n\}$. We call $(\alpha_1, \ldots, \alpha_n)$ a *transcendence basis* of F.

Proposition 18. *Let F be a finite dimensional modular q-field. Then* $\operatorname{trd} F = \dim F$. *Furthermore, all transcendence bases of F have the same cardinality and are invariant under permutation.*

Proof. For any unit $\alpha \in F^\times \smallsetminus \overline{\varepsilon}$, Lemma 5 gives $\dim F \geqslant \dim \partial(\varepsilon(\alpha)) + \dim F/\varepsilon(\alpha)$. By Corollary 2 of Proposition 17, and induction, this yields $\dim F \geqslant \operatorname{trd} F$. Let $e_1 < \cdots < e_{n+1} = 1$ be a maximal chain of prime orders in F. If $n = 0$, then $\overline{\varepsilon} = 1$, and thus $\operatorname{trd} F = 0$. Otherwise, we choose $\alpha \in [e_2] \smallsetminus [e_1]$. By Lemma 6, we have $\varepsilon(\alpha) \leqslant e_2$. Since $\alpha \nleqslant e_1$, Proposition 17 yields $\alpha \notin \overline{\varepsilon}$. Hence, by induction, we can assume that $\dim F = 1 + \dim F/\varepsilon(\alpha) = 1 + \operatorname{trd} F/\varepsilon(\alpha) \leqslant \operatorname{trd} F$. □

Corollary. *Let F be a finite dimensional modular q-field with a rational order e. Then*

$$\dim F = \dim \partial e + \dim F/e.$$

Proof. If $\dim \partial e = 0$, then $e \leqslant \overline{\varepsilon}$. Thus, Proposition 17 implies that every prime order in F contains e, from which the equation follows. Otherwise, we have $\varepsilon(\alpha_1) \leqslant e$ for some $\alpha_1 \nleqslant \overline{\varepsilon}$. By induction, we can assume that $\dim F/\varepsilon(\alpha_1) = \dim \partial\big(e/\varepsilon(\alpha_1)\big) + \dim F/e$. Extending α_1 to a transcendence basis of F, we obtain $\dim F = 1 + \dim F/\varepsilon(\alpha_1) = 1 + \dim \partial\big(e/\varepsilon(\alpha_1)\big) + \dim F/e = \dim \partial e + \dim F/e$. □

4 The connected components of a modular q-field

Let F be a q-field F. We say that $S \subset V(F)$ is *strongly independent* if for any finite collection $e_1, \ldots, e_n \in S$, together with $x_i \in G_{e_i}$ and $\alpha_i \in F^\times$, there is a unit $\alpha \in F^\times$ with $\alpha \vee x_i = \alpha_i \vee x_i$ for all i. For $x_i < e_i\alpha_i$, this implies that $e_i\alpha = e_i\alpha_i$ for all i. A necessary condition for this is that any pair of distinct orders $e, f \in S$ is *independent* in the sense that $ef = 1$ (cf. [3], VI.7.2, Definition 1). Indeed, if there exists a unit $\alpha \notin [ef]$, there is no $\gamma \in F^\times$ with $e\gamma = e\alpha$ and $f\gamma = f\varepsilon$.

Definition 10. Let L be a modular lattice. We call a set S of atoms *independent* if every n-element subset $S' \subset S$ satisfies $h(\bigvee S') = n$. We call $S = \{\alpha_0, \cdots, \alpha_n\}$ with $n \geqslant 2$ a *circuit* if every proper subset is independent and $h(\alpha_0 \vee \cdots \vee \alpha_n) = n$. We say two atoms $\alpha, \beta \in L$ are *connected* if either $\alpha = \beta$ or there is a circuit $\{\gamma, \alpha, \beta\}$ in L.

So the relation of connectedness is reflexive and symmetric.

Proposition 19. *Connectedness in a modular lattice is transitive.*

Proof. Let $\{\beta, \alpha_1, \alpha_2\}$ and $\{\gamma, \alpha_2, \alpha_3\}$ be distinct circuits with $\alpha_2 \nleqslant \alpha_1 \vee \alpha_3$. Then $\alpha_1 \vee \alpha_3 \neq \beta \vee \gamma$. (Otherwise, $\alpha_2 \leqslant \alpha_1 \vee \beta \leqslant \alpha_1 \vee \alpha_3$, which is impossible.) By the modularity, $\beta \vee \big((\alpha_1 \vee \alpha_3) \wedge (\beta \vee \gamma)\big) = (\beta \vee \alpha_1 \vee \alpha_3) \wedge (\beta \vee \gamma) = \beta \vee \gamma > \beta$. (If $\beta = \gamma$, then $\alpha_3 \leqslant \alpha_2 \vee \gamma = \alpha_2 \vee \beta = \alpha_1 \vee \alpha_2$, which yields $\alpha_2 \leqslant \alpha_1 \vee \alpha_3$ by Lemma 1.) Hence $\delta := (\alpha_1 \vee \alpha_3) \wedge (\beta \vee \gamma)$ is of height 1, with $\delta \leqslant \alpha_1 \vee \alpha_3$. ☐

By Proposition 19, the unit group $F^\times$ of a modular q-field F splits into classes of pairwise connected units. We call them the *connected components* of $F^\times$. If $F^\times$ consists of a single component, we say that F is *connected*.

Lemma 7. *Let F be a connected modular q-field. If $\alpha_1, \ldots, \alpha_n \in F^\times$ are independent with $n \geqslant 3$, there exists a circuit $\{\alpha_0, \ldots, \alpha_n\}$ in $F^\times$.*

Proof. By induction, we can assume that there is a circuit $\{\beta, \alpha_2, \ldots, \alpha_n\}$ in $F^\times$. So there is a circuit $\{\alpha_0, \alpha_1, \beta\}$. Hence $\alpha_0 \leqslant \alpha_1 \vee \beta \leqslant \alpha_1 \vee \cdots \vee \alpha_n$. If $\alpha_0 \leqslant \alpha_2 \vee \cdots \vee \alpha_n$, then $\alpha_1 \leqslant \alpha_0 \vee \beta \leqslant \alpha_2 \vee \cdots \vee \alpha_n$, which is impossible. Suppose that $\alpha_0 \leqslant \alpha_1 \vee \cdots \vee \alpha_{n-1}$. Then $\beta \leqslant \alpha_0 \vee \alpha_1 \leqslant \alpha_1 \vee \cdots \vee \alpha_{n-1}$ and $\beta \leqslant \alpha_2 \vee \cdots \vee \alpha_n$, which yields $\beta \leqslant \alpha_2 \vee \cdots \vee \alpha_{n-1}$, a contradiction. By symmetry, this proves that $\{\alpha_0, \ldots, \alpha_n\}$ is a circuit. ☐

Lemma 8. *Let F be a modular q-field, and let $\{\alpha_0, \ldots, \alpha_n\}$ be a circuit. Assume that a prime order $e \in V(F)$ satisfies $e\alpha_1 > e\alpha_i$ for all $i > 1$. Then $e\alpha_0 = e\alpha_1$.*

Proof. By Lemma 1, $e\alpha_0 \leqslant e\alpha_1 \vee \cdots \vee e\alpha_n = e\alpha_1 \leqslant e\alpha_0 \vee e\alpha_2 \vee \cdots \vee e\alpha_n$. Hence $e\alpha_0 = e\alpha_1$. ☐

Theorem 1. *Let F be a connected modular q-field. Then $V(F)$ is strongly independent if and only if* $\dim F \leqslant 1$.

Proof. If $\dim F = 0$, there is nothing to prove. If $V(F)$ is strongly independent, any distinct $e, f \in V(F)$ satisfy $ef = 1$. So the prime orders $e < 1$ are pairwise incomparable, which shows that $\dim F \leqslant 1$. Conversely, assume that $\dim F = 1$. Since $G_1 = \{1\}$, it is enough to verify that $X(F)$ is strongly independent.

Let $e_1, \ldots, e_n \in X(F)$ be pairwise distinct, and let $x_i \in G_{e_i}$ and $\alpha_i \in F^\times$ with $\alpha_i \nleqslant x_i$ be given. We show first that there is a unit $\gamma \in F^\times$ with $e_1\gamma < e_1$ and $\gamma \nleqslant e_i$

for all $i > 1$. For $n = 1$, this is trivial. Assume that $n = 2$. Then there are units $\alpha \in [e_1] \smallsetminus [e_2]$ and $\beta \in [e_2] \smallsetminus [e_1]$. Hence $\gamma := \alpha\beta^{-1}$ meets the requirement. Now assume that $n \geqslant 3$, and that a unit α with $e_1\alpha < e_1$ and $\alpha \nleqslant e_i$ for $1 < i < n$ has been found. If $\alpha \nleqslant e_n$, we are done. So let us assume that $\alpha \leqslant e_n$. Choose a unit $\beta \in F^\times$ with $e_1\beta < e_1$ and $\beta \nleqslant e_n$. By Corollary 1 of Proposition 14, there is an integer $r > 0$ with $e_i\alpha^r > e_i\beta$ for $i \in \{2, \ldots, n-1\}$. Choose a circuit $\{\gamma, \alpha^r, \beta\}$. Then $e_1\gamma \leqslant e_1(\alpha^r \vee \beta) < e_1$, and Lemma 8 gives $e_i\gamma = e_i\alpha^r > e_i$ for $1 < i < n$, and $e_n\gamma = e_n\beta > e_n$.

Replacing γ by a sufficiently large power γ^j, we can assume that $\gamma\alpha_1 \leqslant x_1$. Choose a circuit $\{\gamma_0, \varepsilon, \gamma\}$. Then $\gamma_0 \leqslant \varepsilon \vee \gamma \leqslant \varepsilon \vee x_1\alpha_1^{-1}$ and $\varepsilon \leqslant \gamma_0 \vee \gamma \leqslant \gamma_0 \vee x_1\alpha_1^{-1}$. Hence $\gamma_0 \vee x_1\alpha_1^{-1} = \varepsilon \vee x_1\alpha_1^{-1}$. Since $\alpha_1 \nleqslant x_1$, this implies that $e_1\gamma_0 = e_1$. Therefore, multiplying the equation with $\gamma_1 := \gamma_0^{-1}$ gives $\varepsilon \vee x_1\alpha_1^{-1} = \gamma_1 \vee x_1\alpha_1^{-1}$. Application of Lemma 8 to the circuit $\{\gamma_0, \varepsilon, \gamma\}$ yields $e_i\gamma_0 = e_i\gamma > e_i$ for all $i > 1$. Hence $e_i\gamma_1 < e_i$ for $i > 1$. Similarly, we find $\gamma_i \in F^\times$ with $\gamma_i \vee x_i\alpha_i^{-1} = \varepsilon \vee x_i\alpha_i^{-1}$ and $e_j\gamma_i < e_j$ for $j \neq i$.

For a sufficiently large integer s, we have $e_j\alpha_i\gamma_i^s < e_j\alpha_j \wedge x_j$ for $j \neq i$. Hence $e_j\alpha_i\gamma_i^s < e_j\alpha_j = e_j\alpha_j\gamma_j^s$ whenever $j \neq i$. Let us show that $\{\alpha_1\gamma_1^s, \ldots, \alpha_n\gamma_n^s\}$ is independent. Suppose that $\alpha_1\gamma_1^s \leqslant \alpha_2\gamma_2^s \vee \cdots \vee \alpha_n\gamma_n^s$. Then $e_1\alpha_1\gamma_1^s \leqslant e_1\alpha_2\gamma_2^s \vee \cdots \vee e_1\alpha_n\gamma_n^s < e_1\alpha_1\gamma_1^s$, a contradiction. So Lemma 7 gives a circuit $\{\alpha, \alpha_1\gamma_1^s, \ldots, \alpha_n\gamma_n^s\}$. Since $e_i\gamma_i = e_i$, we have $\alpha_i\gamma_i^s \nleqslant x_i$ and $\alpha_i\gamma_i^s \leqslant x_j$ for $j \neq i$. Hence $\alpha \nleqslant x_i$ for all i. Furthermore, $\alpha \vee x_i \leqslant \alpha_1\gamma_1^s \vee \cdots \vee \alpha_n\gamma_n^s \vee x_i = \alpha_i\gamma_i^s \vee x_i = \alpha_i(\varepsilon \vee x_i\alpha_i^{-1})^s \vee x_i = \alpha_i(\varepsilon \vee x_i\alpha_i^{-1}) \vee x_i = \alpha_i \vee x_i$ for all i. Thus $\alpha \vee x_i = \alpha_i \vee x_i$. □

Definition 11. We define a *function q-field* to be a one-dimensional connected modular q-field F with $\overline{\varepsilon} = \varepsilon$ which is *finitely generated* over ε, that is, $1 = \varepsilon(\alpha_1, \ldots, \alpha_n)$ for some $\alpha_1, \ldots, \alpha_n \in F^\times$. We say that F has *enough constants* if the residue degree $d_e = 1$ for all $e \in X(F)$.

Example 9. Let $K|k$ be a finitely generated field extension of transcendence degree 1, with k algebraically closed in K (i. e. any $\alpha \in K \smallsetminus k$ is transcendental over k). Then $F(K|k)$ is a function q-field. It has enough constants if and only if k is algebraically closed. Indeed, the residue field of any non-trivial valuation ring $V \supset k$ of K is a finite extension of k. Thus $F(K|k)$ has enough constants if k is algebraically closed. Conversely, assume that $F(K|k)$ has enough constants. Choose $x \in K \smallsetminus k$. Then the field $k(x)$ is of finite index in K. Let $f \in k[x]$ be irreducible. Then $k[x]/(f)$ is the residue class field of the localization $k[x]_{(f)}$. Since every non-trivial valuation of K has residue class field k, this implies that f must be linear. So k is algebraically closed.

Recall that a lattice is said to be *atomistic* if each of its elements is a join of atoms.

Lemma 9. *Let L be an atomistic modular lattice. For $a, b > 0$ and an atom γ with $\gamma \leqslant a \vee b$ there are atoms $\alpha, \beta \in L$ with $\alpha \leqslant a$ and $\beta \leqslant b$ such that $\gamma \leqslant \alpha \vee \beta$.*

Proof. Without loss of generality, we can assume that $\gamma \nleqslant a$. Then $a \vee \big(b \wedge (\gamma \vee a)\big) = (a \vee b) \wedge (\gamma \vee a) = \gamma \vee a$ implies that $b \wedge (\gamma \vee a) \nleqslant a$. So there is an atom $\beta \leqslant b \wedge (\gamma \vee a)$ with $\beta \nleqslant a$. By Lemma 1, $\gamma \leqslant a \vee \beta$. Now the result follows by symmetry. □

Proposition 20. *Let F be a modular q-field with $e \in V(F)$ and $\alpha \in F^\times$ transcendental over ε. Let (α) denote the prime ideal $\varepsilon[\alpha]\alpha$ of $\varepsilon[\alpha]$. Then $e\alpha < e$ if and only if $\varepsilon(\alpha) \wedge e = \varepsilon[\alpha]_{(\alpha)}$. If $e\alpha < e$, then $\varepsilon[\alpha]_{(\alpha)} \vee e\alpha = \varepsilon \vee e\alpha$.*

Proof. Assume that $e\alpha < e$. Every $\delta \in [\varepsilon(\alpha)]$ is of the form $\delta = \beta\gamma^{-1}\alpha^n$ with $\beta, \gamma \leqslant \varepsilon[\alpha]$ and $\beta, \gamma \nleqslant (\alpha)$, and $n \in \mathbb{Z}$. Moreover, $\delta \leqslant \varepsilon[\alpha]_{(\alpha)}$ if and only if $n \geqslant 0$. Assume that $\delta \leqslant \varepsilon(\alpha) \wedge e$. Then $\beta \vee (\alpha) = \varepsilon[\alpha]$ implies that $e\beta = e$. Similarly, $e\gamma = e$. Hence $e\alpha^n = e\delta \leqslant e$ gives $n \geqslant 0$. Thus $\delta \leqslant \varepsilon[\alpha]_{(\alpha)}$. Conversely, $n \geqslant 0$ implies that $\delta \leqslant e \wedge \varepsilon(\alpha)$.

Now assume that $\varepsilon(\alpha) \wedge e = \varepsilon[\alpha]_{(\alpha)}$. Suppose that $e\alpha \geqslant e$. Then $\alpha^{-1} \leqslant \varepsilon(\alpha) \wedge e = \varepsilon[\alpha]_{(\alpha)}$. So there are $\beta, \gamma \in F^\times$ with $\beta, \gamma \leqslant \varepsilon[\alpha]$ and $\gamma \nleqslant (\alpha)$ such that $\alpha^{-1} = \beta\gamma^{-1}$. Hence $\gamma = \beta\alpha \leqslant \varepsilon[\alpha]\alpha$, a contradiction. Thus $e\alpha < e$. Every unit $\leqslant \varepsilon[\alpha]_{(\alpha)}$ is of the form $\beta\gamma^{-1}$ with $\beta, \gamma \leqslant \varepsilon[\alpha]$ and $\gamma \nleqslant (\alpha)$. Hence $\gamma \vee e\alpha = \varepsilon \vee e\alpha$ and $e\gamma = e$. Multiplication with γ^{-1} gives $\varepsilon \vee e\alpha = \gamma^{-1} \vee e\alpha$. Thus $\beta\gamma^{-1} \vee e\alpha \leqslant (\varepsilon \vee e\alpha)\gamma^{-1} \vee e\alpha \leqslant \gamma^{-1} \vee e\alpha = \varepsilon \vee e\alpha \leqslant \varepsilon[\alpha]_{(\alpha)} \vee e\alpha$. □

Theorem 2. *Let F be a function q-field. Then every $e \in X(F)$ is discrete, $X(F)$ is infinite, and every $\alpha \in F^\times$ satisfies $e\alpha = e$ for almost all $e \in X(F)$. Furthermore, $\varphi_e([\varepsilon(\alpha)]) = G_e e\alpha$ holds for all $\alpha \in F^\times$ and $e \in X(F)$ with $e\alpha \neq e$, and $F/\varepsilon(\alpha)$ is of finite length $\ell \geqslant \sum_{e\in X(F)} \ell(e/e \wedge e\alpha)$.*

Proof. Let $e \in X(F)$ and $\alpha \in F^\times$ with $e\alpha > e$ be given. For any $\beta \leqslant \varepsilon[\alpha]$, there is a minimal $n \in \mathbb{N}$ with $\beta \leqslant \varepsilon \vee \alpha \vee \cdots \vee \alpha^n$. Hence $e\beta \leqslant e\alpha^n$. By Lemma 1, $\alpha^n \leqslant \varepsilon \vee \alpha \vee \cdots \vee \alpha^{n-1} \vee \beta$, which yields $e\beta = e\alpha^n$. Thus $\varphi_e([\varepsilon(\alpha)]) = G_e e\alpha$. If $e\alpha < e$, then $e\alpha^{-1} > e$, which leads to the same result.

Since F is finitely generated, Proposition 18 implies that $F/\varepsilon(\alpha)$ is of finite length. Assume that $e_i\alpha < e_i$ holds for distinct $e_1, \ldots, e_n \in X(F)$. With $\ell_i := \ell(e_i/e_i\alpha)$, the strong independence of $X(F)$ implies that there are $\alpha_{i,1}, \ldots, \alpha_{i,\ell_i} \in F^\times$ for each $i \in \{1, \ldots, n\}$ such that $e_i\alpha \vee \alpha_{i,1} \vee \cdots \vee \alpha_{i,\ell_i} = e_i$ and $\alpha_{i,k} \leqslant e_j\alpha$ for all $j \neq i$ and $k \in \{1, \ldots, \ell_i\}$. In particular, the $a_i := \alpha_{i,1} \vee \cdots \vee \alpha_{i,\ell_i}$ satisfy

$$e_i\alpha \vee a_i = e_i, \qquad e_i\alpha \wedge a_i = 0.$$

To prove that $\ell \geqslant \sum_{e\in X(F)} \ell(e/e \wedge e\alpha)$, we have to verify that the $\varepsilon(\alpha)\alpha_{i,k} \in F/\varepsilon(\alpha)$ are independent. Let I be the set of all (i,k) with $i \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, \ell_i\}$. Suppose that there is a circuit $\{\varepsilon(\alpha)\alpha_{i,k} \mid (i,k) \in J\}$ for some $J \subset I$. For any $(i,k) \in J$ and $J' := J \smallsetminus \{(i,k)\}$, this implies that $\alpha_{i,k} \leqslant \bigvee\{\varepsilon(\alpha)\alpha_{j,s} \mid (j,s) \in J'\}$. By Lemma 9, there are units $\beta_{j,s} \leqslant \varepsilon(\alpha)$ with $\alpha_{i,k} \leqslant \bigvee\{\beta_{j,s}\alpha_{j,s} \mid (j,s) \in J'\}$. We set $\beta_{i,k} := \varepsilon$. Then the $\beta_{j,s}\alpha_{j,s}$ with $(j,s) \in J$ form a circuit in F. By Proposition 20, the prime order $e := \varepsilon(\alpha) \wedge e_i$ in $\varepsilon(\alpha)$ does not depend on $i \in \{1, \ldots, n\}$. Choose $(j,s) \in J$ with $e\beta_{j,s} \in G_e$ maximal. Multiplying the β's with $\beta_{j,s}^{-1}$ and replacing (j,s) by (i,k), we can assume that $\beta_{i,k} = \varepsilon$ and $\beta_{j,s} \leqslant e\beta_{i,k} = e$ for all $(j,s) \in J$. By Proposition 20, $e \vee e_i\alpha = \varepsilon \vee e_i\alpha$. Let J_0 be the set of all $(i,s) \in J'$ with $\beta_{i,s} \nleqslant e_i\alpha$. Then $\beta_{i,s} \vee e_i\alpha = \varepsilon \vee e_i\alpha$ for

$(i,s)\in J_0$. Hence $\alpha_{i,k}\vee e_i\alpha \leqslant \bigvee\{\beta_{i,s}\alpha_{i,s}\mid (i,s)\in J_0\}\vee e_i\alpha = \bigvee\{\alpha_{i,s}\mid (i,s)\in J_0\}\vee e_i\alpha$. Thus

$$\begin{aligned}\alpha_{i,k} &= \alpha_{i,k}\vee(e_i\alpha\wedge a_i) = (\alpha_{i,k}\vee e_i\alpha)\wedge a_i \leqslant \big(\bigvee\{\alpha_{i,s}\mid (i,s)\in J_0\}\vee e_i\alpha\big)\wedge a_i \\ &= \bigvee\{\alpha_{i,s}\mid (i,s)\in J_0\}\vee(e_i\alpha\wedge a_i) = \bigvee\{\alpha_{i,s}\mid (i,s)\in J_0\},\end{aligned}$$

a contradiction.

In particular, the inequality $\ell \geqslant \sum_{e\in X(F)}\ell(e/e\wedge e\alpha)$ shows that each $\alpha\in F^\times$ satisfies $e\alpha = e$ for almost all $e\in X(F)$. Since every $e\in X(F)$ satisfies $e\alpha < e$ for some $\alpha\in F^\times$, the finiteness of $\ell(e/e\alpha)$ implies that e is discrete.

Finally, suppose that $X(F)$ is finite. By the strong independence of $X(F)$, there is a unit $\alpha\in F^\times$ with $e\alpha < e$ for all $e\in X(F)$. Then $\alpha \leqslant \bigwedge X(F) = \varepsilon$, which is impossible. So $X(F)$ must be infinite. □

Let F be a function q-field. By Theorem 2, the morphism $F^\times \hookrightarrow \prod_{e\in X(F)} G_e$ has its image in the free abelian group

$$J_F := \coprod_{e\in X(F)} G_e.$$

The elements of J_F will be called *replete ideals* of F (cf. [21], III.1.4). The replete ideals form a group with unit element $e_F := (e)_{e\in X(F)}$. By Proposition 8, the radical of each $e\in X(F)$ is of the form $e\pi$ for some $\pi\in F^\times$. Accordingly, there is a replete ideal $\widehat{e}\in J_F$ with components $\widehat{e}_e = e\pi^{-1}$ and $\widehat{e}_f = f$ for all $f\in X(F)\smallsetminus\{e\}$. So the $\widehat{e}$ form a basis of J_F. Every $\xi\in J_F$ has a *positive part* $\xi^+ := \xi\vee e_F$ and a *negative part* $\xi^- := \xi^+\xi^{-1}$. We call $\deg\xi := \ell(\xi^+/e_F) - \ell(\xi^-/e_F)$ the *degree* of ξ.

Corollary. *Let F be a function q-field. Then* $\deg\xi < \infty$ *for all* $\xi\in J_F$.

Proof. It suffices to prove that $\deg\widehat{e}$ is finite for each $e\in X(F)$. Let $\pi\in F^\times$ be a unit with $e\pi < e$ maximal as an e-module. Then Theorem 2 implies that $\varphi_e([\varepsilon(\pi)]) = G_e(e\pi)$. Thus $\deg\widehat{e} = \ell(e\pi^{-1}/e) = \ell(e/e\pi) \leqslant \ell(F/\varepsilon(\pi)) < \infty$. □

5 Structure of modular connected q-fields

Let F be a connected modular q-field of length $\geqslant 4$. By a theorem of Frink ([11], Theorem 7), the lattice F is isomorphic to the lattice of subspaces of a vector space K over a skew-field k. Each $\alpha\in F^\times$ defines a lattice automorphism $a\mapsto \alpha a$ of F. By the fundamental theorem of projective geometry [27], it is induced by a semilinear automorphism of K, that is, an additive bijection $f\colon K\to K$ satisfying $f(\lambda x) = \gamma_f(\lambda)f(x)$ for $x\in K$ and $\lambda\in k$, with an automorphism γ_f of k. The group of semilinear maps is denoted by $\Gamma\mathrm{L}(K)$. Note that f is determined up to a constant in $k^\times$. So we have a group representation

$$\overline{\rho}\colon F^\times \longrightarrow \mathrm{P}\Gamma\mathrm{L}(K)$$

into the *projective semilinear group* $\mathrm{P\Gamma L}(K) := \Gamma\mathrm{L}(K)/k^\times$. Consider the pullback of groups

$$\begin{array}{ccccc} k^\times & \hookrightarrow & K^\times & \xrightarrow{\sigma} & F^\times \\ \| & & \downarrow \rho & \text{PB} & \downarrow \overline{\rho} \\ k^\times & \xhookrightarrow{\iota} & \Gamma\mathrm{L}(K) & \xrightarrow{\pi} & \mathrm{P\Gamma L}(K). \end{array}$$

For $\lambda \in k^\times$, the semilinear map $\iota(\lambda)$ is the left multiplication $x \mapsto \lambda x$. The corresponding automorphism of k is the conjugation with λ. Let us fix a generator $1 \in K$ of the unit element $\varepsilon \in F^\times$. Thus $\varepsilon = k1$. The map ρ defines an action

$$K^\times \times K \longrightarrow K,$$

given by $(x, y) \mapsto x \cdot y := \rho(x)(y)$. So we have

$$\sigma(x) \cdot ky = k(x \cdot y).$$

In particular, $\sigma(x) = k(x \cdot 1)$. Since $\sigma(x)$ determines the semilinear map $\rho(x)$ up to a scalar in k, the pair $\big(\sigma(x), \rho(x)\big)$ and thus x is uniquely determined by $x \cdot 1 \in K \smallsetminus \{0\}$. So the map $x \mapsto x \cdot 1$ gives a natural embedding

$$K^\times \hookrightarrow K.$$

Conversely, each non-zero $y \in K$ gives rise to a unit $ky \in F^\times$. Choose $f \in \Gamma\mathrm{L}(K)$ with $\pi(f) = \overline{\rho}(ky)$ and $f(1) = y$. Then there is an element $x \in K^\times$ with $\rho(x) = f$ and $\sigma(x) = ky$. Thus $x \cdot 1 = f(1) = y$. So the image of $K^\times \hookrightarrow K$ is $K \smallsetminus \{0\}$, which justifies the notation. Identifying $K^\times$ with $K \smallsetminus \{0\}$, the action $K^\times \times K \to K$ extends to a multiplication on K with $0 \cdot x := 0$ and $x \cdot 1 = x$ for all $x \in K$. In particular, $1 \cdot 1 = 1$ and $\sigma(1) = \varepsilon$ implies that $\rho(1)$ is the identity map. Hence $1 \cdot x = x$ for all $x \in K$. Thus K is a monoid with subgroup $K^\times$. Furthermore, K is left distributive:

$$x(y + z) = xy + xz.$$

Recall that a group N with a left distributive associative multiplication is said to be a (left) *near-ring* [24]. If the multiplicative semigroup has a unit element, N is said to be *unital*. If it is a group, N is called a *near-field*. Thus K is a near-field. Note that the additive group of every near-field is abelian [22, 34, 33].

For $\lambda \in k^\times$ and $x, y \in K^\times$, we have $\rho(\lambda x) = \rho(\lambda)\rho(x) = \iota(\lambda)\rho(x)$. Hence

$$\lambda x \cdot y = \lambda(x \cdot y)$$

for all $y \in K$. In particular, $\lambda 1 \cdot y = \lambda y$, that is, $\rho(\lambda 1) = \iota(\lambda)$. So the map $\lambda \mapsto \lambda 1$ embeds k as a sub-skewfield of K, and $k^\times$ is a normal subgroup of $K^\times$.

On the other hand, the semilinearity of the left multiplication in K yields

$$x \cdot \lambda y = \gamma_x(\lambda)(x \cdot y)$$

with $x, y \in K$ and $\lambda, \gamma_x(\lambda) \in k$. We set $\gamma_0 := 0$ and $\gamma_x(0) := 0$. Since $\gamma_{xy}(\lambda)xy \cdot z = xy \cdot \lambda z = x \cdot \gamma_y(\lambda)yz = \gamma_x\gamma_y(\lambda)xyz$, it follows that γ is a monoid homomorphism

$$\gamma \colon K \longrightarrow \mathrm{End}(k).$$

With $y = 1$, the equation $x \cdot \lambda y = \gamma_x(\lambda)(x \cdot y)$ gives the commutation rule

$$x \cdot \lambda = \gamma_x(\lambda) \cdot x.$$

Using ideas of Karzel [17, 18], we show now that K has to be a field.

Theorem 3. *Let F be a connected modular q-field of length $\geqslant 4$. Then $F = F(K|k)$ with a field extension $K|k$.*

Proof. For $x, y \in K^\times$, we have $kxy = kx \cdot ky = ky \cdot kx = kyx$. So there is a map $\lambda \colon K^\times \times K^\times \to k^\times$ (a 2-cocycle) with $xy = \lambda(x, y)yx$. Thus $xyx^{-1}y^{-1} = \lambda(x, y) \in k^\times$. We will show that K is commutative.

Assume that $\dim(k+kx+ky+kxy) = 4$. Then $(1+x)(1+y) = (1+x)1+(1+x)y = 1 + x + \lambda(1 + x, y)y(1 + x) = 1 + x + \lambda(1 + x, y)y + \lambda(1 + x, y)yx$. Hence

$$(1 + x)(1 + y) = 1 + x + \lambda(1 + x, y)y + \lambda(1 + x, y)\lambda(y, x)xy.$$

On the other hand, $(1 + x)(1 + y) = \lambda(1 + x, 1 + y)(1 + y)(1 + x)$, which yields

$$(1 + x)(1 + y) = \lambda(1 + x, 1 + y)\bigl(1 + y + \lambda(1 + y, x)x + \lambda(1 + y, x)xy\bigr).$$

Comparing the coefficients of 1, we get $1 = \lambda(1 + x, 1 + y)$. Similarly, the coefficients of x, y, xy give $1 = \lambda(1 + y, x) = \lambda(1 + x, y)$ and $\lambda(1 + x, y)\lambda(y, x) = \lambda(1 + y, x)$. Thus $\lambda(y, x) = 1$, which shows that $xy = yx$.

So we have to deal with the case $\dim(k+kx+ky+kxy) \leqslant 3$. Assume that $x, y \notin k$. As F is of length $\geqslant 4$, there is a maximal subspace $H \subsetneq K$ with $k+kx+ky+kxy \subset H$. Consider the subspace

$$N := \bigcap \{h^{-1}H \mid h \in H\}.$$

of H. For $z \in K^\times$, we have

$$z \in N \Leftrightarrow \forall\, h \in H \colon hz \in H \Leftrightarrow \forall\, h \in H \smallsetminus \{0\} \colon \lambda(h, z)zh \in H \Leftrightarrow zH \subset H \Leftrightarrow zH = H.$$

So N is a sub-nearfield of K. Choose $t \in K \smallsetminus H$. Since $K = kt+H = tk+H$, any $z \in N$ satisfies $tz = t\mu + h$ for some $\mu \in k$ and $h \in H$. Hence $(z - \mu)t = \lambda(z - \mu, t)t(z - \mu) = \lambda(z - \mu, t)h \in H$, and thus $z - \mu = 0$. So we obtain $N = k$.

Since $x \notin k = N$, the preceding argument shows that $xH \neq H$. So we find an element $z \in H \smallsetminus x^{-1}H$. Assume first that z does not belong to $k + kx$ or $ky + kxy$. Then $\dim(k + kx + kz) = \dim(ky + kxy + kz) = 3$. Since $k + kx + kz \subset H$ and $ky + kxy + kz \subset H$, it follows that xz does not belong to these subspaces. Hence $\dim(k + kx + kz + kxz) = \dim(ky + kxy + kz + kxz) = 4$. Left multiplication with

$t := z^{-1}$ yields $\dim(kt + kxt + k + kx) = \dim(kyt + kxyt + k + kx) = 4$. So we obtain $xt = tx$ and $x(yt) = (yt)x$. Hence $xyt = ytx = yxt$, and thus $xy = yx$.

For $x, y \in K^\times \smallsetminus k$ and $\lambda, \mu \in k^\times$, we infer that $\lambda xy = \lambda yx = (\lambda y)x = x(\lambda y) = (x\lambda)y$. Thus $\lambda x = x\lambda$. Hence $\lambda(\mu x) = (\mu x)\lambda = \mu(x\lambda) = \mu(\lambda x)$, which implies that $\lambda\mu = \mu\lambda$. Thus K is a field.

It remains to consider the case where the join of the three subspaces $A := k + kx$, $B := ky + kxy$ and $C := H \cap x^{-1}H$ is H, so that the element z does not exist. This can only happen if k is finite and $\dim H = 3$. Then $C = k + ky$. If $q := |k|$, then $|H| = q^3$, and $|A| = |B| = |C| = q^2$. As the join of two proper subspaces of H cannot reach H, the spaces A, B, C have to be distinct. Hence $A \cap B \cap C = 0$. (Otherwise, $ky = k$.) Thus $|A \cup B \cup C| = 1 + 3(q-1) + 3(q-1)^2 = 3q^2 - 3q + 1 < q^3$ for all $q \geqslant 2$. □

As an immediate consequence, we obtain

Corollary. *Every function q-field is of the form $F(K|k)$ with a field extension $K|k$.*

6 The double of a function q-field

By Theorem 3, a function q-field F is given by a field extension $K|k$. So the product $K^{X(F)}$ is a commutative k-algebra. Accordingly, the lattice $\Pi(F)$ of all subspaces of $K^{X(F)}$ is a modular commutative unital quantale. The diagonal embedding

$$E(F) \hookrightarrow \Pi(F)$$

is length-preserving. Furthermore, there is a componentwise embedding

$$J_F \hookrightarrow \Pi(F)$$

which maps $\xi \in J_F$ to the $k^{X(F)}$-submodule $\prod_{e \in X(F)} \xi_e$. Both embeddings respect the multiplication. So the sublattice

$$R(F) := \{a \in \Pi(F) \mid \exists \xi \in J_F \colon \xi \leqslant a \leqslant \xi^{-1}\}$$

of $\Pi(F)$ is multiplicatively closed. Its elements will be called *regular*. As a diagonal of $K^{X(F)}$, the field K can be regarded as an element of $\Pi(F)$, that is, $K = \bigvee F^\times = \bigvee E(F) \in \Pi(F)$.

Definition 12. Let F be a function q-field. We define the *double* of F to be the sublattice $D(F)$ of $\Pi(F)$ generated by $R(F) \sqcup \{K\}$.

Thus $D(F)$ is modular. For every $a \in R(F)$, the length $\ell(e_F a/a)$ is finite by the corollary of Theorem 2. Therefore, the degree function extends to $R(F)$ by the formula

$$\deg a := \deg e_F a - \ell(e_F a/a).$$

For any $a \in R(F)$ we define

$$L(a) := a \wedge K, \qquad \ell(a) := h(L(a)).$$

If a is a replete ideal $\xi \in J_F$, the $\alpha \in F^\times$ with $\alpha \leqslant L(\xi)$ correspond to the functions in K whose poles are bounded by the divisor of ξ (see [19]).

Proposition 21. *Let F be a function q-field. Every interval $[b, a]$ in $R(F)$ is of finite length*

$$\ell(a/b) = \ell(a \vee K/b \vee K) + \ell(a \wedge K/b \wedge K).$$

Moreover, each $a \geqslant e_F$ in $R(F)$ satisfies $\ell(a) \leqslant 1 + \deg a$.

Proof. Since $D(F)$ is modular, the element $c := (b \vee K) \wedge a$ is equal to $b \vee (K \wedge a)$. So we have $\ell(a/c) = \ell(a \vee K/b \vee K)$ and $\ell(c/b) = \ell(a \wedge K/b \wedge K)$. This proves the length formula. By Proposition 17, $L(e_F) = \bigwedge X(F) = \varepsilon$. Hence $\ell(e_F) = 1$. Any interval $[b, a]$ of length 1 in $R(F)$ satisfies $0 \leqslant \ell(a) - \ell(b) = \ell(a \wedge K/b \wedge K) \leqslant \ell(a/b) = 1$. So the inequality $\ell(a) \leqslant 1 + \deg a$ follows by induction. □

Since $\ell(a/b) = \deg a - \deg b$ and $\ell(a \wedge K/b \wedge K) = \ell(L(a)/L(b)) = \ell(a) - \ell(b)$, the formula in Proposition 21 can be rewritten as

$$\deg a - \ell(a) = \deg b - \ell(b) + \ell(a \vee K/b \vee K).$$

Hence $a \mapsto \deg a - \ell(a)$ is a monotone function $R(F) \to \mathbb{Z}$.

Theorem 4. *Let F be a function q-field. For a unit $\alpha \neq \varepsilon$ of F, the length of $F/\varepsilon(\alpha)$ is equal to the degree of the replete ideal $\xi := (e_F\alpha)^+$. Furthermore, $\deg \xi^t - \ell(\xi^t) = \ell(\xi^t \vee K/e_F \vee K) - 1$ is bounded for $t \in \mathbb{N}$.*

Proof. Since F is finitely generated, Proposition 18 implies that $n := \ell(F/\varepsilon(\alpha))$ is finite. So there are $\beta_1, \dots, \beta_n \in F^\times$ with $\varepsilon(\alpha)\beta_1 \vee \dots \vee \varepsilon(\alpha)\beta_n = 1$. In particular, there is an integer $r > 0$ with $\beta_i^{r+1} \leqslant \varepsilon(\alpha)(\varepsilon \vee \beta_i \vee \dots \vee \beta_i^r)$ for all $i \in \{1, \dots, n\}$. Hence $\beta_i^{r+1} \leqslant \gamma^{-1}\varepsilon[\alpha](\varepsilon \vee \beta_i \vee \dots \vee \beta_i^r)$ for some unit $\gamma \leqslant \varepsilon[\alpha]$ and all i. Replacing β_i by $\beta_i\gamma$, we can assume that $\beta_i^{r+1} \leqslant \varepsilon[\alpha] \vee \varepsilon[\alpha]\beta_i \vee \dots \vee \varepsilon[\alpha]\beta_i^r = \varepsilon[\alpha](\varepsilon \vee \beta_i \vee \dots \vee \beta_i^r)$ for all i. Thus $\varepsilon \leqslant \varepsilon[\alpha](\beta_i^{-r-1} \vee \dots \vee \beta_i^{-1})$ for all i. If $e \in X(F)$ satisfies $e\alpha \leqslant e$, it follows that $e \leqslant e\beta_i^{-1}$, hence $\beta_i \leqslant e$. So there exists an integer $k > 0$ with $\beta_i \leqslant \xi^k$ for all i.

By Theorem 2, $n \geqslant \sum_{e \in X(F)} \ell(e/e \wedge e\alpha^{-1})$. Since $\ell(e/e \wedge e\alpha^{-1}) = \ell(e \vee e\alpha^{-1}/e\alpha^{-1}) = \ell(e\alpha \vee e/e)$, it follows that $n \geqslant \deg \xi$.

Now let s, t be integers with $0 \leqslant s \leqslant t - k$. Then $\alpha^s\beta_i \leqslant \xi^{s+k} \leqslant \xi^t$ holds for $i \in \{1, \dots, n\}$. The height of $(\varepsilon \vee \alpha \vee \dots \vee \alpha^{t-k})(\beta_1 \vee \dots \vee \beta_n)$ in F is $(t - k + 1)n$. Hence $\ell(\xi^t) \geqslant (t - k + 1)n$. With $n_t := \ell(\xi^t \vee K/e_F \vee K)$, Proposition 21 yields

$$t \deg \xi = \deg \xi^t = \ell(\xi^t/e_F) = n_t + \ell(\xi^t) - 1 \geqslant n_t - 1 + (t - k + 1)n. \tag{3}$$

Thus $\deg \xi \geqslant \frac{n_t-1}{t} + \bigl(1 - \frac{k-1}{t}\bigr)n$. For $t \to \infty$, this implies that $\deg \xi \geqslant n$. Whence $\deg \xi = n$. Therefore, the inequality (3) gives $n_t - 1 \leqslant (k-1)n$. Thus $\deg \xi^t - \ell(\xi^t) = n_t - 1$ is bounded. $\square$

Corollary 1. *Let F be a function q-field. Every unit $\alpha \in F^\times$ satisfies $\deg e_F\alpha = 0$.*

Proof. For $\alpha = \varepsilon$, this is obvious. Otherwise, it follows since $\varepsilon(\alpha) = \varepsilon(\alpha^{-1})$. $\square$

Since multiplication by a unit is a lattice automorphism of $D(F)$, Corollary 1 shows that $\deg c\alpha = \deg c$ holds for $c \in R(F)$ and $\alpha \in F^\times$. Similarly, $c\alpha \wedge K = (c \wedge K)\alpha$ implies that $\ell(c\alpha) = \ell(c)$. So the functions $c \mapsto \deg c$ and $c \mapsto \ell(c)$ are invariant under multiplication with units. The next corollary shows that a single pole of $\alpha \in F^\times$ suffices to realize any finite collection of zeros.

Corollary 2. *Let F be a function q-field, and let $\eta \in J_F$ be a replete ideal. For any $e \in X(F)$ there exists an integer $r \in \mathbb{N}$ with $L(\eta\hat{e}^r) > 0$.*

Proof. Choose $\alpha \in F^\times$ with $e\alpha > e$ and $f\alpha > f$ for all $f \in X(F)$ with $\eta_f > f$. Define $\xi := (e_F\alpha)^+$. By Theorem 4, there is an integer $k \in \mathbb{N}$ with $\deg \xi^t - \ell(\xi^t) \leqslant k$ for all $t \in \mathbb{N}$. Choose $r \in \mathbb{N}$ with $r \deg \widehat{e} > k - \deg \eta$. So there exists an integer $t > 0$ with $\xi^t \geqslant \eta\hat{e}^r$. Thus $\deg \eta\hat{e}^r - \ell(\eta\hat{e}^r) \leqslant \deg \xi^t - \ell(\xi^t)$, which implies that $\ell(\eta\hat{e}^r) \geqslant \deg \eta + r \deg \widehat{e} - \deg \xi^t + \ell(\xi^t) \geqslant \deg \eta + r \deg \widehat{e} - k > 0$. $\square$

Corollary 3. *Let F be a function q-field with $\alpha \in F^\times \smallsetminus \{\varepsilon\}$, and $\xi := (e_F\alpha)^+$.*

(a) *For any $\eta \in J_F$ there is a unit $\beta \in F^\times$ with $\beta\eta \leqslant \xi^t$ for some $t \in \mathbb{N}$.*
(b) *There is a greatest element $1 \in D(F)$, given by $1 = \eta \vee K$ for some $\eta \in J_F$.*
(c) *The length $\ell(1/a \vee K)$ is finite for every $a \in R(F)$.*

Proof. a) Since $\alpha \neq \varepsilon$, Corollary 1 implies that there is a prime order e with $\xi_e > e$. By Corollary 2, we find a unit $\beta \in F^\times$ with $\beta \leqslant \eta_f^{-1}$ for all $f \in X(F) \smallsetminus \{e\}$. Furthermore, $\beta\eta_e \leqslant \xi_e^t$ for some $t \in \mathbb{N}$, which yields $\beta\eta \leqslant \xi^t$.

b) By Theorem 4, $\deg \eta - \ell(\eta) \leqslant \deg \beta^{-1}\xi^t - \ell(\beta^{-1}\xi^t) = \deg \xi^t - \ell(\xi^t)$ is bounded. Thus, for a fixed $\eta_0 \in J_F$ and $\eta \geqslant \eta_0$, Proposition 21 shows that $\ell(\eta \vee K/\eta_0 \vee K) = \ell(\eta/\eta_0) - \ell(\eta \wedge K/\eta_0 \wedge K) = \deg \eta - \deg \eta_0 - \bigl(\ell(\eta) - \ell(\eta_0)\bigr) = \deg \eta - \ell(\eta) - \deg \eta_0 + \ell(\eta_0)$ is bounded for increasing η. Hence $\eta \vee K = 1$ for large η.

c) For $\eta_0 \leqslant a$, Proposition 21 implies that $\ell(1/a \vee K)$ is finite. $\square$

Corollary 4. *Let F be a function q-field. Then $E(F) = \{a \in D(F) \mid a < K\} = \{a \wedge K \mid a \in R(F)\}$ and $E^*(F) := \{a \in D(F) \mid a > K\} = \{a \vee K \mid a \in R(F)\}$. The double $D(F)$ is a disjoint union*

$$D(F) = R(F) \sqcup E(F) \sqcup E^*(F) \sqcup \{K\}.$$

Proof. Since $e_F\alpha \wedge K = \alpha$ for each $\alpha \in F^\times$, we have $E(F) \subset D(F)$. Hence $R(F) \sqcup E(F) \sqcup E^*(F) \sqcup \{K\} \subset D(F)$. Clearly, $E(F) \sqcup E^*(F) \sqcup \{K\}$ is a sublattice of $D(F)$. Assume that $a \in R(F)$. By Proposition 21, $a \wedge K \in E(F)$. For $b \in D(F)$ with $b \not\leqslant K$, there is a replete ideal $\xi \leqslant b$. This follows immediately by induction. In particular, any $b \in E^*(F)$ admits a replete ideal $\xi \leqslant b$. Hence $a \wedge b \in R(F)$. On the other hand, $b \in E(F)$ implies that $a \vee b \leqslant a \vee e_F b$, which yields $a \vee b \in R(F)$. Since $a \vee K \in E^*(F)$, we have shown that $R(F) \sqcup E(F) \sqcup E^*(F) \sqcup \{K\}$ is a sublattice of $D(F)$, hence equal to $D(F)$.

For any $\xi \in J_F$ with $\xi < e_F$ we have $\xi \wedge K = 0$. So every $a \in E(F)$ satisfies $a = a \vee (\xi \wedge K) = (a \vee \xi) \wedge K$. Similarly, Corollary 3 gives a replete ideal η with $\eta \vee K = 1$. So the dual argument shows that every $a \in E^*(F)$ is of the form $a = K \vee (\eta \wedge a)$. □

For $a \in R(F)$, we call $i(a) := \ell(1/a \vee K)$ the *speciality index* of a. For an interval $[a, b]$ in $R(F)$, we thus have

$$\ell(b \vee K/a \vee K) = i(a) - i(b). \tag{4}$$

The invariant

$$g_F := i(e_F)$$

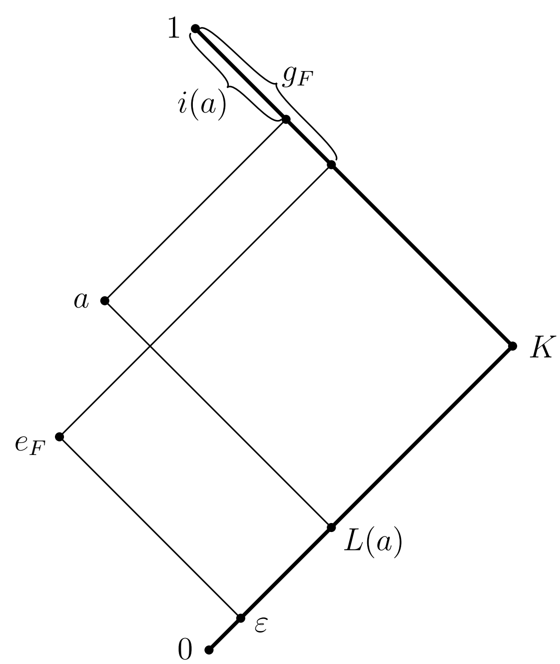
will be called the *genus* of F.

Proposition 22 (Riemann-Roch). *Let F be a function q-field. Every $a \in R(F)$ satisfies*

$$\ell(a) - i(a) = \deg a + 1 - g_F. \tag{5}$$

Proof. With $b \leqslant a \wedge e_F$, Proposition 21 and Eq. (4), applied to $[b, a]$ and $[b, e_F]$, give $\deg a = g_F - i(a) + \ell(a) - 1$. Whence Eq. (5) follows. □

We illustrate Proposition 22 by the following picture:



Corollary 1. *Let F be a function q-field. There exists an integer $k > 0$ such that every $\xi \in J_F$ with $\deg \xi \geqslant k$ satisfies $\xi \vee K = 1$. Furthermore, $\xi \vee K < 1$ if $\deg \xi < g_F - 1$.*

Proof. By Corollary 3 of Theorem 4, there is a replete ideal η with $\eta \vee K = 1$. For any $\xi \in J_F$ with $\deg \xi \geqslant \deg \eta + g_F$, Proposition 22 implies that $\ell(\xi\eta^{-1}) = \deg \xi\eta^{-1} + i(\xi\eta^{-1}) + 1 - g_F \geqslant g_F + 1 - g_F > 0$. So there is a unit $\alpha \in F^\times$ with $\alpha \leqslant \xi\eta^{-1}$. Thus $\xi \vee K \geqslant \alpha\eta \vee K = \alpha(\eta \vee K) = 1$. For $\deg \xi < g_F - 1$, Eq. (5) yields $i(\xi) \geqslant -\deg \xi + g_F - 1 > 0$. □

Remark. Corollary 1 does not hold for regular elements instead of replete ideals. Indeed, if $\deg \xi < g_F - 1$ and $a \in E(F)$, the degree $\deg(\xi \vee a)$ is unbounded, but $\xi \vee a \leqslant \xi \vee K < 1$. Similarly, $\deg \xi < 0$ implies that $\ell(\xi) = 0$, but there are regular elements $a > \varepsilon$ of arbitrarily small degree.

Corollary 2. *The double $D(F)$ of a function q-field F is a modular unital commutative quantale. For $a \in D(F)$ with $a \nleqslant K$, we have $Ka = 1$ and $K^2 = K$.*

Proof. Let $A \subset D(F)$ be a subset. If $A = \varnothing$, then $\bigvee A = 0$. Thus, let A be non-empty. If A is bounded by K, then $\bigvee A$ exists by Corollary 4 of Theorem 4. Otherwise, Corollary 3 of Theorem 4 implies that there is a smallest $b \in E^*(F)$ with $a \leqslant b$ for all $a \in A$. If A contains an element $a \geqslant K$, then $\bigvee A = b$. Otherwise, there exists an element $a \in A \cap R(F)$ with $a \vee K = b$. If A has a maximal element c with this property, then $c = \bigvee A$. If not, then $\ell(a)$ is unbounded for these $a \in A$, which yields $b = \bigvee A$. Thus $D(F)$ is a complete lattice.

Since $K = \bigvee F^\times$, we have $K^2 = K$. Assume that $\xi \in J_F$. By Corollary 1, there is a unit $\alpha \in F^\times$ with $\xi(\alpha \vee \varepsilon) \geqslant e_F$ and $\xi(\alpha \vee \varepsilon) \vee K = 1$. Hence $\xi K = \xi K K \geqslant \xi(\alpha \vee \varepsilon)K \geqslant \xi(\alpha \vee \varepsilon) \vee K = 1$. Thus $Ka = 1$ holds for all $a \nleqslant K$ in $D(F)$. Now let $a, b \in D(F)$ be given. Assume first that $a, b \notin R(F)$. Since multiplication by a unit is a lattice automorphism, $ab \in D(F)$ if $a \in E(F)$ or $b \in E(F)$. Otherwise, $a, b \geqslant K$. So either $ab = 1$ or $ab = K$. It remains to consider the case $a \in R(F)$. If $b \geqslant K$, then $ab = 1$. Otherwise, if $b > 0$, then $ab \in R(F)$. Thus $D(F)$ is a quantale. □

7 Differentials and Serre duality

Our next aim is to show that the lattice $E^*(F)$ is dual to $E(F)$. Recall that a maximal element $\lambda < 1$ in a quantale is said to be a *co-atom*. By Corollary 1 of Proposition 22, there are replete ideals ξ with $i(\xi) > 0$. Hence co-atoms exist in $D(F)$. Note that the atoms in $D(F)$ coincide with the units in F. So it is natural to ask how the co-atoms in $D(F)$ can be described in terms of F.

Definition 13. Let F be a function q-field. We define a *differential* of F to be a co-atom of $D(F)$. The set of differentials will be denoted by $\Omega(F)$.

Since multiplication by a unit $\alpha \in F^\times$ is a lattice automorphism of $D(F)$, there is an action

$$\Omega(F) \times F^\times \to \Omega(F), \tag{6}$$

given by $(\alpha, \omega) \mapsto \omega\alpha$. By Corollary 1 of Proposition 22, any differential $\omega \in \Omega(F)$ admits a greatest $\xi \in J_F$ with $\xi \leqslant \omega$. We write $c_\omega := \xi$.

There is a close connection between differentials of F and Weil's differentials [31, 19].

Definition 14. Let F be a function q-field. We call $x \in K^{X(F)}$ an *adèle* if $x_e \in e$ for almost all $e \in X(F)$ and write $A(F)$ for the subspace of adèles. A *Weil differential* is a k-linear form $\delta\colon A(F) \to k$ with $\delta(K \vee \xi) = 0$ for some $\xi \in J_F$.

Note that $A(F) = 1_{D(F)}$. Indeed, every $a \in D(F)$ consists of adèles, and every adèle is contained in some $\xi \in J_F$. In contrast to classical adèles [19], no completions are involved in the definition of $A(F)$.

Proposition 23. *Let F be a function q-field. There is a one-to-one correspondence between differentials in $\Omega(F)$ and Weil differentials (up to a factor in k). For any $a \in E^*(F)$ there are differentials $\omega_1, \ldots, \omega_n \in \Omega(F)$ with $a = \omega_1 \wedge \cdots \wedge \omega_n$.*

Proof. Let $[b, a]$ be an interval of length 1 in $E^*(F)$. By Corollary 3 of Theorem 4, there is a maximal element $\omega \in E^*(F)$ with $a \wedge \omega = b$. Choose $\xi \in J_F$ with $\xi \leqslant \omega$. Suppose that $a \vee \omega < 1$. Then there is a non-zero adèle $x \nleqslant a \vee \omega$. So we have $kx + \xi \in R(F)$. Hence $kx + \omega = (kx + \xi) \vee \omega \in D(F)$ and $kx + \omega > \omega$. Thus $(kx + \omega) \wedge a \leqslant (kx + \omega) \wedge (a \vee \omega) = \omega$, which yields $(kx + \omega) \wedge a = b$, a contradiction. So $a \vee \omega = 1$, which shows that ω is a differential. By induction, we infer that each element of $E^*(F)$ is of the form $\omega_1 \wedge \cdots \wedge \omega_n$ with $\omega_i \in \Omega(F)$.

Since every $\omega \in \Omega(F)$ is of codimension 1 in $A(F)$, the k-linear form $\delta\colon A(F) \to A(F)/\omega \cong k$ is a Weil differential with $\operatorname{Ker} \delta = \omega$. Conversely, let ω be the kernel of a Weil differential. By Corollary 3 of Theorem 4, there is a replete ideal η with $\eta \vee K = 1$. Hence $\eta \wedge \omega \in \Pi(F)$ and $\ell(\eta/\eta \wedge \omega) = 1$. Choose $\xi \in J_F$ with $\delta(\xi) = 0$. Then $\xi \wedge \eta \leqslant \eta \wedge \omega$, which shows that $\eta \wedge \omega \in R(F)$. Hence $\omega = (K \vee \eta) \wedge \omega = K \vee (\eta \wedge \omega) \in D(F)$, and thus $\omega \in \Omega(F)$. $\square$

Corollary. *Let F be a function q-field. Every right k-invariant maximal k-submodule ω of $A(F)$ with $\omega \geqslant a$ for some $a \in E^*(F)$ is a differential.*

Proof. This follows since ω is the kernel of a Weil differential. $\square$

Remarks. 1. For $\omega \in \Omega(F)$, let δ be the corresponding Weil differential. Choose a replete ideal $\xi \leqslant \omega$. Since every adèle $x \in A(F)$ satisfies $x_e \leqslant \xi_e$ for almost all $e \in X(F)$, we have $A(F)/\xi \cong \coprod_{e \in X(F)} K/\xi_e$, and the Weil differential δ induces a linear form on this space. Thus, δ is given by a sequence $(\delta_e)_{e \in X(F)}$ of linear forms $\delta_e\colon K \to k$ with $\delta_e(\xi_e) = 0$. We call δ_e the *local component* at e.

2. The definition of differentials as co-atoms in $D(F)$ already hints at a duality between $\Omega(F)$ and $F^\times$. The relationship to Weil differentials exhibits them as linear forms, which partly justifies their name. In the simplest case $F = F(\mathbb{C}(x)|\mathbb{C})$, units $\alpha \in F^\times$ are non-zero rational functions on $\mathbb{C}$, that is, differentiable maps $f\colon \widetilde{\mathbb{C}} \to \widetilde{\mathbb{C}}$ from the Riemann sphere $\widetilde{\mathbb{C}} := \mathbb{C} \sqcup \{\infty\}$ to itself. By Proposition 12, prime orders correspond to valuations, hence to the points of the Riemann sphere. For $\lambda \in \mathbb{C}$, the corresponding prime order is the localization $e_\lambda := \mathbb{C}[x]_{(x-\lambda)}$ with radical is $e_\lambda(x-\lambda)$, while $e_\infty := \mathbb{C}[x^{-1}]_{(x^{-1})}$. Note that $\mathbb{C}[x^{-1}]_{(x^{-1}-\lambda)} = \mathbb{C}[x]_{(x-\lambda^{-1})}$.

The Weil differentials of F are expressions $\delta = g(x)dx$ with $g \in \mathbb{C}(x)^\times$. Their local components $\delta_\lambda := \delta_{e_\lambda}$ are given by $\delta_\lambda(f) = \frac{1}{2\pi i}\oint_\lambda f\omega = \frac{1}{2\pi i}\oint_\lambda f(x)g(x)dx$, the residue of fg at λ. (To get δ_∞, one has to change the variable x to x^{-1}.) The residue theorem says that $\delta(f) = 0$ for all $f \in \mathbb{C}(x)$, as stated in the definition of a Weil differential.

Proposition 24. *Let F be a function q-field. Assume that $\alpha_1, \dots, \alpha_n \in F^\times$ and $\omega, \omega' \in \Omega(F)$ satisfy $\omega\alpha_1^{-1} \wedge \cdots \wedge \omega\alpha_n^{-1} \leqslant \omega'$. Then $\omega' = \omega\alpha^{-1}$ for some $\alpha \in F^\times$ with $\alpha \leqslant \alpha_1 \vee \cdots \vee \alpha_n$.*

Proof. Let $\delta, \delta'\colon A(F) \to k$ be Weil differentials with $\operatorname{Ker}\delta = \omega$ and $\operatorname{Ker}\delta' = \omega'$. Then $\operatorname{Ker}\alpha_i\delta = \omega\alpha_i^{-1}$. So $\Delta := (\alpha_1\delta, \dots, \alpha_n\delta)^T$ is a k-linear map $A(F) \to k^n$ with kernel $\omega\alpha_1^{-1} \wedge \cdots \wedge \omega\alpha_n^{-1}$. Hence δ' factors through Δ. So there is a k-linear map $(\lambda_1, \dots, \lambda_n)\colon k^n \to k$ with $\delta'(x) = \sum_{i=1}^n \delta(x\alpha_i)\lambda_i = \delta\big(\sum_{i=1}^n x\alpha_i\lambda_i\big) = \delta\big(x\sum_{i=1}^n \alpha_i\lambda_i\big)$ for all $x \in A(F)$. Thus $\alpha := k(\alpha_1\lambda_1 + \cdots + \alpha_n\lambda_n)$ satisfies $\alpha \leqslant \alpha_1 \vee \cdots \vee \alpha_n$ and $x \in \omega' \iff x\alpha \in \omega$. Whence $\alpha \in F^\times$ and $\omega' = \omega\alpha^{-1}$. □

Theorem 5. *Let F be a function q-field. The action (6) of $F^\times$ on $\Omega(F)$ is transitive and free.*

Proof. Suppose that ω and ω' are differentials which belong to different $F^\times$-orbits. Then $\eta := c_\omega \wedge c_{\omega'}$ is the greatest replete ideal $\leqslant \omega \wedge \omega'$. Choose $\xi \in J_F$ with $\deg\xi > \deg\eta$. By Proposition 21, there are units $\alpha_1, \dots, \alpha_n$ with $L(\xi) = \alpha_1 \vee \cdots \vee \alpha_n$ and $n = \ell(\xi)$. So $\alpha_i\eta^{-1} \leqslant \xi\eta^{-1}$ for all i, which gives $\eta\xi^{-1} \leqslant \eta\alpha_1^{-1} \wedge \cdots \wedge \eta\alpha_n^{-1} \leqslant \omega\alpha_1^{-1} \wedge \cdots \wedge \omega\alpha_n^{-1} \wedge \omega'\alpha_1^{-1} \wedge \cdots \wedge \omega'\alpha_n^{-1}$. Since $\{\alpha_1, \dots, \alpha_n\}$ is independent, Proposition 24 implies that $\ell(1/\omega\alpha_1^{-1} \wedge \cdots \wedge \omega\alpha_n^{-1}) = n$. Similarly, $\ell(1/\omega'\alpha_1^{-1} \wedge \cdots \wedge \omega'\alpha_n^{-1}) = n$. If $(\omega\alpha_1^{-1} \wedge \cdots \wedge \omega\alpha_n^{-1}) \vee (\omega'\alpha_1^{-1} \wedge \cdots \wedge \omega'\alpha_n^{-1}) < 1$, there exists a differential $\omega'' \geqslant (\omega\alpha_1^{-1} \wedge \cdots \wedge \omega\alpha_n^{-1}) \vee (\omega'\alpha_1^{-1} \wedge \cdots \wedge \omega'\alpha_n^{-1})$, and Proposition 24 shows that $\omega'' = \omega\alpha = \omega'\alpha'$ with $\alpha, \alpha' \in F^\times$, contrary to our assumption. So we obtain

$$\ell(1/\omega\alpha_1^{-1} \wedge \cdots \wedge \omega\alpha_n^{-1} \wedge \omega'\alpha_1^{-1} \wedge \cdots \wedge \omega'\alpha_n^{-1}) = 2n,$$

which implies that $i(\eta\xi^{-1}) \geqslant 2n$.

Since $\deg\eta\xi^{-1} = \deg\eta - \deg\xi < 0$, we have $\ell(\eta\xi^{-1}) = 0$. So Proposition 22 gives $-2n \geqslant -i(\eta\xi^{-1}) = \deg\eta\xi^{-1} + 1 - g_F = \deg\eta - \deg\xi + 1 - g_F$ and $n \geqslant n - i(\xi) = \ell(\xi) - i(\xi) = \deg\xi + 1 - g_F$. Hence $2n + \deg\eta + 1 - g_F \leqslant \deg\xi \leqslant n + g_F - 1$, and thus $n \leqslant -\deg\eta + 2g_F - 2$. So $\ell(\xi) = n$ is bounded, a contradiction.

This proves that the action (6) is transitive. To show that the action is free, assume that $\omega\alpha = \omega\beta$ for some $\alpha, \beta \in F^\times$. Then $c_\omega\alpha\beta^{-1} = c_\omega$, which gives $\alpha\beta^{-1} \leqslant e_F$. Hence $\alpha\beta^{-1} = \varepsilon$, and thus $\alpha = \beta$. □

Corollary. *Let F be a function q-field. For any $\omega \in \Omega(F)$, the greatest replete ideal $c_\omega \leqslant \omega$ satisfies $c_\omega \vee K = \omega$.*

Proof. Suppose that $a := c_\omega \vee K < \omega$. By Proposition 23, there is a differential $\omega' \neq \omega$ with $a \leqslant \omega'$. So $\omega' = \omega\alpha^{-1}$ for some $\alpha \in F^\times$. Hence $c_\omega \leqslant \omega\alpha^{-1}$, and thus $c_\omega\alpha \leqslant \omega$. Therefore, $c_\omega\alpha \leqslant c_\omega$, which yields $\alpha \leqslant e_F\alpha \leqslant e_F$. Thus $\alpha = \varepsilon$, a contradiction. □

For any quantale Q, there are binary operations

$$a \to b := \bigvee\{c \in Q \mid ca \leqslant b\}, \qquad a \rightsquigarrow b := \bigvee\{c \in Q \mid ac \leqslant b\}$$

which satisfy

$$a \leqslant b \to c \iff ab \leqslant c \iff b \leqslant a \rightsquigarrow c \tag{7}$$

for all $a, b, c \in Q$. If Q is commutative, the two operations coincide.

Theorem 6 (Serre duality). *Let F be a function q-field with a differential ω. Then*

$$(a \to \omega) \to \omega = a \tag{8}$$

holds for all $a \in D(F)$. Every $\xi \in J_F$ satisfies $\xi \to \omega = \xi^{-1}c_\omega$.

Proof. The inequality $a \leqslant (a \to \omega) \to \omega$ follows by (7). Assume first that $a = \alpha_1 \vee \cdots \vee \alpha_n$ with $\alpha_i \in F^\times$ and $h(a) = n$. Then

$$a \to \omega = \omega\alpha_1^{-1} \wedge \cdots \wedge \omega\alpha_n^{-1} \geqslant K.$$

If $b := (a \to \omega) \to \omega \nleqslant K$, then Corollary 2 of Proposition 22 implies that $1 = Kb \leqslant (a \to \omega)b \leqslant \omega$, a contradiction. Thus $b \leqslant K$. Now $\beta \leqslant (a \to \omega) \to \omega$ with $\beta \in F^\times$ gives $(a \to \omega)\beta \leqslant \omega$, that is, $\omega\alpha_1^{-1} \wedge \cdots \wedge \omega\alpha_n^{-1} \leqslant \omega\beta^{-1}$. By Proposition 24, this implies that $\omega\beta^{-1} = \omega\alpha^{-1}$ for some $\alpha \leqslant a$. So Theorem 5 gives $\alpha = \beta$. Hence $(a \to \omega) \to \omega \leqslant a$, and thus $(a \to \omega) \to \omega = a$.

Next assume that $b > K$. Then Theorem 5 and Proposition 23 imply that $b = \omega\alpha_1^{-1} \wedge \cdots \wedge \omega\alpha_n^{-1}$ for some $\alpha_i \in F^\times$. Hence $b = a \to \omega$ with $a := \alpha_1 \vee \cdots \vee \alpha_n$. So we obtain $a \leqslant b \to \omega$, which gives $(b \to \omega) \to \omega \leqslant a \to \omega = b \leqslant (b \to \omega) \to \omega$. If $a \leqslant K \to \omega$, then $aK \leqslant \omega$. Hence $a \leqslant K$ by Corollary 2 of Proposition 22. Since $K^2 = K$, we get $K \to \omega = K$, which proves Eqs. (8) for $a = K$.

Now assume that $a \in R(F)$. For $x, y \in A(F)$ with x invertible, $y \leqslant xa \to \omega \Leftrightarrow yx \leqslant a \to \omega \Leftrightarrow y \leqslant (a \to \omega)x^{-1}$, which gives $xa \to \omega = (a \to \omega)x^{-1}$. Thus $(xa \to \omega) \to \omega = x\big((a \to \omega) \to \omega\big)$. Choose a replete ideal $\xi \geqslant (a \to \omega) \to \omega$ and $x \in A(F)^\times$ with $\deg x\xi < 0$. Then $x\xi \wedge K = 0$. Hence $b := (a \to \omega) \to \omega$ satisfies $xb = (xa \to \omega) \to \omega$ and $xb \wedge K = 0$. So we can assume without loss of generality that $b \wedge K = 0$. Since $b \leqslant \big((a \vee K) \to \omega\big) \to \omega = a \vee K$, it follows that $a = a \vee (K \wedge b) = (a \vee K) \wedge b = b$.

Finally, let $\xi \in J_F$ be a replete ideal. Then (7) gives $a \leqslant \xi \to \omega \Leftrightarrow a\xi \leqslant \omega \Leftrightarrow ae_F \leqslant \xi \to \omega$. By Corollary 2 of Proposition 22, $\xi \to \omega \in J_F$. Hence $(\xi \to \omega)\xi \leqslant c_\omega$, which yields $\xi \to \omega = c_\omega\xi^{-1}$. □

Corollary. *Let F be a function q-field with $\omega \in \Omega(F)$. Every $a \in R(F)$ satisfies $i(a) = \ell(a \to \omega)$ and $\deg(a \to \omega) = 2g_F - 2 - \deg a$. In particular, $\deg c_\omega = 2g_F - 2$.*

Proof. Since $(a \vee K) \to \omega = (a \to \omega) \wedge (K \to \omega) = (a \to \omega) \wedge K$, we have $i(a) = \ell(1/a \vee K) = h\big((a \to \omega) \wedge K\big) = \ell(a \to \omega)$. Now Proposition 22 yields

$$\ell(a) - \ell(a \to \omega) = \ell(a) - i(a) = \deg a + 1 - g_F$$

and $\ell(a \to \omega) - i(a \to \omega) = \deg(a \to \omega) + 1 - g_F$. So we obtain

$$\begin{aligned}\deg(a \to \omega) &= g_F - 1 + \big(\ell(a) - \deg a + g_F - 1\big) - i(a \to \omega) \\ &= 2g_F - 2 - \deg a + \ell(a) - \ell\big((a \to \omega) \to \omega\big) = 2g_F - 2 - \deg a.\end{aligned}$$

Since $c_\omega = e_F \to \omega$ and $\deg e_F = 0$, the last statement follows. ☐

As an immediate consequence, we get the Riemann-Roch theorem for $a \in R(F)$:

$$\boxed{\ell(a) - \ell(a \to \omega) = \deg a + 1 - g_F}$$

8 Serre quantales and Chu's construction

Our next aim is to characterize the quantales $D(F)$ arising from function q-fields F. In particular, we analyse the nature of Serre duality, in a slightly more general, non-commutative, framework.

For a complete lattice L, we write $\mathrm{At}(L)$ for the set of atoms, and $\mathrm{Co}(L)$ for the set of co-atoms. We call $a \in L$ *semisimple* if $a = \bigvee S$ for a finite subset $S \subset \mathrm{At}(L)$. Similarly, a finite meet of co-atoms will be called *co-semisimple*.

Definition 15. Let L be a complete modular lattice. We call $K \in L$ a *birational unit* if $\ell(K/0) = \ell(1/K) = \infty$, every $a < K$ is semisimple, and every $b > K$ is co-semisimple.

Birational units are unique by the following

Proposition 25. *A birational unit K of a complete modular lattice L satisfies*

$$K = \bigvee \mathrm{At}(L) = \bigwedge \mathrm{Co}(L).$$

Proof. Suppose that $p \not\leqslant K$ holds for some $p \in \mathrm{At}(L)$. Then $\ell(1/p \vee K) < \infty$. Since L is modular, $\ell(p \vee K/K) = 1$. Thus $\ell(1/K) < \infty$, which is impossible. So we obtain $\bigvee \mathrm{At}(L) \leqslant K$. Suppose that $\bigvee \mathrm{At}(L) < K$. Then $\ell(K/\bigvee \mathrm{At}(L)) = \infty$. So there is an element $a \in L$ with $\bigvee \mathrm{At}(L) < a < K$. Hence a is semisimple, a contradiction. Thus $K = \bigvee \mathrm{At}(L)$, and similarly, $K = \bigwedge \mathrm{Co}(L)$. ☐

Now let Q be a unital quantale. With $a = \varepsilon$ or $b = \varepsilon$, the equivalence (7) gives

$$\varepsilon \to a = \varepsilon \rightsquigarrow a = a. \tag{9}$$

Furthermore, $a \to a$ is the greatest element $b \in Q$ with $ba \leqslant a$. By Definition 4, $a \to a$ is an order. We call it the *left order* of a. Similarly, $a \rightsquigarrow a$ is the *right order* of a. Note that every order $e \in Q$ satisfies $e \to e = e \rightsquigarrow e = e$. For $a, b \in Q$ and $\gamma \in Q^\times$, we have

$$\gamma(a \to b) = a \to \gamma b, \qquad (a \to b)\gamma = \gamma^{-1}a \to b. \tag{10}$$

For $a, b, c, d \in Q$, we have $d \leqslant ab \to c \Leftrightarrow dab \leqslant c \Leftrightarrow da \leqslant b \to c \Leftrightarrow d \leqslant a \to (b \to c)$. Thus

$$ab \to c = a \to (b \to c). \tag{11}$$

Definition 16. We define a *Serre duality* of a unital quantale Q to be a bijection $D\colon Q \to Q$ with $D(a) \rightsquigarrow D(b) = b \to a$ for all $a, b \in Q$. We call $D(\varepsilon)$ the *counit* of D.

Since $a \leqslant b \Longleftrightarrow \varepsilon \leqslant a \to b \Longleftrightarrow \varepsilon \leqslant a \rightsquigarrow b$, the partial order of Q is inverted by D:

$$a \leqslant b \Longleftrightarrow D(a) \geqslant D(b).$$

For $a, b, c \in Q$, the equivalences

$$ab \leqslant c \Leftrightarrow a \leqslant b \to c \Leftrightarrow a \leqslant D(c) \rightsquigarrow D(b) \Leftrightarrow D(c) \leqslant a \to D(b) \Leftrightarrow D^{-1}(a \to D(b)) \leqslant c$$

give the equation $D^{-1}(a \to D(b)) = ab$, that is,

$$D(ab) = a \to D(b). \tag{12}$$

For $b = \varepsilon$, this implies that D is determined by the counit:

$$D(a) = a \to D(\varepsilon). \tag{13}$$

By Eqs. (10), this gives

$$D(\gamma a) = D(a)\gamma^{-1}$$

for $a \in Q$ and $\gamma \in Q^\times$. Similarly to Eq. (12), we have

$$D^{-1}(ab) = b \rightsquigarrow D^{-1}(a), \tag{14}$$

and in particular,

$$D^{-1}(b) = b \rightsquigarrow D^{-1}(\varepsilon).$$

Furthermore, Eqs. (9) give $D^{-1}(\varepsilon) = \varepsilon \to D^{-1}(\varepsilon) = D(D^{-1}(\varepsilon)) \rightsquigarrow D(\varepsilon) = \varepsilon \rightsquigarrow D(\varepsilon)$. Thus

$$D^{-1}(\varepsilon) = D(\varepsilon).$$

For a Serre duality D and a unit $\gamma \in Q^\times$, we define the map $\gamma D\colon Q \to Q$ as usual:

$$(\gamma D)(a) := D(a\gamma).$$

Proposition 26. *Let Q be a unital quantale. If $D\colon Q \to Q$ is a Serre duality, then γD is a Serre duality for each $\gamma \in Q^\times$, and any Serre duality is of this form.*

Proof. Assume that D_1 and D_2 are Serre dualities. Then Eq. (12) implies that $D_2\big(D_1^{-1}D_2(\varepsilon) \cdot D_2^{-1}D_1^{-1}(\varepsilon)\big) = D_1^{-1}D_2(\varepsilon) \to D_1^{-1}(\varepsilon) = \varepsilon \rightsquigarrow D_2(\varepsilon) = D_2(\varepsilon)$. Hence $D_1^{-1}D_2(\varepsilon) \cdot D_2^{-1}D_1(\varepsilon) = D_1^{-1}D_2(\varepsilon) \cdot D_2^{-1}D_1^{-1}(\varepsilon) = \varepsilon$, which shows that $\gamma := D_1^{-1}D_2(\varepsilon)$ is invertible with $\gamma^{-1} = D_2^{-1}D_1(\varepsilon)$. For all $a \in Q$, Eqs. (12) and (13) give $D_2(a) = a \to D_2(\varepsilon) = D_1(a \cdot D_1^{-1}D_2(\varepsilon)) = D_1(a\gamma)$. Thus $D_2 = \gamma D_1$. Conversely, let D be a Serre duality and $\gamma \in Q^\times$. Then the second of Eqs. (10) gives $\gamma \to a\gamma = \gamma\varepsilon \to a\gamma = (\varepsilon \to a\gamma)\gamma^{-1} = a\gamma\gamma^{-1} = a$. By Eq. (11) this yields $D(a\gamma) \rightsquigarrow D(b\gamma) = b\gamma \to a\gamma = b \to (\gamma \to a\gamma) = b \to a$. Thus γD is a Serre duality. □

The connection to Theorem 6 is given by the following

Corollary. *A unital quantale Q has Serre duality if and only if there is an element $\omega \in Q$ with $(a \to \omega) \rightsquigarrow \omega = (a \rightsquigarrow \omega) \to \omega = a$ for all $a \in Q$.*

Proof. Let D be a Serre duality. Then Eqs. (12) and (14) give $D(a) = a \to D(\varepsilon)$ and $D^{-1}(a) = a \rightsquigarrow D^{-1}(\varepsilon) = a \rightsquigarrow D(\varepsilon)$. Thus, $\omega := D(\varepsilon)$ meets the requirement.

Conversely, assume that $(a \to \omega) \rightsquigarrow \omega = (a \rightsquigarrow \omega) \to \omega = a$ holds for all $a \in Q$. For all $a, b, c, d \in Q$, the equivalences

$$d \leqslant (a \rightsquigarrow (b \to c)) \Leftrightarrow ad \leqslant b \to c \Leftrightarrow adb \leqslant c \Leftrightarrow db \leqslant a \rightsquigarrow c \Leftrightarrow d \leqslant b \to (a \rightsquigarrow c)$$

give

$$a \rightsquigarrow (b \to c) = b \to (a \rightsquigarrow c).$$

Hence $D(a) := a \to \omega$ is a Serre duality. □

Remark. An element ω of a quantale Q satisfying the equations of the corollary is said to be *dualizing* [32]. By Proposition 26, a dualizing element is unique up to right multiplication by a unit. If Q has a dualizing element ω with $a \to \omega = a \rightsquigarrow \omega$ for all $a \in Q$, then Q is said to be a *Girard quantale* [25, 26].

Proposition 27. *Let Q be a unital quantale with Serre duality D. Then D^2 is an automorphism of Q.*

Proof. By Eqs. (12) and (14), we have $D(ab) = a \to D(b) = D^2(b) \rightsquigarrow D(a) = D^{-1}(D^2(a)D^2(b))$. Hence $D^2(ab) = D^2(a)D^2(b)$. □

The following corollary gives a connection between left and right multiplication of counits by units.

Corollary. *Let Q be a unital quantale with Serre duality D. Then $D(\varepsilon)\gamma = D^2(\gamma)D(\varepsilon)$ holds for all $\gamma \in Q^\times$.*

Proof. Since $\varepsilon = D^2(\varepsilon) = D(\gamma D(\varepsilon))\gamma$, we have $\gamma^{-1} = D(\gamma D(\varepsilon))$. Hence $D(\varepsilon)\gamma = D(\gamma^{-1}) = D^2(\gamma D(\varepsilon)) = D^2(\gamma)D^3(\varepsilon) = D^2(\gamma)D(\varepsilon)$. □

Definition 17. We define a *Serre quantale* to be a unital quantale Q such that the following are satisfied:

(a) Every atom of Q is invertible.
(b) As a lattice, Q is modular and has a birational unit.
(c) Q has Serre duality.

Proposition 28. *Let Q be a Serre quantale. Then $\mathrm{At}(Q) = Q^{\times}$, and $\mathrm{Co}(Q)$ is the set of counits of the Serre dualities. They form an orbit under $Q^{\times}$. The birational unit K is an order which is fixed under every Serre duality.*

Proof. By Proposition 25, there are infinitely many atoms in Q. Hence there is an invertible $\alpha \in \mathrm{At}(Q)$. For each $\beta \in Q^{\times}$, the map $x \mapsto x\beta$ is a bijection $\mathrm{At}(Q) \to \mathrm{At}(Q)$. Hence $\alpha\beta \in \mathrm{At}(Q)$, and thus $Q^{\times} \subset \mathrm{At}(Q)$. Furthermore, Proposition 25 implies that $K^2 = \left(\bigvee Q^{\times}\right)\left(\bigvee Q^{\times}\right) = \bigvee Q^{\times} = K$. Since a Serre duality D carries Q into its dual lattice, $D(K) = K$, and every counit $D(\varepsilon)$ is a co-atom. Conversely, each $c \in \mathrm{Co}(Q)$ satisfies $D^{-1}(c) \in \mathrm{At}(Q)$. Hence $\gamma := D^{-1}(c) \in Q^{\times}$, and thus $c = D(\gamma) = (\gamma D)(\varepsilon)$. By Proposition 26, the co-units form an orbit under $Q^{\times}$. □

Definition 18. Let Q be a Serre quantale. We call an element $a \in Q$ *left (right) rational* if $a = Ka$ (respectively, $a = aK$). A left and right rational element will be called *birational.*

Proposition 29. *Let Q be a Serre quantale. Every left or right rational element $a \in Q \smallsetminus \{0, K, 1\}$ is a complement of K.*

Proof. Assume that $a = Ka$. If $a > K$, there is a co-atom $\omega \geqslant a$. So $\gamma\omega \geqslant a$ for all $\gamma \in Q^{\times}$. Since $K = \bigwedge \mathrm{Co}(Q)$, this is impossible. Thus $a \ngeqslant K$. Consequently, there is no $\alpha \in Q^{\times}$ with $\alpha \leqslant a$. Hence $a \wedge K = 0$. So $a \nleqslant K$, which gives $a \vee K = K(a \vee K) > K$. Thus $a \vee K = K$. The case $a = aK$ is treated similarly. □

Corollary 1. *Let Q be a Serre quantale with non-zero elements a, b satisfying $ab = 0$. Then $1a$ and $b1$ are complements of K, and $1a = D(b1)$ for any Serre duality D.*

Proof. By assumption, $1a \cdot b1 = 0$. So $1a, b1 \notin \{0, K, 1\}$, which shows that $1a$ and $b1$ are complements of K. Furthermore, Eq. (12) gives $1 = D(1a \cdot b1) = 1a \to D(b1)$. Hence $1a \leqslant D(b1)$, and thus $1a = D(b1)$. □

Corollary 2. *Let Q be a Serre quantale. A complement a of K is a left zero divisor if and only if $a = 1a$.*

Proof. Assume that $a = 1a$. For a Serre duality D, Eq. (12) gives $D(a \cdot D^{-1}(a)) = a \to a = 1$. Hence $a \cdot D^{-1}(a) = 0$. The converse is trivial. □

Corollary 3. *Let Q be a Serre quantale. The birational elements $a < 1$ with $1a = a1 = 1$ form a multiplicative subgroup G of Q with unit element K. For any Serre duality D, the inverse of $a \in G$ is $D(a)$.*

Proof. The elements of $G \setminus \{K\}$ are complements of K. By Corollary 2, there are no zero-divisors in G. By definition, $aK = Ka = a$ for all $a \in G$. Let D be a Serre duality of Q. For $a \in G$ and $\gamma \in Q^\times$, we have $D(a)\gamma^{-1} = D(\gamma a) = D(a)$. Hence $D(a)K = D(a)$. Furthermore, Eqs. (10) and (14) imply that $D^{-1}(\gamma D(a)) = D(a) \rightsquigarrow D^{-1}(\gamma) = D(a) \rightsquigarrow D(D^{-2}(\gamma)) = D^{-2}(\gamma) \to a = (\varepsilon \to a)D^{-2}(\gamma)^{-1} = a$. Hence $\gamma D(a) = D(a)$. Thus $D(a)$ is birational. By Eq. (14), we get $D^{-1}(D(a)a) = a \rightsquigarrow a = K$, which yields $D(a) \cdot a = K$. Similarly, Eq. (12) gives $D(aD^{-1}(a)) = a \to a = K$. Thus $a \cdot D^{-1}(a) = K$. So we obtain $K \cdot D^{-1}(a) = D(a) \cdot a \cdot D^{-1}(a) = D(a)$. Hence $D(a) \cdot a = a \cdot D(a) = K$. Since $D(a)$ is a complement of K, Corollary 2 implies that $1D(a) > D(a)$. Hence $1D(a) = 1$, and similarly, $D(a)1 = 1$. So $D(a) \in G$.

It remains to verify that $G \subset Q$ is multiplicatively closed. Suppose that $ab = 1$ with $a, b \in G$. Then $1 = D(a)1 = D(a)ab = Kb = b$, which is impossible. Thus $ab \in G$. □

If a Serre quantale Q admits a greatest nilpotent element, we call it the *radical* $\mathrm{rad}(Q)$ of Q. By Proposition 29, the radical is a birational complement of K. There is a close relationship between the radical and Chu's construction of $*$-autonomous categories [1, 6]. For a unital quantale F, Chu's construction yields a quantale $\mathrm{Chu}(F)$ which is obtained as follows [26]. Define $\mathrm{Chu}(F) := F \times F$ with partial order

$$(a, a') \leqslant (b, b') \;:\Longleftrightarrow\; a \leqslant b \text{ and } a' \geqslant b'$$

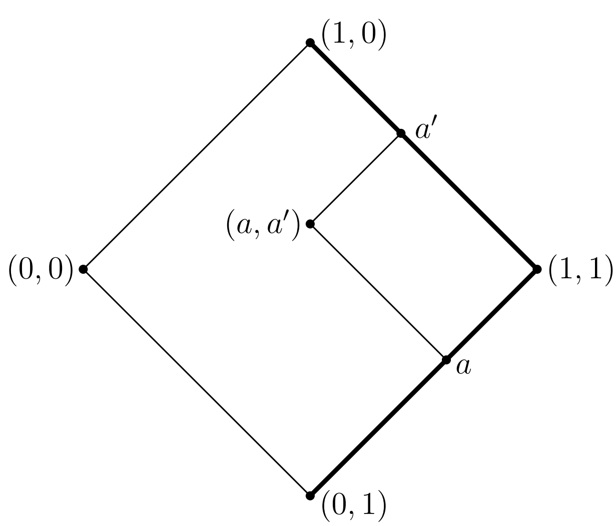
and multiplication

$$(a, a') \cdot (b, b') := \big(ab, (a \to b') \wedge (b \rightsquigarrow a')\big).$$

It is easily checked that $\mathrm{Chu}(F)$ is a quantale with unit element $(\varepsilon, 1)$ and involutive Serre duality $(a, a') \mapsto (a', a)$. The unit group of $\mathrm{Chu}(F)$ is given by

$$\mathrm{Chu}(F)^\times = \{(\alpha, 1) \mid \alpha \in F^\times\}.$$

The greatest element of $\mathrm{Chu}(F)$ is $1 = (1, 0)$, and the smallest element is $0 = (0, 1)$. Moreover, $\mathrm{Chu}(F)$ has a non-zero radical $(0, 0)$ with $(0, 0)^2 = (0, 1) = 0$. As a lattice, $\mathrm{Chu}(F)$ can be represented as follows:



Proposition 30. *Let F be a modular quantale of infinite length such that each $a < 1$ is semisimple. Then* $\mathrm{Chu}(F)$ *is a Serre quantale with birational unit* $K := (1,1)$ *and non-zero radical* $\mathrm{rad}(\mathrm{Chu}(F)) = (0,0)$. *Conversely, any Serre quantale Q with involutive Serre duality and non-zero radical admits a natural embedding* $\mathrm{Chu}([0,K]) \hookrightarrow Q$.

Proof. Let F is a modular quantale of infinite length such that each $a < 1$ is semisimple. Then $K := (1,1)$ is a birational unit in $\mathrm{Chu}(F)$ which splits $\mathrm{Chu}(F)$ into the intervals $[0,K] = F\times 1$ and $[K,1] = 1\times K$. So $\mathrm{At}(\mathrm{Chu}(F)) = \mathrm{At}(F)\times 1 = \mathrm{Chu}(F)^{\times}$. It is easily checked that $\mathrm{Chu}(F)$ is modular. Hence $\mathrm{Chu}(F)$ is a Serre quantale. Since $(0,0)^2 = 0$ and $(1,1)(0,0) = (0,0)$, Proposition 29 implies that $\mathrm{rad}(\mathrm{Chu}(F)) = (0,0)$.

Conversely, every Serre quantale Q gives rise to a subquantale $F := [0,K]$. Let D be an involutive Serre duality and assume that $\mathrm{rad}(Q) > 0$. Define a map $f\colon \mathrm{Chu}(F) \to Q$ by

$$f(a,a') := a \vee \big(D(a') \wedge \mathrm{rad}(Q)\big).$$

Proposition 29 implies that $r := \mathrm{rad}(Q)$ is a complement of K. By Corollary 1 of Proposition 29, $r^n r = 0$ with $r^n \neq 0$ implies that $r^n = D(r)$. Hence $D(r) \leqslant r$, and thus $D(r) = r$. So we obtain $r^2 = 0$.

We show first that

$$(a \to b) \vee r = a \to (b \vee r) \tag{15}$$

holds for $a, b \in F$. For $a = K$, we have

$$r \leqslant (K \to b) \vee r \leqslant K \to (b \vee r) = \big(K \to (b \vee r)\big)K.$$

Thus, to verify Eq. (15), we can assume that $K \to (b\vee r) = 1$. So $b\vee r = 1$, which yields $b = K$. So Eq. (15) trivially holds. Thus, assume that $a < K$. Then $a = \alpha_1 \vee \cdots \vee \alpha_n$ with $\alpha_i \in Q^{\times}$. Hence $a \to b = b\alpha_1^{-1} \wedge \cdots \wedge b\alpha_n^{-1}$. Since r is a complement of K, this gives $(a \to b) \vee r = \bigwedge_{i=1}^n (b\alpha_i^{-1} \vee r) = \bigwedge_{i=1}^n (b \vee r)\alpha_i^{-1} = a \to (b \vee r)$. Similarly,

$$(a \rightsquigarrow b) \vee r = a \rightsquigarrow (b \vee r).$$

Now let $(a,a'),(b,b') \in \mathrm{Chu}(F)$ be given. Since $r^2 = 0$, we have $f(a,a')f(b,b') = \big(a \vee D(a' \vee r)\big)\big(b \vee D(b' \vee r)\big) = ab \vee aD(b' \vee r) \vee D(a' \vee r)b$. By Eq. (14),

$$\begin{aligned} D^{-1}\big(aD(b' \vee r) \vee D(a' \vee r)b\big) &= D^{-1}\big(aD(b' \vee r)\big) \wedge D^{-1}\big(D(a' \vee r)b\big) \\ &= \big(D(b' \vee r) \rightsquigarrow D^{-1}(a)\big) \wedge \big(b \rightsquigarrow (a' \vee r)\big) \\ &= \big(D^{-2}(a) \to (b' \vee r)\big) \wedge \big(b \rightsquigarrow (a' \vee r)\big). \end{aligned}$$

On the other hand,

$$f\big((a,a')(b,b')\big) = f\big(ab, (a \to b') \wedge (b \rightsquigarrow a')\big) = ab \vee D\big(((a \to b') \wedge (b \rightsquigarrow a')) \vee r\big),$$

and the above equations give

$$\begin{aligned} \big((a \to b') \wedge (b \rightsquigarrow a')\big) \vee r &= \big((a \to b') \vee r\big) \wedge \big((b \rightsquigarrow a') \vee r\big) \\ &= \big(a \to (b' \vee r)\big) \wedge \big(b \rightsquigarrow (a' \vee r)\big). \end{aligned}$$

Since $D^2 = 1$, this shows that f is a homomorphism of semigroups. Furthermore, $f(\varepsilon, 1) = \varepsilon \vee (D(1) \wedge r) = \varepsilon$.

For a family of elements $(a_i, a_i') \in \mathrm{Chu}(F)$, the natural isomorphism between the intervals $[0, r]$ and $[K, 1]$ implies that $\big(\bigvee D(a_i')\big) \wedge r = \bigvee (D(a_i') \wedge r)$. Hence $f(\bigvee (a_i, a_i')) = f(\bigvee a_i, \bigwedge a_i') = \bigvee a_i \vee \big(D(\bigwedge a_i') \wedge r\big) = \bigvee a_i \vee \big(\big(\bigvee D(a_i')\big) \wedge r\big) = \bigvee a_i \vee \bigvee \big(D(a_i') \wedge r\big) = \bigvee f(a_i, a_i')$, which shows that f is a morphism of unital quantales.

It remains to show that f is injective. Let $a, a' \in F$ be given. Then $f(a, a') \wedge K = a \vee \big(D(a') \wedge r \wedge K\big) = a$ and $D(f(a, a')) \wedge K = D(a) \wedge (a' \vee r) \wedge K = a' \vee (r \wedge K) = a'$. □

9 Non-degenerate Serre quantales

Proposition 30 gives a construction of Serre quantales with non-zero radical. In this section, we deal with a class of Serre quantales Q with zero radical. We define

$$R(Q) := \{a \in Q \mid K \not\leqslant a \not\leqslant K\}$$

and call the elements of $R(Q)$ *regular.* With the same proof as Proposition 21, we have

Proposition 31. *Let Q be a Serre quantale. Then each interval $[b, a]$ in $R(Q)$ is of finite length*

$$\ell(a/b) = \ell(a \vee K / b \vee K) + \ell(a \wedge K / b \wedge K).$$

Clearly, $R(Q)$ is closed under Serre duality and multiplication with units.

Definition 19. We say that a Serre quantale Q is *non-degenerate* if $R(Q)$ is non-empty and closed with respect to multiplication, and ε is the only non-zero idempotent $< K$.

By Corollary 1 of Proposition 29, non-degenerate Serre quantales Q have no zero divisors $a > 0$.

Proposition 32. *Let Q be a non-degenerate Serre quantale. Then $R(Q)$ is a sublattice of Q. If $a, b \in R(Q)$, then $a \to b \in R(Q)$ and $a \rightsquigarrow b \in R(Q)$.*

Proof. Let $a, b \in R(Q)$ be given. We show first that $a \vee b \in R(Q)$. Since $a \wedge K$ is semisimple, $(\varepsilon \vee a) \wedge K = \varepsilon \vee (a \wedge K)$ is semisimple, too. Hence $\varepsilon \vee a, \varepsilon \vee b \in R(Q)$, and thus $(\varepsilon \vee a)(\varepsilon \vee b) \in R(Q)$. Since $a \vee b \leqslant (\varepsilon \vee a)(\varepsilon \vee b)$, this proves that $a \vee b \in R(Q)$. By Serre duality, it follows that $R(Q)$ is a sublattice of Q. Since $R(Q)$ is closed with respect to any Serre duality and its inverse, Eq. (12) yields $a \to b = D(aD^{-1}(b)) \in R(Q)$. Similarly, $a \rightsquigarrow b = D^{-1}(D(b)a) \in R(Q)$. □

Proposition 33. *Let Q be a non-degenerate Serre quantale. There is a maximal order $e_Q \in R(Q)$ of Q. Moreover, $e_Q \wedge K = \varepsilon$.*

Proof. Since Q is non-degenerate, there exists a regular element $a \in R(Q)$. Hence $e := a \to a \in R(Q)$ is an order in Q. Thus $e \wedge K$ is an order $< K$, which yields $e \wedge K = \varepsilon$. Since $\ell(1/e \vee K)$ is finite, there is an order $e_Q \geqslant e$ with $\ell(1/e_Q \vee K)$ minimal. By Proposition 31, e_Q is a maximal order in $R(Q)$. $\square$

For a non-degenerate Serre quantale Q, consider the set

$$\mathrm{Div}(Q) := \{a \in R(Q) \mid a = (a \to e_Q) \rightsquigarrow e_Q = (a \rightsquigarrow e_Q) \to e_Q\}.$$

We call the elements of $\mathrm{Div}(Q)$ *divisors*. From now on, we fix a Serre duality D.

Theorem 7. *Let Q be a non-degenerate Serre quantale with maximal order $e_Q \in R(Q)$ such that $D(e_Q) \in \mathrm{Div}(Q)$. Then the divisors form a group*

$$\mathrm{Div}(Q) = \{a \in R(Q) \mid e_Q a = a e_Q = a\}$$

with neutral element $e_Q = D^2(e_Q)$. The inverse of $a \in \mathrm{Div}(Q)$ is $a \to e_Q = a \rightsquigarrow e_Q$.

Proof. For any $a \in R(Q)$ we have

$$e_Q(a \to e_Q) = a \to e_Q, \qquad (a \rightsquigarrow e_Q)e_Q = a \rightsquigarrow e_Q. \tag{16}$$

Indeed, $(a \to e_Q)a \leqslant e_Q$ implies that $e_Q(a \to e_Q)a \leqslant e_Q$, which yields $a \to e_Q \leqslant e_Q(a \to e_Q) \leqslant a \to e_Q$. The second equation is proved similarly. Since $D(e_Q)$ is a divisor, this implies that $e_Q D(e_Q) = D(e_Q)e_Q = D(e_Q)$. By Eq. (14), we obtain $e_Q = D^{-1}(e_Q D(e_Q)) = D(e_Q) \rightsquigarrow D^{-1}(e_Q)$. Hence $D(e_Q) = D(e_Q)e_Q \leqslant D^{-1}(e_Q)$, and thus $e_Q \leqslant D^2(e_Q)$. Now Proposition 27 implies that $D^2(e_Q)$ is an order. Since e_Q is maximal, we get $D^2(e_Q) = e_Q$.

Next we show that $D(e_Q)$ is invertible in $\mathrm{Div}(Q)$. By Eq. (14), we have

$$D^{-1}\big(D(e_Q)(D(e_Q) \to e_Q)\big) = (D(e_Q) \to e_Q) \rightsquigarrow e_Q = D(e_Q).$$

Hence $D(e_Q)(D(e_Q) \to e_Q) = D^2(e_Q) = e_Q$. Similarly, Eq. (12) gives

$$D\big((D(e_Q) \rightsquigarrow e_Q)D(e_Q)\big) = (D(e_Q) \rightsquigarrow e_Q) \to D^2(e_Q) = (D(e_Q) \rightsquigarrow e_Q) \to e_Q = D(e_Q).$$

Thus $(D(e_Q) \rightsquigarrow e_Q)D(e_Q) = e_Q$. By Eqs. (16), $D(e_Q) \rightsquigarrow e_Q = (D(e_Q) \rightsquigarrow e_Q)e_Q = (D(e_Q) \rightsquigarrow e_Q)D(e_Q)(D(e_Q) \to e_Q) = e_Q(D(e_Q) \to e_Q) = D(e_Q) \to e_Q$ is an inverse of $D(e_Q)$ in $\mathrm{Div}(Q)$.

Now assume that $a \in R(Q)$ belongs to the submonoid

$$M := \{a \in R(Q) \mid e_Q a = a e_Q = a\}$$

of $R(Q)$. Then $e_Q \leqslant a \rightsquigarrow a$, which yields $a \rightsquigarrow a = e_Q$. So Eq. (14) implies that $D^{-1}(D(a)a) = a \rightsquigarrow a = e_Q$. Hence $D(a)a = D(e_Q)$. In particular, $D(e_Q)^{-1}D(a) \leqslant a \to e_Q$, which yields $e_Q = D(e_Q)^{-1}D(a)a \leqslant (a \to e_Q)a \leqslant e_Q$. Thus $(a \to e_Q)a = e_Q$. On the other hand, $a \to a = e_Q$, and Eq. (12) gives $D(aD^{-1}(a)) = a \to a = e_Q$. So $aD^{-1}(a) = D^{-1}(e_Q) = D(e_Q)$. Therefore, $D^{-1}(a)D(e_Q)^{-1} \leqslant a \rightsquigarrow e_Q$, which gives $e_Q = aD^{-1}(a)D(e_Q)^{-1} \leqslant a(a \rightsquigarrow e_Q) \leqslant e_Q$. Hence $a(a \rightsquigarrow e_Q) = e_Q$. Since $a \to e_Q$ and

$a \rightsquigarrow e_Q$ belong to M, this shows that a is invertible in M with inverse $a \to e_Q = a \rightsquigarrow e_Q$. In particular, $a \in \mathrm{Div}(Q)$. $\square$

In the following corollaries, let Q be a non-degenerate Serre quantale with a maximal order $e_Q \in R(Q)$ and $D(e_Q) \in \mathrm{Div}(Q)$. We call $D(e_Q)$ the *canonical divisor*. By $P(Q)$ we denote the set of upper neighbours of e_Q in $\mathrm{Div}(Q)$. The elements of $P(Q)$ will be called *primes* of Q.

Corollary 1. *The divisor group* $\mathrm{Div}(Q)$ *is abelian and a sublattice of* $R(Q)$*. Every* $a \in \mathrm{Div}(Q)$ *has a unique factorization* $a = \pi_1^{n_1} \cdots \pi_r^{n_r}$ *with* $\pi_i \in P(Q)$ *and* $n_i \in \mathbb{Z}$*.*

Proof. By Theorem 7, $\mathrm{Div}(Q)$ is closed with respect to finite joins. If $a, b \in \mathrm{Div}(Q)$, then $e_Q(a \wedge b) \leqslant e_Q a \wedge e_Q b = a \wedge b$. Hence $\mathrm{Div}(Q)$ is a sublattice of $R(Q)$. Proposition 31 implies that $\mathrm{Div}(Q)$ is an ℓ-group with intervals of finite length. By [2], Theorem 37, $\mathrm{Div}(Q)$ is a free abelian group with basis $P(Q)$. $\square$

As in Section 6, every regular element $a \in R(Q)$ has a *degree*

$$\deg(a) := \ell\big((a \vee e_Q)/e_Q\big) - \ell\big((a \vee e_Q)/a\big).$$

For any $a \in R(Q)$, we consider the finite integers

$$\ell(a) := \ell\big((a \wedge K)/0\big), \qquad i(a) := \ell\big(1/(a \vee K)\big)$$

and call $i(a)$ the *speciality index* of a. The integer $g_Q := i(e_Q) \in \mathbb{N}$ will be called the *genus* of Q. The Riemann-Roch theorem easily follows:

Corollary 2. *Assume that* $a \in R(Q)$*. Then* $i(a) = \ell(D(a))$ *and*

$$\ell(a) - i(a) = \deg(a) + 1 - g_Q. \tag{17}$$

If ξ *is a divisor, then* $D(\xi) = D^{-1}(\xi) = \xi^{-1} D(e_Q)$*.*

Proof. With $b := a \vee e_Q$, Proposition 31 gives $\deg(b) - \deg(a) = \ell(b/a) = i(a) - i(b) + \ell(b) - \ell(a)$. Replacing a by e_Q, we obtain $\deg(b) = g_Q - i(b) + \ell(b) - 1$. Subtracting both equations, this proves Eq. (17). Since $D(a \vee K) = D(a) \wedge K$, we have $i(a) = \ell(D(a))$. Let $\xi \in \mathrm{Div}(Q)$ be a divisor. Then $D(\xi) \rightsquigarrow D(\xi) = \xi \to \xi = e_Q$. By Proposition 27, $D^2(e_Q) = e_Q$ gives $D^2(\xi) \in \mathrm{Div}(Q)$. Hence $D(\xi) \to D(\xi) = D^2(\xi) \rightsquigarrow D^2(\xi) = e_Q$. Thus $D(\xi) \in \mathrm{Div}(Q)$. So Eq. (12) gives $D(D(\xi)\xi) = D(\xi) \to D(\xi) = e_Q$, which yields $D(\xi)\xi = D(e_Q)$. Thus $D(\xi) = D(e_Q)\xi^{-1} = \xi^{-1} D(e_Q)$. Furthermore, we obtain $e_Q = D^2(e_Q) = D(\xi D(\xi)) = \xi \to D^2(\xi)$, which implies that $\xi = e_Q \xi \leqslant D^2(\xi)$. By Proposition 27, $\deg \xi = \deg D^2(\xi)$. Whence $\xi = D^2(\xi)$. $\square$

Note that a unit $\gamma \in Q^\times$ need not commute with the maximal order e_Q, that is, $e_Q \gamma$ need not be a divisor. Nevertheless, we have

Corollary 3. $\deg e_Q \gamma = 0$ *for all* $\gamma \in Q^\times$*.*

Proof. Since $e_Q\gamma \wedge K = (e_Q \wedge K)\gamma$ and $e_Q\gamma \vee K = (e_Q \vee K)\gamma$, the corollary follows immediately by the Riemann-Roch formula Eq. (17). □

Corollary 4. *The order e_Q is the smallest divisor $\xi \in \mathrm{Div}(Q)$ with $\varepsilon \leqslant \xi$.*

Proof. If $\varepsilon \leqslant \xi$, then $e_Q = e_Q\varepsilon \leqslant e_Q\xi = \xi$. □

10 Riemann-Roch quantales

In this section, we focus upon commutative non-degenerate Serre quantales. The unit group $Q^\times$ of such a quantale Q is a modular discrete q-group. So its completion F_Q is a modular q-field. Since any two orders $e, e' \in R(Q)$ have a common overorder $ee' \in R(Q)$, Proposition 33 implies that Q has a unique maximal order $e_Q \in R(Q)$. As before, we fix a Serre duality D.

Definition 20. We define a *Riemann-Roch quantale* to be a commutative non-degenerate Serre quantale Q with $D(e_Q) \in \mathrm{Div}(Q)$.

Since Q is commutative, Eqs. (13) and (14) give $D(a) = a \to D(\varepsilon) = a \rightsquigarrow D^{-1}(\varepsilon) = D^{-1}(a)$ for all $a \in Q$. Hence D is involutive. Therefore, we simply write $a^* := D(a)$ for the dual. Thus

$$(ab)^* = a \to b^*, \qquad a^* = a \to \varepsilon^*$$

holds for $a, b \in Q$. Furthermore, Eqs. (10) give

$$\gamma(a \to b) = a \to \gamma b = \gamma^{-1}a \to b,$$

and in particular,

$$(\gamma a)^* = \gamma^{-1}a^*$$

for $a, b \in Q$ and $\gamma \in Q^\times$. For $a \in R(Q)$, the Riemann-Roch theorem states that

$$\ell(a) - \ell(a^*) = \deg a + 1 - g_Q. \tag{18}$$

For the canonical divisor $a = e_Q^*$, this gives

$$\deg e_Q^* = 2g_Q - 2. \tag{19}$$

By Corollary 1 of Theorem 7, every divisor $\xi \in \mathrm{Div}(Q)$ has a canonical representation

$$\xi = \prod_{\pi \in P(Q)} \pi^{n_\pi(\xi)}$$

with $n_\pi(\xi) \in \mathbb{Z}$ such that $n_\pi(\xi) = 0$ for almost all $\pi \in P(Q)$. Hence

$$e_\pi := \bigvee\{\alpha \in Q^\times \mid e_Q\alpha \wedge \pi \leqslant e_Q\} = \bigvee\{\alpha \in Q^\times \mid n_\pi(e_Q\alpha) \leqslant 0\}$$

is a prime order in F_Q for all $\pi \in P(Q)$. So we have a canonical map

$$P(Q) \longrightarrow V(F_Q). \tag{20}$$

Proposition 34. *Let Q be a Riemann-Roch quantale. The map* (20) *is injective. Its image consists of discrete prime orders.*

Proof. For distinct $\pi, \sigma \in P(Q)$, the Riemann-Roch formula gives $\ell(\pi^{g_Q+1}\sigma^{-1}) \geqslant \ell(\pi^{g_Q+1}\sigma^{-1}) - i(\pi^{g_Q+1}\sigma^{-1}) = \deg(\pi^{g_Q+1}\sigma^{-1}) + 1 - g_Q = 1$. So there is a unit $\alpha \in Q^\times$ with $\alpha \leqslant \pi^{g_Q+1}\sigma^{-1}$, that is, $e_Q\alpha^{-1} \wedge \sigma \nleqslant e_Q$ and $e_Q\alpha \wedge \pi \nleqslant e_Q$. Hence $\alpha \nleqslant e_\pi$ and $\alpha^{-1} \nleqslant e_\sigma$. Thus $\alpha \leqslant e_\sigma$, which proves that $e_\pi \neq e_\sigma$.

For $\alpha, \beta \in Q^\times$, we have $\alpha \leqslant e_\pi\beta \Leftrightarrow \alpha\beta^{-1} \leqslant e_\pi \Leftrightarrow n_\pi(e_Q\alpha\beta^{-1}) \leqslant 0 \Leftrightarrow n_\pi(e_Q\alpha) \leqslant n_\pi(e_Q\beta)$. Hence $\bigvee\{\alpha \in Q^\times \mid n_\pi(e_Q\alpha) < 0\}$ is the radical of e_π. Thus e_π is discrete. □

For a Riemann-Roch quantale Q we consider the following condition:

(∗) For each $\pi \in P(Q)$, the atoms in the modular lattice $[\pi^{-1}, e_Q]$ are connected, and $|P(Q)| \geqslant 3$.

Theorem 8. *Let Q be a Riemann-Roch quantale satisfying* (∗). *Then F_Q is a connected modular q-field. Moreover, the following are equivalent:*

(a) *F_Q is a function q-field.*
(b) *The image of the map* (20) *consists of the minimal prime orders in F_Q.*
(c) $\dim F_Q = 1$.

Conversely, the double of any function q-field F is a Riemann-Roch quantale with $J_F = \mathrm{Div}(D(F))$, *satisfying* (∗) *and* $F_{D(F)} = F$.

Proof. As a sublattice of Q, the q-monoid $E(F_Q)$ is modular. So F_Q is modular by Proposition 2. Let us show that F_Q is connected. Thus, let $\alpha, \beta \in Q^\times$ be distinct units. Then $\deg e_Q\alpha = \deg e_Q\beta = 0$ by Corollary 3 of Theorem 7. With $\eta := e_Q\alpha \vee e_Q\beta$, there are distinct primes $\pi_1, \pi_2 \in P(Q)$ with $e_Q\alpha \leqslant \eta\pi_1^{-1} < \eta$ and $e_Q\beta \leqslant \eta\pi_2^{-1} < \eta$. Choose $\pi \in P(Q)$ with $\pi_1 \neq \pi \neq \pi_2$. Define $d' := e_Q\alpha \wedge e_Q\beta$ and $d := d' \wedge \eta\pi^{-1}$. Then $\ell(\alpha \vee d/d) = \ell(\beta \vee d/d) = 1$. By Corollary 4 of Theorem 7, $e_Q\alpha$ is the smallest divisor $\geqslant \alpha$. Hence $\alpha \vee d \nleqslant d'$ and $\alpha \vee d \nleqslant e_Q\alpha \wedge \eta\pi^{-1}$. So there is a circuit $\{\alpha \vee d, (\alpha \vee \eta\pi^{-1}) \wedge d', (\alpha \vee d') \wedge \eta\pi^{-1}\}$ in $L := [d, \eta]$. Indeed,

$$\begin{aligned}\alpha \vee d &\leqslant (\eta\pi^{-1} \vee d') \wedge (\alpha \vee \eta\pi^{-1}) \wedge (\alpha \vee d') = \big(\eta\pi^{-1} \vee (d' \wedge (\alpha \vee \eta\pi^{-1}))\big) \wedge (\alpha \vee d') \\ &= \big(((\alpha \vee \eta\pi^{-1}) \wedge d') \vee \eta\pi^{-1}\big) \wedge (\alpha \vee d') = \big(\alpha \vee \eta\pi^{-1}) \wedge d'\big) \vee \big(\eta\pi^{-1} \wedge (\alpha \vee d')\big).\end{aligned}$$

So $\alpha \vee d$ is connected with $a := (\alpha \vee \eta\pi^{-1}) \wedge d'$. Similarly, $\beta \vee d$ is connected with $b := (\beta \vee \eta\pi^{-1}) \wedge d'$. Now the intervall $[d, d']$ is isomorphic to $[e_Q\alpha \wedge \eta\pi^{-1}, e_Q\alpha] = [\pi^{-1}\alpha, e_Q\alpha]$, which is isomorphic to $[\pi^{-1}, e_Q]$. So condition (∗) implies that a and b are connected in $[d, d']$. Hence there is a circuit $\{\alpha \vee d, \beta \vee d, c\}$ in $[d, \eta]$.

Since $\beta \nleqslant \alpha \vee d \leqslant e_Q\alpha$, we have $\alpha = \alpha \vee \big(\beta \wedge (\alpha \vee d)\big) = (\alpha \vee \beta) \wedge (\alpha \vee d)$. Hence $(\alpha \vee \beta) \wedge d \leqslant (\alpha \vee \beta) \wedge (\alpha \vee d) = \alpha$, and thus $(\alpha \vee \beta) \wedge d = 0$. So the intervals $[d, \alpha \vee \beta \vee d]$ and $[0, \alpha \vee \beta]$ are isomorphic, which shows that $\{\alpha, \beta, c \wedge (\alpha \vee \beta)\}$ is a circuit in F_Q. Whence F_Q is connected.

(a) ⇒ (b): Since $\dim F_Q = 1$, the image of the map (20) consists of minimal prime orders. Suppose that $e \in X(F_Q)$ does not belong to the image of this map. Choose any $\pi \in P(Q)$. By Corollary 2 of Theorem 4, $L(\widehat{e_\pi}^{-1}\hat{e}^r) > 0$ holds in $D(F_Q)$ for some $r \in \mathbb{N}$. So there is a unit $\alpha \in Q^\times$ with $\alpha \leqslant \pi^{-1}$. Since $\deg \pi^{-1} < 0$, this is impossible. Thus $X(F_Q)$ is the image of (20).

(b) ⇒ (c): By Proposition 34, the e_π are discrete, hence maximal by Corollary 1 of Proposition 14. Thus $\dim F_Q = 1$.

(c) ⇒ (a): Since ε is the only order $e < K$, it follows that $\overline{\varepsilon} = \varepsilon$ holds in F_Q. So it remains to verify that F_Q is finitely generated. Let $\alpha \neq \varepsilon$ be a unit in $Q^\times$. We use the argument in the proof of Theorem 4. Let $\beta_1, \ldots, \beta_n \in Q^\times$ be units which are independent over $\varepsilon(\alpha)$. So there is an integer $r > 0$ with $\beta_i^{r+1} \leqslant \varepsilon(\alpha)(\varepsilon \vee \beta_i \vee \cdots \vee \beta_i^r)$ for all $i \in \{1, \ldots, n\}$. Hence $\beta_i^{r+1} \leqslant \gamma^{-1}\varepsilon[\alpha](\varepsilon \vee \beta_i \vee \cdots \vee \beta_i^r)$ for some unit $\gamma \leqslant \varepsilon[\alpha]$ and all i. Replacing β_i by $\beta_i\gamma$, we can assume that $\beta_i^{r+1} \leqslant \varepsilon[\alpha] \vee \varepsilon[\alpha]\beta_i \vee \cdots \vee \varepsilon[\alpha]\beta_i^r = \varepsilon[\alpha](\varepsilon \vee \beta_i \vee \cdots \vee \beta_i^r)$ for all i. Thus $\varepsilon \leqslant \varepsilon[\alpha](\beta_i^{-r-1} \vee \cdots \vee \beta_i^{-1})$ for all i. If $\alpha \leqslant e_\pi$ with $\pi \in P(Q)$, then $e_\pi \leqslant e_\pi(\beta_i^{-r-1} \vee \cdots \vee \beta_i^{-1})$, which implies that $\beta_i \leqslant e_\pi$. With $\xi := e_Q\alpha \vee e_Q$, it follows that there is an integer $k > 0$ with $\beta_i \leqslant \xi^k$ for all i.

Now let s, t be integers with $0 \leqslant s \leqslant t - k$. Then $\alpha^s\beta_i \leqslant \xi^{s+k} \leqslant \xi^t$ holds for $i \in \{1, \ldots, n\}$. The height of $(\varepsilon \vee \alpha \vee \cdots \vee \alpha^{t-k})(\beta_1 \vee \cdots \vee \beta_n)$ in F_Q is $(t - k + 1)n$. Hence $\ell(\xi^t) \geqslant (t - k + 1)n$. For large t, we have $i(\xi^t) = 0$. So the Riemann-Roch formula gives $(t - k + 1)n \leqslant \ell(\xi^t) = \deg \xi^t + 1 - g_Q = t \deg \xi + 1 - g_Q$. Thus $t(n - \deg \xi) \leqslant (k - 1)n + 1 - g_Q$ holds for large t, which yields $n \leqslant \deg \xi$. This proves that $\ell(K/\varepsilon(\alpha)) < \infty$. So F_Q is finitely generated.

Finally, let F is a function q-field. The atoms of $D(F)$ are invertible. So condition (a) of Definition 17 holds. By Theorem 6 and Corollary 4 of Theorem 4, $K \in D(F)$ is birational. So Theorem 6 implies that $D(F)$ is a Serre quantale. Furthermore, $D(F)$ is non-degenerate, and e_F is the unique maximal order in $R(D(F)) = R(F)$. For a fixed Serre duality D in $D(F)$, Theorem 7 implies that $D(e_F) \in J_F \subset \mathrm{Div}(D(F))$. So $D(F)$ is a Riemann-Roch quantale, and $F_{D(F)} = F$. For $\pi \in P(D(F))$ and $\alpha \in F^\times$, we have $e_F\alpha \wedge \widehat{e_\pi} \leqslant e_F \Leftrightarrow \alpha \leqslant e_\pi \Leftrightarrow e_F\alpha \wedge \pi \leqslant e_F$. Thus, by the approximation theorem (Theorem 1), $\pi = \widehat{e_\pi} \in J_F$. So Corollary 1 of Theorem 7 yields $\mathrm{Div}(D(F)) \subset J_F$. □

Example 10. Let $C := \langle \gamma \rangle$ be an infinite cyclic group. Then $F := \mathfrak{P}(C)$ is a modular q-field (Example 5). We regard F as an algebra over the field $k := \mathbb{F}_2$ with two elements. Though F is not connected, the construction of the double $D(F)$ still applies in a modified form. There are two prime orders

$$e_+ := \{\gamma^n \mid n \geqslant 0\}, \qquad e_- := \{\gamma^n \mid n \leqslant 0\}$$

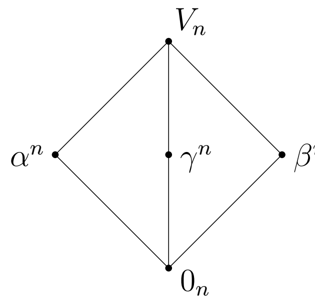
in F with $e_+ \wedge e_- = \{\gamma^0\} = \varepsilon$. The corresponding value groups are $G_{e_+} = \langle \pi_+ \rangle$ and $G_{e_-} = \langle \pi_- \rangle$, where

$$\pi_+ := \{\gamma^n \mid n > 0\}, \qquad \pi_- := \{\gamma^n \mid n < 0\}.$$

The group of replete ideals is

$$J_F := \{\pi_+^n \pi_-^m \mid m, n \in \mathbb{Z}\} \cong \mathbb{Z} \times \mathbb{Z}.$$

To define $\Pi(F)$, let V be the quantale of subspaces of $k \times k$, the multiplication of V being induced from the ring structure of $k \times k$. For example, $k(1,0) \cdot k(1,1) = k(1,0)$ and $k(1,0) \cdot k(0,1) = 0$. So the product $\Pi(F) := V^C$ is a quantale. For each $\gamma^n \in C$, we denote the elements of the corresponding component V_n of V^C as follows:



$$\alpha^n := k(\gamma^n, 0)$$
$$\beta^n := k(0, \gamma^n)$$
$$\gamma^n := k(\gamma^n, \gamma^n)$$

So the elements of $\Pi(F)$ can be regarded as subsets of $\bigsqcup_{n\in\mathbb{Z}}(V_n \smallsetminus \{0\})$. Consider the componentwise embedding

$$J_F \hookrightarrow \Pi(F),$$

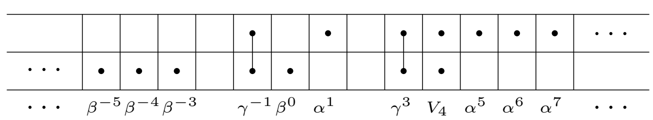
given by $\pi_+^n \pi_-^m \mapsto \{\alpha^i \mid i \geqslant n\} \vee \{\beta^i \mid i \leqslant m\}$, and the multiplicatively closed sublattice of regular elements

$$R(F) := \{a \in \Pi(F) \mid \exists \xi \in J_F \colon \xi \leqslant a \leqslant \xi^{-1}\}.$$

Furthermore, we consider the diagonal

$$K := \{\gamma^n \mid n \in \mathbb{Z}\} \in \Pi(F),$$

and define the double of F to be the sublattice $D(F)$ of $\Pi(F)$ generated by $R(F)$ and K. It is convenient to represent the elements of $D(F)$ graphically. For example, a typical element of $R(F)$ looks as follows:

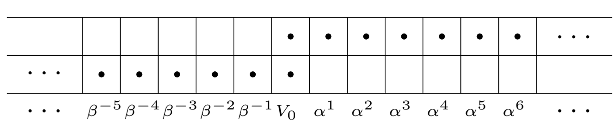


(21)

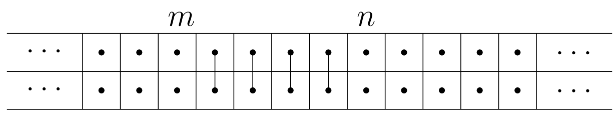
As in Corollary 4 of Theorem 4, we will obtain

$$D(F) = R(F) \sqcup E(F) \sqcup E^*(F) \sqcup \{K\}. \tag{22}$$

The order $e_F := \pi_+^0 \pi_-^0$ satisfies $e_F \wedge K = \varepsilon$:



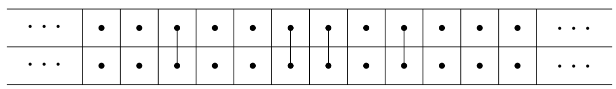
For a unit $\gamma^n \in K$, we have $e_F\gamma = \pi_+^n \pi_-^n \in J_F$ (by abuse, we identify γ^n with $\{\gamma^n\} \in F^\times$), and thus $e_F\gamma^n \wedge K = \gamma^n$. So $E(F)$ belongs to $D(F)$. The elements $\pi_+^n \pi_-^m \vee K$ with $\pi_+^n \pi_-^m \in J_F$ are of the form



In particular, $\pi_+^n\pi_-^m \vee K = 1$ if $n \leqslant m+1$. So a typical element of

$$E^*(F) := \{a \in D(F) \mid a > K\}$$

looks as follows:



Thus Eq. (22) is easily verified. All the arguments of Sections 6 and 7 remain valid in a simplified form. In particular, $D(F)$ is a complete lattice, and $aK = 1$ for $a \in R(F)$. So $D(F)$ is a commutative quantale. There is a canonical differential

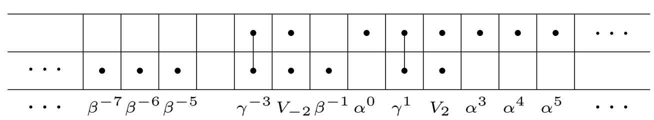
$$\omega := \pi_+\pi_-^{-1} \vee K,$$

and the set $\Omega(F)$ of differentials is a single $F^\times$-orbit. Each element of $E^*(F)$ is a finite meet of differentials.

To calculate the Serre dual $a^* = a \to \omega$ for $a \in D(F)$, each component V_n has to be replaced by a *dual component*:

$$V_n^* = 0_{-n}, \qquad (\alpha^n)^* = \beta^{-n}, \qquad (\gamma^n)^* = \gamma^{-n}.$$

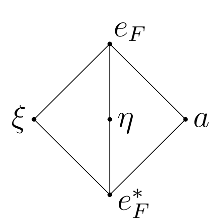
For example, the Serre dual of the element (21) is



Thus $D(F)$ is a Riemann-Roch quantale with canonical divisor $e_F^* = \pi_+\pi_-^{-1}$. The genus is $g_F = 0$. For divisors $\xi = \pi_+^n\pi_-^m$, we have

$$\deg\xi = m - n, \qquad \ell(\xi) = \max\{m - n + 1, 0\}, \qquad i(\xi) = \max\{n - m - 1, 0\}.$$

Between e_F^* and e_F, there are two divisors $\xi := \pi_+^0\pi_-^{-1}$ and $\eta := \pi_+\pi_-^0$, and a regular element $a \in R(F)$ with $\ell(a) = i(a) = 1$:



Remarks. 1. Example 10 describes a curve with two points which is "non-connected" in the sense that the corresponding q-field F is not connected. The example suggests that the construction of the double $D(F)$ extends to more general (perhaps all) non-connected modular q-fields.

2. In Example 10, the double $D(F)$ is generated by J_F and K.

11 The category of q-fields

In Section 1, we introduced the category $\mathbf{Gr}^q$ of complete q-groups. The q-fields form a full subcategory $\mathbf{Fld}^q$. Valuations of a q-field have been identified as morphisms into an ordered q-group. In this section, we study the category of q-fields.

Our first observation is that $\mathbf{Fld}^q$ has a *zero object* $\mathbb{F}_1$, that is, for every q-field F, there is a unique morphism $\mathbb{F}_1 \to F$, and a unique morphism $F \to \mathbb{F}_1$. The q-field $\mathbb{F}_1$ has two elements, 0 and ε. Recall that every morphism $f\colon F \to G$ in $\mathbf{Fld}^q$ is determined by the morphism $f^\times\colon F^\times \to G^\times$ of q-groups. Since $F^\times$ consists of the atoms of F, no non-zero element is mapped to 0. Thus $F \to \mathbb{F}_1$ maps all non-zero elements to ε. Note that $\mathbb{F}_1$ is a sub-q-field of every q-field.

The existence of a zero object gives rise to a zero morphism between arbitrary q-fields F and G, namely $0\colon F \to \mathbb{F}_1 \to G$.

For any morphism $f\colon F \to G$ in $\mathbf{Fld}^q$, the join $e := \bigvee \operatorname{Ker} f^\times = \bigvee\{a \in F \mid f(a) = \varepsilon\}$ is a rational order in F. So $\operatorname{Ker} f := \partial e = [0, e]$ is a sub-q-field of F, the *kernel* of f. The inclusion $\ker f\colon \operatorname{Ker} f \hookrightarrow F$ is a kernel in a categorical sense: $f(\ker f) = 0$, and every morphism $f_0\colon F_0 \to F$ with $ff_0 = 0$ factors uniquely through $\ker f$.

In categories with a zero object, monomorphisms need not be characterized by means of the kernel. In $\mathbf{Fld}^q$, such a problem does not arise.

Proposition 35. *A morphism $f\colon F \to G$ in $\mathbf{Fld}^q$ is monic if and only if* $\ker f = 0$.

Proof. Assume that $\ker f = 0$. Let $f_1, f_2\colon F_0 \to F$ be morphisms with $ff_1 = ff_2$. For any $\alpha \in F_0^\times$, this gives $ff_1(\alpha) = ff_2(\alpha)$. Hence $f(f_1(\alpha)f_2(\alpha)^{-1}) = \varepsilon$, and thus $f_1(\alpha)f_2(\alpha)^{-1} \in \operatorname{Ker} f = \mathbb{F}_1$. So we obtain $f_1(\alpha) = f_2(\alpha)$ for all $\alpha \in F_0^\times$. Thus $f_1 = f_2$. The converse is trivial. □

By Proposition 35, $f\colon F \to G$ is monic if and only if $f^\times\colon F^\times \to G^\times$ is injective. However, f itself need not be injective. Indeed, every q-field F admits a natural morphism $\mathfrak{P}(F^\times) \twoheadrightarrow F$ which maps $A \subset F^\times$ to $\bigvee A$. This morphism is monic, but usually far from injective.

For a q-field F, any sub-q-field G gives rise to a solid order $e := \bigvee G^\times$ in F. Conversely, each solid order e in F determines a sub-q-field with unit group $e^\times$.

Definition 21. We call a sub-q-field G of a q-field F *solid* if $G^\times$ is a solid subgroup of $F^\times$. The sub-q-fields ∂e with a rational order e in F will be called *rational*.

Thus rational sub-q-fields coincide with the kernels of morphisms. In the special case $F = F(K|k)$ with a field extension $K|k$, every intermediate field $k \subset L \subset K$ gives rise to a rational sub-q-field $F(L|k)$, and conversely, every rational sub-q-field of F is of this form. Moreover, the solid orders in $F = F(K|k)$ are the k-subalgebras of K which are k-linearly generated by their units.

The image $f(F)$ of a morphism $f\colon F \to G$ in $\mathbf{Fld}^q$ is a sub-q-field of G. So there is a solid order $e := \bigvee f(F^\times) = f(\bigvee F^\times) = f(1)$ in G. Its quotient order e_0 (Section 2) is a rational order in G. We call $\operatorname{Im} f := \partial e_0$ the *(categorical) image* of f. Every morphism $g\colon G \to H$ in $\mathbf{Fld}^q$ with $gf = 0$ maps e_0 into $\mathbb{F}_1$. So there is a *cokernel* $\operatorname{Cok} f := G/e_0$ of f. The natural morphism $\operatorname{cok} f\colon G \twoheadrightarrow \operatorname{Cok} f$ is a cokernel in the categorical sense. If F, G are modular, then the kernel, the image, and the cokernel of f are again modular. For the cokernel $\operatorname{Cok} f$ this follows since $G/e_0 = \operatorname{Mod}(e_0)$ is a sublattice of G.

By the preceding results, the morphisms in the category $\mathbf{Fld}^q$ of q-fields behave similar to morphisms of topological groups. To be more closely to classical field extensions, morphisms have to be restricted. Otherwise, we would have at least four types of subobject: monomorphisms, sub-q-fields (given by injective monomorphisms), solid sub-q-fields, and rational sub-q-fields. Only the last type occurs in field theory, the others are "irrational".

Definition 22. We call a morphism $f\colon F \to G$ of q-fields *rational* if $f(\downarrow a) = \downarrow f(a)$ holds for all $a \in F$.

Definition 22 shows that the class of rational morphisms is closed with respect to composition. Note that zero morphisms are rational.

Proposition 36. *A morphism $f\colon F \to G$ of q-fields is rational if and only if it satisfies $f([a]) = [f(a)]$ for all $a \in E(F)$.*

Proof. The necessity is trivial. Assume that the condition of the proposition holds. By Definition 22, we have to verify

$$b \leqslant f(a) \implies \exists\, c \leqslant a\colon f(c) = b$$

for $a \in F$ and $b \in G$. Since $a = \bigvee [a]$, any $\beta \leqslant b$ belongs to $[f(\alpha_1) \vee \cdots \vee f(\alpha_n)] = [f(\alpha_1 \vee \cdots \vee \alpha_n)] = f([\alpha_1 \vee \cdots \vee \alpha_n])$ for some units $\alpha_i \leqslant a$. Hence $\beta = f(\alpha)$ with $\alpha \in [\alpha_1 \vee \cdots \vee \alpha_n] \subset [a]$. Thus $b = f(c)$ for some $c \leqslant a$. □

For morphisms $f\colon F \to G$ into a modular q-field G, Proposition 36 can be simplified. The following corollary characterizes rational morphisms f by the geometric property that f maps lines in F to lines in G.

Corollary 1. *Let G be a modular q-field. A morphism $f\colon F \to G$ in $\mathbf{Fld}^q$ is rational if and only if*

$$f([\alpha \vee \beta]) = [f(\alpha) \vee f(\beta)] \tag{23}$$

holds for $\alpha, \beta \in F^\times$.

Proof. We apply Proposition 36, using induction on the height of a. Assume that Eq. (23) holds and $f([a]) = [f(a)]$ has been proved for some $a > 0$ in $E(F)$. We have to verify $f([a \vee \beta]) = [f(a \vee \beta)]$ for any $\beta \in F^\times$. By Lemma 9, each $\gamma \in [f(a) \vee f(\beta)]$ satisfies $\gamma \leqslant f(\alpha) \vee f(\beta)$ for some $\alpha \in [a]$. Hence $\gamma \in [f(\alpha) \vee f(\beta)] = f([\alpha \vee \beta]) \subset f([a \vee \beta])$, and thus $[f(a \vee \beta)] = [f(a) \vee f(\beta)] \subset f([a \vee \beta]) \subset [f(a \vee \beta)]$. So f is rational. The converse is trivial. □

Every morphism $f\colon F \to G$ in $\mathbf{Fld}^q$ gives rise to a commutative diagram

$$\begin{array}{ccccccc} \operatorname{Ker} f & \hookrightarrow & F & \xrightarrow{\qquad f \qquad} & G & \twoheadrightarrow & \operatorname{Cok} f \\ & & \searrow & & \nearrow & & \\ & & & \operatorname{Coim} f \xrightarrow{\dot{f}} \operatorname{Im} f & & & \end{array}$$

with short exact sequences $\operatorname{Ker} f \hookrightarrow F \twoheadrightarrow \operatorname{Coim} f$ and $\operatorname{Im} f \hookrightarrow G \twoheadrightarrow \operatorname{Cok} f$. Like in arbitrary categories, the kernel $\operatorname{Im} f$ of $\operatorname{cok} f$ is the categorical image of f. The cokernel $\operatorname{Coim} f$ of $\ker f$ is said to be the *coimage* of f. If e denotes the rational order in F with $\partial e = \operatorname{Ker} f$, then $\operatorname{Coim} f = \operatorname{Mod}(e)$, and the induced morphism $\dot{f}$ maps $a = ea \in \operatorname{Coim} f$ to $f(a)$. Thus $\dot{f}$ is monic and epic.

Corollary 2. *For a rational morphism $f\colon F \to G$ of q-fields, the induced map $\dot{f}$ is invertible.*

Proof. By Definition 22, $f(F) = {\downarrow} f(1)$. Furthermore, $f(1)$ is an order in G. Assume that $\gamma \in [f(1)]$. Then $\gamma \leqslant f(\bigvee F^\times) = \bigvee f(F^\times)$ implies that $\gamma \leqslant f(a)$ for some $a \in E(F)$. By Proposition 36, it follows that $\gamma = f(\alpha)$ for some $\alpha \in [a]$. Hence $\gamma^{-1} = f(\alpha^{-1}) \leqslant f(1)$, and thus $f(1)$ is a rational order. So $\operatorname{Im} f = f(F)$. Let e be the rational order in F with $\partial e = \operatorname{Ker} f$. For any $a \in \operatorname{Coim} f = \operatorname{Mod}(e)$, we have $\dot{f}(a) = f(a)$. Thus $\dot{f}$ is surjective. On the other hand, $a = \bigvee f^{-1}([f(a)])$. Indeed, this holds for $a = 0$. For $a > 0$, let $\alpha \in f^{-1}([f(a)])$. Then $\varepsilon \leqslant f(\alpha)^{-1} f(a) = f(\alpha^{-1} a)$. Since f is rational, there exists a unit $\beta \leqslant \alpha^{-1} a$ with $f(\beta) = \varepsilon$. Hence $\alpha \leqslant \beta^{-1} a \leqslant ea = a$, and thus $\bigvee f^{-1}([f(a)]) \leqslant a = \bigvee [a] \leqslant \bigvee f^{-1}([f(a)])$. □

Remark. The subcategory of $\mathbf{Fld}^q$ with the same objects and rational morphisms is not closed with respect to cokernels. Indeed, if F is a connected modular q-field with a rational order e such that $\varepsilon < e < 1$, then the cokernel $p\colon F \twoheadrightarrow F/e$ is not rational. Indeed, let $\alpha, \beta \in F^\times$ be units with $\varepsilon \neq \alpha \leqslant e$ and $\beta \nleqslant e$. Choose a circuit $\{\varepsilon, \beta, \gamma\}$. Then $\alpha^{-1} \leqslant e$ implies that $e\alpha = e$. Hence $e(\gamma \vee \alpha\beta) = e(\gamma \vee \beta) \geqslant e$ holds in F/e. Thus, if p would be rational, there would be a unit $\delta \leqslant \gamma \vee \alpha\beta$ with $e\delta = e$. So $\delta \leqslant \varepsilon \vee \beta \vee \alpha\beta = \varepsilon \vee (\varepsilon \vee \alpha)\beta$. By Lemma 9, this gives $\delta \leqslant \varepsilon \vee \alpha'\beta$ for some $\alpha' \in [\varepsilon \vee \alpha]$. If $\delta \neq \varepsilon$, then Lemma 1 implies that $\alpha'\beta \leqslant \varepsilon \vee \delta \leqslant e$. Since $e\alpha' = e$, this is impossible. Thus $\varepsilon \leqslant \gamma \vee \alpha\beta$, which yields $\alpha\beta \leqslant \varepsilon \vee \gamma$. Hence $\beta(\varepsilon \vee \alpha) = \beta \vee \alpha\beta \leqslant \varepsilon \vee \gamma = \varepsilon \vee \beta$. So we obtain $\varepsilon \vee \alpha = \beta^{-1}(\varepsilon \vee \beta) = \beta^{-1} \vee \varepsilon$, a contradiction.

Proposition 37. *Every family $(F_i)_{i \in I}$ of q-fields (I a set) admits a product in $\mathbf{Fld}^q$.*

Proof. Let $f_i \colon F_i \to \mathbb{F}_1$ be the unique morphisms into the zero object. Consider the pullback M of the q-monoid morphisms $f_i|_{E(F_i)} \colon E(F_i) \to \mathbb{F}_1$, that is,

$$M := \{0\} \sqcup \prod_{i\in I} \big(E(F_i) \smallsetminus \{0\}\big).$$

So M is a q-monoid with unit group $M^\times = \prod_{i\in I} F_i^\times$. Hence $M^\times$ is a commutative discrete q-group. Its completion $\widehat{M}$ is a q-field with morphisms $p_i \colon \widehat{M} \to F_i$ of q-fields, induced by the canonical morphisms $Ep_i \colon M \to E(F_i)$.

Now let $g_i \colon F \to F_i$ be arbitrary morphisms of q-fields. Since $(Ef_i)(Ep_i) = 0$ for all $i \in I$, there is a unique morphism $g \colon E(F) \to M$ with $(Ep_i)g = Ef_i$ for all i. So $p_i\widehat{g} = f_i$ for all $i \in I$. This proves that $\widehat{M}$ with the projection maps p_i is a product of the F_i in $\mathbf{Fld}^q$. □

In what follows, we write $\prod_{i\in I} F_i$ for the product of the F_i. If I is finite, the passage through $E(F_i)$ can be avoided: $\prod_{i\in I} F_i$ is just the pullback of the morphisms $F_i \to \mathbb{F}_1$. Note that the product of standard q-fields $F_i = F(K_i|k_i)$ cannot be represented by a field extension. It is a genuine q-field.

Remark. The morphisms $p_i \colon \prod_{i\in I} F_i \twoheadrightarrow F_i$ of a product are rational. Indeed, let $\beta_i \in F_i^\times$ be a unit with $\beta_i \leqslant (Ep_i)(a)$ for some $a \in M$. Then all components of a are non-zero. So there is a unit $\beta \leqslant a$ in $M^\times$ with $(Ep_i)(\beta) = \beta_i$. By Proposition 36, this implies that the p_i are rational. However, $\prod_{i\in I} F_i$ need not be a product in the subcategory with rational morphisms. For example, let $K|k$ be a quadratic field extension, generated by $x \in K$, and $F := F(K|k)$. With $I = \{1, 2\}$ and $F_1 = F_2 = F$, the identity morphisms $F \to F_i$ induce a diagonal morphism $d \colon F \to F_1 \times F_2$. But d is not rational: With $\alpha := kx$, we have $d(\varepsilon \vee \alpha) = (\varepsilon \times \varepsilon) \vee (\alpha \times \alpha) = K \times K$. However, the subspace $\varepsilon \times \alpha$ of $K \times K$ is not of the form $d(b)$ with $b \leqslant \varepsilon \vee \alpha$.

12 Coproducts of q-fields

To introduce coproducts of q-fields, we need some basic facts on the category **Sup** of *sup-lattices*. Objects in **Sup** are complete lattices, morphisms are maps $f \colon L \to M$ with $f(\bigvee A) = \bigvee f(A)$ for each subset $A \subset L$. The morphisms $L \to M$ between sup-lattices L, M form a sup-lattice $Q(L, M)$ with pointwise supremum $(\bigvee F)(x) := \bigvee\{f(x) \mid f \in F\}$ for $F \subset Q(L, M)$. The composition of morphisms distributes over joins:

$$\big(\bigvee f_i\big)g = \bigvee(f_i g), \qquad f\big(\bigvee g_i\big) = \bigvee(f g_i).$$

So the endomorphisms of a sup-lattice L form a quantale $Q(L) := Q(L, L)$. If a map $f \colon L \to M$ between complete lattices respects meets, that is, $f(\bigwedge A) = \bigwedge f(A)$ for all $A \subset L$, we can express this by the *dual* sup-lattices L° and M° which are equipped with the opposite order. So $f \in Q(L^\circ, M^\circ)$. Every morphism $f \in Q(L, M)$ has a "mirror image", the *adjoint morphism* $f^\circ \colon M^\circ \to L^\circ$ given by the relation

$$f(x) \leqslant y \iff x \leqslant f^\circ(y)$$

for $x \in L$ and $y \in M$. Thus $(f^\circ)^\circ = f$, which gives a natural isomorphism

$$Q(L, M) \cong Q(M^\circ, L^\circ). \tag{24}$$

Viewed as a sup-lattice, the q-field $\mathbb{F}_1$ satisfies

$$Q(\mathbb{F}_1, L) \cong L, \qquad Q(L, \mathbb{F}_1) \cong L^\circ.$$

Just as q-fields generalize field extensions, sup-lattices form the quantalic analogue to abelian groups.

An important property of the category **Ab** of abelian groups is that finite products and finite coproducts coincide. For sup-lattices, we can do better. Consider the cartesian product $\prod_{i \in I} L_i$ of sup-lattices L_i with the canonical *projections* $p_j \in Q\big(\prod_{i \in I} L_i, L_j\big)$ on the j-th component. The *injections* $e_j \in Q\big(L_j, \prod_{i \in I} L_i\big)$ are given by $e_j(a) := (a_i)$ with $a_j = a$ and $a_i = 0$ for $i \neq j$. Then the p_i and e_i are completely determined by the equations

$$p_i e_j = \delta_{i,j}, \qquad \bigvee_{i \in I} e_i p_i = 1.$$

Moreover, it is easily verified that the p_i define a product and the e_i define a coproduct of the sup-lattices L. Therefore, we call $\bigoplus_{i \in I} L_i := \prod_{i \in I} L_i$ the *biproduct* of the L_i. As in the case of additive categories, we can use this to represent morphisms $\bigoplus_{i \in I} L_i \to \bigoplus_{i \in I} M_i$ as matrices with morphisms $L_i \to M_j$ as entries.

Our special concern here is to introduce tensor products of sup-lattices. The easy way to do this is by using the exchange rule

$$Q(L, Q(M, N)) \cong Q(M, Q(L, N)). \tag{25}$$

Both sides represent a *bimorphism* $f\colon L \times M \to N$, that is, a map for which the partial maps $f(x, -)$ and $f(-, y)$ are morphisms of sup-lattices. Explicitly:

$$f\big(x, \bigvee y_i\big) = \bigvee f(x, y_i), \qquad f\big(\bigvee x_i, y\big) = \bigvee f(x_i, y).$$

Now the isomorphisms (24) and (25) give $Q(L, Q(M, N)) \cong Q(L, Q(N^\circ, M^\circ)) \cong Q(N^\circ, Q(L, M^\circ)) \cong Q(Q(L, M^\circ)^\circ, N)$. So by defining

$$L \otimes M := Q(L, M^\circ)^\circ, \tag{26}$$

we have the adjointness property

$$Q(L \otimes M, N) \cong Q(L, Q(M, N)).$$

By (24), the tensor product is commutative:

$$L \otimes M \cong M \otimes L.$$

If $\mathrm{Bi}(L \times M, N)$ denotes the set of bimorphisms, we obtain a universal description of the tensor product:

$$Q(L \otimes M, N) \cong \mathrm{Bi}(L \times M, N).$$

In particular, the identity morphism 1_N of $N = L \otimes M$ gives a canonical bimorphism

$$t\colon L \times M \to L \otimes M.$$

For $x \in L$ and $y \in M$, we write $x \otimes y := t(x, y)$. Following the above isomorphisms with $N := L \otimes M$, the identity morphism of N corresponds to the bimorphism $t\colon L \times M \to N$ and to the morphism $x \mapsto t_x$ in $Q(L, Q(M, N))$ with $t_x(y) := x \otimes y$. For $z \in N$, define $g(z) \in Q(L, M^\circ))$ by $g(z)(x) := t_x^\circ(z)$. So we have the correspondences

$$\begin{array}{ccccccc} Q(L, Q(M, N)) & \cong & Q(L, Q(N^\circ, M^\circ)) & \cong & Q(N^\circ, Q(L, M^\circ)) & \cong & Q(Q(L, M^\circ)^\circ, N) \\ (x \mapsto t_x) & \mapsto & (x \mapsto t_x^\circ) & \mapsto & g & \mapsto & g^\circ \end{array}$$

with $g^\circ = 1_N$. For $f, z \in N$ we have $g^\circ(f) \leqslant z \Leftrightarrow f \leqslant g(z) \Leftrightarrow \forall\, x \in L\colon f(x) \leqslant t_x^\circ(z) \Leftrightarrow \forall\, x \in L\colon t_x(f(x)) \leqslant z \Leftrightarrow \bigvee_{x \in L} x \otimes f(x) \leqslant z$. So any $f \in L \otimes M = Q(L, M^\circ)^\circ$ has a canonical representation

$$f = \bigvee_{x \in L} x \otimes f(x) = \bigvee_{y \in M} f^\circ(y) \otimes y. \tag{27}$$

On the other hand, $g^\circ = 1_N$ implies that $g(f) = f$. So $y \leqslant f(x) \Leftrightarrow y \leqslant t_x^\circ(f) \Leftrightarrow t_x(y) \leqslant f$ holds for $x \in L$ and $y \in M$. Hence

$$x \otimes y \leqslant f \iff y \leqslant f(x). \tag{28}$$

The map $f \in Q(L, M^\circ)^\circ$ corresponding to a simple tensor $x \otimes y \in L \otimes M$ can be calculated explicitly. For $x' \in L$ we have:

$$(x \otimes y)(x') := \begin{cases} 1 & \text{for } x' = 0 \\ y & \text{for } 0 < x' \leqslant x \\ 0 & \text{for } x' \nleqslant x. \end{cases} \tag{29}$$

Note first that Eq. (29) defines a sup-lattice morphism. For $x_i \in L$, we have to verify

$$(x \otimes y)\big(\bigvee x_i\big) = \bigvee (x \otimes y)(x_i).$$

If all $x_i = 0$, then $\bigvee x_i = 0$, and both sides of the equation are 1. Otherwise, if all $x_i \leqslant x$, both sides are y. (Note that the ordering in M° is inverted.) If $x_i \nleqslant x$ for at least one i, then both sides are zero. So the map (29) belongs to $Q(L, M^\circ)^\circ$.

To show that the map (29) represents $x \otimes y$, we have to verify the equivalence (28). Assume that $x \in L$, $y \in M$, and $f \in Q(L, M^\circ)^\circ$ satisfy $y \leqslant f(x)$. We have to show that $(x \otimes y)(x') \leqslant f(x')$ holds for all $x' \in L$. For $x' = 0$, we have $f(x') = 1$, and $x' \nleqslant x$ implies that $(x \otimes y)(x') = 0$. So $(x \otimes y)(x') \leqslant f(x')$ in these cases. If $0 < x' \leqslant x$, then $(x \otimes y)(x') = y \leqslant f(x) \leqslant f(x')$. Conversely, assume that $x \otimes y \leqslant f$. If $x = 0$, then $y \leqslant 1 = f(x)$. Otherwise, $y = (x \otimes y)(x) \leqslant f(x)$. Thus Eq. (29) is proved.

Now let F and G be q-fields. Then $(a, b, a', b') \mapsto aa' \otimes bb'$ defines a map $F \times G \times F \times G \to F \otimes G$ which is a bimorphism in the first two and the last two variables. So it induces a multiplication $(F \otimes G) \times (F \otimes G) \to (F \otimes G)$, given by

$$\big(\bigvee a_i \otimes b_i\big)\big(\bigvee a'_j \otimes b'_j\big) = \bigvee a_i a'_j \otimes b_i b'_j,$$

which is again a bimorphism. Thus $F \otimes G$ is a quantale with unit element $\varepsilon \otimes \varepsilon$.

Proposition 38. *Let F and G be q-fields. Then $F \otimes G$ is a q-field.*

Proof. For $\alpha \in F^\times$ and $\beta \in G^\times$, the simple tensor $\alpha \otimes \beta \in F \otimes G$ is invertible. Conversely, let $f = \bigvee_{i\in I} a_i \otimes a'_i \in F \otimes G$ be invertible with inverse $g = \bigvee_{j\in J} b_j \otimes b'_j$. Without loss of generality, we assume that the a_i, b_i, a'_i, b'_i are all non-zero. By Eqs. (27), we can also assume that the a'_i are pairwise distinct. Then $fg = \bigvee a_i b_j \otimes a'_i b'_j = \varepsilon \otimes \varepsilon$. Hence $a_i b_j \otimes a'_i b'_j \leqslant \varepsilon \otimes \varepsilon$ for all $i \in I$ and $j \in J$. By (28), this implies that $a'_i b'_j \leqslant (\varepsilon \otimes \varepsilon)(a_i b_j)$. So Eq. (29) gives $a'_i b'_j \leqslant \varepsilon$. Hence $a'_i b'_j = \varepsilon$, which implies that the index set I is a singleton. Thus $f = \alpha \otimes \beta$ with $\alpha \in F^\times$ and $\beta \in G^\times$. If $0 < a \otimes b \leqslant \alpha \otimes \beta$, then (28) and (29) give $b \leqslant (\alpha \otimes \beta)(a) \leqslant \beta$. By symmetry, $a \leqslant \alpha$. So the units in $F \otimes G$ coincide with the atoms, and $F \otimes G$ is atomistic.

Finally, assume that $\varepsilon \otimes \varepsilon \leqslant \bigvee_{i\in I} f_i$ with $f_i \in F \otimes G$. Then $\varepsilon \leqslant \big(\bigvee_{i\in I} f_i\big)(\varepsilon) = \bigvee_{i\in I} f_i(\varepsilon)$. Hence $\varepsilon \leqslant \bigvee_{i\in I'} f_i(\varepsilon)$ for a finite subset I' of I, and thus $\varepsilon \otimes \varepsilon \leqslant \bigvee_{i\in I'} f_i$. This proves that $\varepsilon \otimes \varepsilon \in F \otimes G$ is compact. Whence $F \otimes G$ is a q-field. ☐

The map $a \mapsto \varepsilon \otimes a$ gives a morphism $G \to F \otimes G$ of q-fields. Since $(\varepsilon \otimes a)(\varepsilon) = a$ by Eq. (29), this makes G into a sub-q-field of $F \otimes G$. By the preceding proof, the units in $F \otimes G$ are of the form $\alpha \otimes \beta$ with $\alpha \in F^\times$ and $\beta \in G^\times$. Hence G is a rational sub-q-field. So we have rational embeddings:

$$F \hookrightarrow F \otimes G \hookleftarrow G. \tag{30}$$

Corollary. *Let F and G be q-fields. Then $F \otimes G$ is a coproduct of F and G in* $\mathbf{Fld}^q$.

Proof. Let $f\colon F \to H$ and $g\colon G \to H$ be morphisms of q-fields. Then $(a,b) \mapsto f(a)g(b)$ is a bimorphism $F \times G \to H$. So there is a unique morphism $h\colon F \otimes G \to H$ with $h(a \otimes b) = f(a)g(b)$ for all $a \in F$ and $b \in G$, that is, $h|_F = f$ and $h|_G = g$. ☐

Let us consider the special case $F = F(K|k)$ and $G = F(L|k)$ with field extensions $K|k$ and $L|k$. By $F(K, L|k)$ we denote the quantale of k-subspaces of $K \otimes_k L$ generated by simple tensors. The following result shows that $F(K, L|k)$ is a q-field.

Proposition 39. *Let K and L be fields with a common subfield k. Then*

$$F(K, L|k) \cong F(K|k) \otimes F(L|k). \tag{31}$$

Proof. For any $U \in F(K, L|k)$, we define $f_U \in F(K|k) \otimes F(L|k)$ by

$$f_U := \bigvee\{kx \otimes ky \mid x \otimes y \in U\} = \bigvee_{x\in K^\times} \Big(kx \otimes \sum\{ky \mid x \otimes y \in U\}\Big).$$

Assume that $kx \otimes ky \leqslant f_U$. Then $x \otimes y = \sum_{i=1}^n x_i \otimes y_i$ with $x_i \otimes y_i \in U$. Choose a linear form $\varphi \in K^*$ with $\varphi(x) = 1$. Then $y = (\varphi \otimes 1_L)(x \otimes y) = \sum_{i=1}^n \varphi(x_i) y_i \in \sum\{ky \mid x \otimes y \in U\}$. So (28) yields

$$f_U(kx) = \sum\{ky \mid x \otimes y \in U\}.$$

Conversely, every $f \in F(K|k) \otimes F(L|k)$ defines a subspace $U_f \in F(K, L|k)$ of $K \otimes_k L$:

$$U_f := \sum\{k(x \otimes y) \mid ky \leqslant f(kx)\}.$$

Hence $U_{f_U} = \sum\{k(x \otimes y) \mid x \otimes y \in U\} = U$ and $f_{U_f} = \bigvee\{kx \otimes ky \mid ky \in f(kx)\} = f$. So there is a bijection (31). By (28), we have $x \otimes y \in U \Rightarrow kx \otimes ky \leqslant f_U \Leftrightarrow ky \leqslant f_U(kx) \Rightarrow k(x \otimes y) \subset U$. So we obtain

$$x \otimes y \in U \iff kx \otimes ky \leqslant f_U,$$

which shows that the bijection (31) is multiplicative. Hence it is an isomorphism of q-fields. $\square$

Remark. If $K|k$ and $L|k$ are separable of finite degree $(K : k) = n$ and $(L : k) = m$, respectively, then $K \otimes_k L$ is a semisimple k-algebra $\prod_{i=1}^{r} E_i$, where $E_1, \ldots, E_r$ is the collection of compounds $E_i = KL$ generated by K and L. The following example shows that in general, the q-field (31) is neither connected nor modular.

Example 11. We consider the special case $K = L = \mathbb{C}$ and $k = \mathbb{R}$ of the preceding remark. The bilinear map $\mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$, given by $(z, w) \mapsto (zw, z\overline{w})$, induces an isomorphism of $\mathbb{R}$-algebras

$$\sigma\colon \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} \mathbb{C} \times \mathbb{C}. \tag{32}$$

Indeed,

$$e := \tfrac{1}{2}(1 \otimes 1 - i \otimes i)$$

is an idempotent with $\sigma(e) = \frac{1}{2}\big((1, 1) - (-1, 1)\big) = (1, 0)$.

The simple tensors in $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ correspond to the pairs $(a, b) \in \mathbb{C} \times \mathbb{C}$ satisfying

$$|a| = |b|. \tag{33}$$

The necessity follows since $|zw| = |z\overline{w}|$. Assume that (33) holds for $(a, b) \neq 0$.

Case 1: $a + b \neq 0$. Then $\frac{a-b}{a+b}$ is purely imaginary. Indeed,

$$\frac{a - b}{a + b} + \frac{\overline{a} - \overline{b}}{\overline{a} + \overline{b}} = \frac{(a - b)(\overline{a} + \overline{b}) + (a + b)(\overline{a} - \overline{b})}{(a + b)(\overline{a} + \overline{b})} = \frac{2(a\overline{a} - b\overline{b})}{(a + b)(\overline{a} + \overline{b})} = 0.$$

We set $z := \frac{a+b}{2}$ and $w := 1 + \frac{a-b}{a+b}$. Then $zw = \frac{a+b}{2}\big(1 + \frac{a-b}{a+b}\big) = \frac{a+b}{2} + \frac{a-b}{2} = a$ and $z\overline{w} = \frac{a+b}{2}\big(1 - \frac{a-b}{a+b}\big) = \frac{a+b}{2} - \frac{a-b}{2} = b$.

Case 2: $a + b = 0$. With $z := -ai$ and $w := i$, we obtain $zw = a$ and $z\overline{w} = -a = b$. So the condition (33) gives exactly the simple tensors.

Next we show that $F(\mathbb{C}|\mathbb{R}) \otimes F(\mathbb{C}|\mathbb{R})$ is not connected. Let (a_1, b_1) and (a_2, b_2) be distinct non-zero elements of $\mathbb{C} \times \mathbb{C}$ with $|a_i| = |b_i|$. Suppose that $\lambda(a_1, b_1) + \mu(a_2, b_2)$ satisfies Eq. (33), where $\lambda, \mu \in \mathbb{R}^\times$. Then $(\lambda a_1 + \mu a_2)(\lambda\overline{a_1} + \mu\overline{a_2}) = (\lambda b_1 + \mu b_2)(\lambda\overline{b_1} + \mu\overline{b_2})$, that is, $\lambda\mu(a_1\overline{a_2} + a_2\overline{a_1}) = \lambda\mu(b_1\overline{b_2} + b_2\overline{b_1})$. For $(a_1, b_1) = (1, 1)$ and $(a_2, b_2) = (i, 1)$, this equation cannot be satisfied. Hence $F(\mathbb{C}|\mathbb{R}) \otimes F(\mathbb{C}|\mathbb{R})$ is not connected.

For $a := \mathbb{R}(1, 1)$, $b := \mathbb{R}(1, i) \oplus \mathbb{R}(1, -i)$, and $c := \mathbb{R}(1, 1) \oplus \mathbb{R}(1, -1)$, we have $a \subset c$ in the lattice of $\mathbb{R}$-subspaces of $\mathbb{C} \times \mathbb{C}$. However, $b \cap c = \mathbb{R}(1, 0)$, which is not generated by pairs (a, b) satisfying Eq. (33). Now the map (32) induces a lattice structure on the subspaces of $\mathbb{C} \times \mathbb{C}$. So we have $b \wedge c = 0$. A straightforward calculation shows that $(a + b) \cap c = c$. Thus, $(a \vee b) \wedge c = c$ and $a \vee (b \wedge c) = a$, which shows that $F(\mathbb{C}|\mathbb{R}) \otimes F(\mathbb{C}|\mathbb{R})$ is not modular.

13 Lax morphisms

We have seen that rational and prime orders play an important rôle for the structure of q-fields. While rational orders arise as kernels of morphisms, prime orders have not been encountered in $\mathbf{Fld}^q$. In this section, we consider a larger class of morphisms where also prime orders occur. Let us first characterize the radical of a prime order.

Definition 23. Let F be a q-field. We call $p \in F$ *prime* if the following are satisfied:

(a) $p^2 \leqslant p$.
(b) $\varepsilon \nleqslant p$.
(c) $\alpha\beta \leqslant p \implies (\alpha \leqslant p$ or $\beta \leqslant p)$ holds for $\alpha, \beta \in F^\times$.

We write $P(F)$ for the set of primes of F.

Proposition 40. *Let F be a q-field. The map which associates the radical to a prime order gives a bijection $V(F) \xrightarrow{\sim} P(F)$.*

Proof. Let e be a prime order with radical p. Then $p^2 \leqslant ep = p$ and $p < e$. Hence (a) and (b) hold. For $\alpha, \beta \in F^\times \smallsetminus [p]$ we have $e \leqslant e\alpha \leqslant e\alpha\beta$, which gives (c). So p is a prime. Every $\alpha \in [p \to p]$ satisfies $\alpha p < e$. Hence $\alpha e \leqslant e$, which yields $e = p \to p$.

Conversely, let $p \in F$ be a prime. Then $e := p \to p$ is an order. For any unit $\alpha \in F^\times$ with $\alpha \nleqslant e$ there is a unit $\beta \leqslant p$ with $\alpha\beta \nleqslant p$. Since $\beta = \alpha^{-1} \cdot \alpha\beta$, this implies that $\alpha^{-1} \leqslant p \leqslant e$. Thus $e \in V(F)$. For a unit $\alpha \in F^\times$ with $e\alpha < e$, we have $\alpha^{-1} \nleqslant e$. Hence $\alpha^{-1}\beta \nleqslant p$ for some $\beta \in [p]$. Thus $\beta = \alpha \cdot \alpha^{-1}\beta$, which yields $\alpha \leqslant p$. So p is the radical of e. ☐

In what follows, we augment each q-field F by an external element ∞ with $\infty > a$ for all $a \in F$ and write $\widetilde{F} := F \sqcup \{\infty\}$. For $a \in \widetilde{F} \smallsetminus \{0\}$ we define

$$a \cdot \infty = \infty, \qquad 0 \cdot \infty = 0, \qquad \infty^{-1} = 0, \qquad 0^{-1} = \infty.$$

Definition 24. We define a *lax morphism* of q-fields to be a morphism $f \colon \widetilde{F} \to \widetilde{G}$ of sup-lattices such that for $\alpha, \beta \in F^\times \cup \{0, \infty\}$,

(a) $f(\alpha)f(\beta) \leqslant f(\alpha\beta)$.
(b) $f(\alpha^{-1}) = f(\alpha)^{-1}$.

We call $\mathrm{rad}(f) := \bigvee\{\alpha \in F^\times \mid f(\alpha) = 0\} \in F$ the *radical* of f.

Since $f(0) = 0$, axiom (b) gives $f(\infty) = \infty$ and $f(\varepsilon) = \varepsilon$. Thus lax morphisms generalize morphisms in $\mathbf{Fld}^q$. For $\alpha \in F^\times$, at most one of $f(\alpha)$ and $f(\alpha^{-1})$ can be zero. If both are non-zero, then $f(\alpha) \in G^\times$. Note that (a) extends to $f(a)f(b) \leqslant f(ab)$ for all $a, b \in \widetilde{F}$. With lax morphisms, q-fields form a category $\mathbf{Fl}^q$. It contains $\mathbf{Fld}^q$ as a subcategory with the same objects.

Proposition 41. *Let $f\colon F \to G$ be a lax morphism of q-fields. Then the radical p of f is a prime of F with corresponding prime order $p \to p = \bigvee\{\alpha \in F^\times \mid f(\alpha) < \infty\}$. Moreover,* $\mathrm{rad}(f) = 0$ *if and only if $f \in \mathbf{Fld}^q$.*

Proof. Assume that $\alpha, \beta \in F^\times \cup \{0, \infty\}$ satisfy $f(\alpha) = f(\beta) = 0$. Then $\infty \cdot f(\alpha\beta) = f(\alpha^{-1})f(\alpha\beta) \leqslant f(\beta) = 0$, which yields $f(\alpha\beta) = 0$. Thus $p^2 \leqslant p$. Furthermore, $\varepsilon \nleqslant p$. If $\alpha\beta \leqslant p$, then $f(\alpha)f(\beta) \leqslant f(\alpha\beta) = 0$, which gives $f(\alpha) = 0$ or $f(\beta) = 0$. So p is a prime of F.

Now assume that $f(\alpha) < \infty$. Then $f(\alpha^{-1})f(\alpha\beta) \leqslant f(\beta) = 0$ for all $\beta \in [p]$, which yields $f(\alpha\beta) = 0$. Thus $\alpha p \leqslant p$, that is, $\alpha \leqslant p \to p$. Conversely, assume that $f(\alpha) = \infty$. Then $\alpha^{-1} \leqslant p$. So $\alpha\alpha^{-1} = \varepsilon$ implies that $\alpha \nleqslant p \to p$. The last statement is obvious. □

Proposition 42. *Let F be a modular q-field. Each prime order e with radical p determines a canonical lax morphism $q_e\colon F \twoheadrightarrow \partial e$ with* $\mathrm{rad}(q_e) = p$*, given by*

$$q_e(a) = \begin{cases} \infty & \text{for } a \nleqslant e \\ a \vee p & \text{for } a \leqslant e. \end{cases} \tag{34}$$

Proof. Note first that q_e is a sup-lattice morphism. Condition (a) of Definition 24 is obvious for $\alpha\beta \nleqslant e$. Thus, assume that $\alpha\beta \leqslant e$. Then $\alpha \leqslant e$ or $\beta \leqslant e$. By symmetry, we can assume that $\alpha \leqslant e$. If $\beta \leqslant e$, then $q_e(\alpha)q_e(\beta) = (\alpha \vee e)(\beta \vee e) \leqslant \alpha\beta \vee e = q_e(\alpha\beta)$. Otherwise, $e\beta > e$, which yields $\alpha \leqslant p$, and thus $q_e(\alpha)q_e(\beta) = p \leqslant q_e(\alpha\beta)$. This proves (a). Condition (b) is trivial. Thus q_e is a lax morphism. By definition, $\mathrm{rad}(q_e) = p$. □

Proposition 43. *Let $f\colon F \to G$ be a lax morphism of q-fields with radical p and $e := p \to p$. There is a unique factorization $f = gq_e$ with $g \in \mathbf{Fld}^q$.*

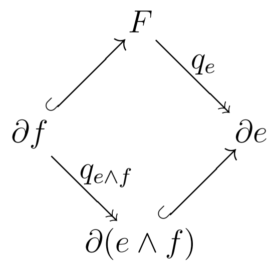
Proof. By Proposition 41 and Eq. (34), an element $a \in F$ satisfies $f(a) = \infty \Leftrightarrow a \nleqslant e \Leftrightarrow q_e(a) = \infty$. Otherwise, $f(a) = f(a \vee p)$, which establishes the factorization $f = gq_e$ which is unique since q_e is surjective. Furthermore, $g \in \mathbf{Fld}^q$ since $\mathrm{rad}(g) = 0$. □

The counterpart to the epimorphism q_e is a rational embedding. So it is natural to consider a class of morphisms which are built from a morphism (34) and a rational embedding. In Sections 2 and 3 we found a close interaction between rational orders and prime orders. The next result adds to the interplay between these remarkable two types of orders.

Proposition 44. *Let e be a prime order with radical p, and let f be a rational order in a modular q-field F. Then $e \wedge f$ is a prime order in ∂f with radical $p \wedge f$, and $\partial(e \wedge f)$ is isomorphic to a rational sub-q-field of ∂e.*

Proof. Since $(e \wedge f)(e \wedge f) \leqslant e \wedge f$ and $\varepsilon \leqslant e \wedge f$, it follows that $e \wedge f$ is an order in ∂f. Since $e \in V(F)$, we infer that $e \wedge f$ is a prime order in ∂f. A unit $\alpha \leqslant e \wedge f$ satisfies $\alpha^{-1} \leqslant e \wedge f$ if and only if $\alpha^{-1} \leqslant e$, that is, $\alpha \notin p$. Hence $p \wedge f$ is the radical of $e \wedge f$. In particular, $\partial(e \wedge f) = [p \wedge f, e \wedge f] \hookrightarrow [p, e] = \partial e$ is a rational embedding. $\square$

Proposition 44 can be interpreted as a factorization of lax morphisms:

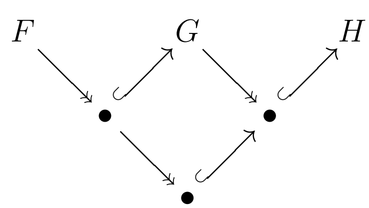


(35)

With regard to classical field theory, F can be viewed as an "unramified extension" of ∂f, with corresponding residue q-field extension $\partial(e \wedge f) \hookrightarrow \partial e$. In what follows, a lax morphism (34) will be called a *prime surjection*.

Definition 25. We define a *rigid morphism* of modular q-fields to be a lax morphism $f \colon F \to G$ which admits a factorization $f = iq$ into a prime surjection q followed by a rational embedding i.

Since prime surjections are surjective, and rational embeddings are injective, the factorization in Definition 25 is unique. The commutative diagram (35) shows that the class of rigid morphisms is closed under composition:



So the modular q-fields with rigid morphisms form a subcategory $\mathbf{Flr}^q$ of $\mathbf{Fl}^q$. For field extensions, a similar category has been considered by Deck and Harrison [10].

14 Simple transcendental extensions

As in classical field theory, there are two types of simple extensions, namely, algebraic and transcendental ones. Let G be a rational sub-q-field of a q-field F, that is, $G = \partial e$ for a rational order e in F. Assume that $F = \partial f$ with $f = e(\gamma)$ for a unit $\gamma \in F^\times$. Then we call F a *simple extension* of G and write $F = G(\gamma)$. Assume that γ is transcendental over e. If G is modular and connected of length $\geqslant 4$, then up to isomorphism, F is uniquely determined by G (Theorem 3). Otherwise, this need not be the case. To investigate the structure of $G(\gamma)$, we can assume that $G = \mathbb{F}_1$.

In what follows, we study the order $\varepsilon[\gamma]$ in a modular q-field $\mathbb{F}_1(\gamma)$ with γ transcendental over ε. We call such a q-field $\mathbb{F}_1(\gamma)$ *simply transcendental*. For a unit $\alpha \leqslant \varepsilon[\gamma]$, the smallest $n \in \mathbb{N}$ with $\alpha \leqslant \varepsilon \vee \gamma \vee \cdots \vee \gamma^n$ will be called the *degree* $\deg \alpha$ of α in $\varepsilon[\gamma]$.

Proposition 45. *Let $\mathbb{F}_1(\gamma)$ be a simply transcendental modular q-field. For units $\alpha, \beta \leqslant \varepsilon[\gamma]$, we have*

$$\deg \alpha\beta = \deg \alpha + \deg \beta.$$

Proof. Assume that $\deg \alpha = n$ and $\deg \beta = m$. Without loss of generality, $n, m > 0$. By Lemma 1, $\gamma^n \leqslant \varepsilon \vee \cdots \vee \gamma^{n-1} \vee \alpha$ and $\gamma^m \leqslant \varepsilon \vee \cdots \vee \gamma^{m-1} \vee \beta$. Hence $\gamma^{n+m} \leqslant \varepsilon \vee \cdots \vee \gamma^{n+m-2} \vee \alpha(\varepsilon \vee \cdots \vee \gamma^{m-1}) \vee \beta(\varepsilon \vee \cdots \vee \gamma^{n-1}) \vee \alpha\beta$. Since $\gamma^{n+m} \nleqslant \varepsilon \vee \cdots \vee \gamma^{n+m-1}$, this shows that $\deg \alpha\beta = n + m$. □

Corollary 1. *Let $\mathbb{F}_1(\gamma)$ be a simply transcendental modular q-field. Then $\varepsilon[\gamma]^\times = \{\varepsilon\}$.*

Proof. If $\alpha \in \varepsilon[\gamma]^\times$, that is, $\alpha, \alpha^{-1} \leqslant \varepsilon[\gamma]$, then $\deg \alpha + \deg \alpha^{-1} = \deg \varepsilon = 0$. Thus $\alpha = \varepsilon$. □

Definition 26. Let e be an order in a q-field F. For $\alpha, \beta \in F^\times$, we say that α *divides* β (with respect to e) and write $\alpha | \beta$ if $\alpha\gamma = \beta$ for some $\gamma \in [e]$.

Corollary 2. *Let $\mathbb{F}_1(\gamma)$ be a simply transcendental modular q-field. Divisibility with respect to $\varepsilon[\gamma]$ is a partial order, given by*

$$\alpha | \beta \iff \varepsilon[\gamma]\beta \leqslant \varepsilon[\gamma]\alpha.$$

In particular, $\varepsilon[\gamma]\alpha = \varepsilon[\gamma]\beta$ if and only if $\alpha = \beta$.

Proof. The equivalence follows since $\varepsilon[\gamma]\beta \leqslant \varepsilon[\gamma]\alpha \Leftrightarrow \beta\alpha^{-1} \in \varepsilon[\gamma]$. So divisibility is reflexive and transitive. If $\varepsilon[\gamma]\alpha = \varepsilon[\gamma]\beta$, then $\alpha\beta^{-1} \leqslant \varepsilon[\gamma]$ and $\beta\alpha^{-1} \leqslant \varepsilon[\gamma]$, that is, $\alpha\beta^{-1} \in \varepsilon[\gamma]^\times = \{\varepsilon\}$ by Corollary 1. □

Example 12. Note that Proposition 45 does not require $\mathbb{F}_1(\gamma)$ to be connected. For example, the q-field $\mathfrak{P}(C)$ with an infinite cyclic group $C = \langle\gamma\rangle$ (Example 10) is a distributive q-field $\mathbb{F}_1(\gamma)$, with a prime order $\varepsilon[\gamma]$. Here the degree function is injective.

Definition 27. Let e be an order in a q-field F. We define an *ideal* of e to be an e-module $a \leqslant e$. If a is maximal among the ideals $< e$, we speak of a *maximal ideal*. The units $\pi \in F^\times$ generating a maximal ideal $e\pi$ of e will be called *irreducibles* of e.

For an order e in a q-field, we set $G_e := \{e\alpha \mid \alpha \in F^\times\}$ (cf. Section 3). The next result extends the classical fact that polynomial rings $k[x]$ are principal ideal domains.

Proposition 46. *Let $\mathbb{F}_1(\gamma)$ be a simply transcendental modular q-field. Every non-zero ideal $a \leqslant \varepsilon[\gamma]$ belongs to $G_{\varepsilon[\gamma]}$.*

Proof. Choose $\alpha \in [a]$ with $m := \deg \alpha$ minimal. For any $\beta \in [a]$, we thus have $\beta \leqslant \varepsilon \vee \gamma \vee \cdots \vee \gamma^n$ with $n \geqslant m$. With

$$c_i := \varepsilon \vee \gamma \vee \cdots \vee \gamma^i, \qquad a_i := \alpha(\gamma^{n-m-i+1} \vee \cdots \vee \gamma^{n-m}),$$

we prove

$$\beta \leqslant c_{n-i} \vee a_i, \qquad (0 \leqslant i \leqslant n-m+1). \tag{36}$$

For $i = 0$, this says that $\beta \leqslant c_n \vee 0 = c_n$. Assume that (36) has been verified for some $i \leqslant n-m$. Then $\alpha\gamma^{n-m-i} \leqslant c_m\gamma^{n-m-i} \leqslant c_{n-i} = c_{n-i-1} \vee \gamma^{n-i}$. Since $\alpha \not\leqslant c_{m-1}$, Lemma 1 implies that $\gamma^{n-i} \leqslant c_{n-i-1} \vee \alpha\gamma^{n-m-i}$. Hence $\beta \leqslant c_{n-i-1} \vee \gamma^{n-i} \vee a_i = c_{n-i-1} \vee \alpha\gamma^{n-m-i} \vee a_i = c_{n-i-1} \vee a_{i+1}$. This proves (36). With $i = n-m+1$, we obtain

$$\beta \leqslant c_{m-1} \vee a_{n-m+1} = c_{m-1} \vee \alpha c_{n-m}.$$

If $m = 0$, then $\beta \leqslant \alpha c_{n-m}$. Otherwise, Lemma 9 gives $\beta \leqslant \delta \vee \zeta$ with $\delta \in [c_{m-1}]$ and $\zeta \in [\alpha c_{n-m}]$. If $\beta \neq \zeta$, Lemma 1 implies that $\delta \leqslant \beta \vee \zeta \leqslant a$. Since $\delta \leqslant c_{m-1}$, this contradicts the minimality of m. Thus, in any case, $\beta \leqslant \alpha c_{n-m}$. Whence $a = \alpha\varepsilon[\gamma]$. □

It is easily checked that every maximal ideal is prime. Conversely,

Corollary 1. *Let $\mathbb{F}_1(\gamma)$ be a simply transcendental modular q-field. Every non-zero prime ideal of $\varepsilon[\gamma]$ is maximal.*

Proof. Let $\varepsilon[\gamma]\pi$ be a non-zero prime ideal, with a unit π of $\mathbb{F}_1(\gamma)$. Assume that $\varepsilon[\gamma]\pi < \varepsilon[\gamma]\sigma \leqslant \varepsilon[\gamma]$ with $\sigma \in \mathbb{F}_1(\gamma)^\times$. Then $\pi\sigma^{-1} \cdot \sigma = \pi$ implies that $\pi\sigma^{-1} \leqslant \varepsilon[\gamma]\pi$. Whence $\varepsilon[\gamma] \leqslant \varepsilon[\gamma]\sigma \leqslant \varepsilon[\gamma]$. □

So the irreducibles π of $\varepsilon[\gamma]$ correspond bijectively to the maximal ideals $(\pi) := \varepsilon[\gamma]\pi$. For brevity, we write (α) for an $\varepsilon[\gamma]$-module $\varepsilon[\gamma]\alpha \in G_{\varepsilon[\gamma]}$.

Corollary 2. *Let $\mathbb{F}_1(\gamma)$ be a simply transcendental modular q-field. Then $\mathbb{F}_1(\gamma)^\times$ is a free abelian group with the irreducibles of $\varepsilon[\gamma]$ as basis.*

Proof. Let $\alpha, \beta \in \mathbb{F}_1(\gamma)^\times$ be given. Then $\alpha = \alpha_1\alpha_2^{-1}$ and $\beta = \beta_1\beta_2^{-1}$ with units $\alpha_i, \beta_i \leqslant \varepsilon[\gamma]$. So $(\alpha_1\beta_1) \leqslant (\alpha) \wedge (\beta) \leqslant (\alpha) \vee (\beta) \leqslant (\alpha_2^{-1}\beta_2^{-1})$. Since multiplication with $\alpha_2\beta_2$ is a lattice automorphism of $\mathbb{F}_1(\gamma)$, this shows that $G_{\varepsilon[\gamma]}$ is a lattice, hence an abelian ℓ-group.

Now let $(\alpha_0) \leqslant (\alpha_1) \leqslant (\alpha_2) \leqslant (\alpha_3) \leqslant \cdots$ be an ascending chain of ideals in $\varepsilon[\gamma]$. Since $a := \bigvee\{\alpha_i \mid i \in \mathbb{N}\}$ is an ideal, there is a unit $\pi \in \mathbb{F}_1(\gamma)^\times$ with $a = \varepsilon[\gamma]\pi$. Since π is compact, we obtain $\pi \leqslant (\alpha_i)$ for some $i \in \mathbb{N}$. Hence $(\pi) = (\alpha_i)$. By Birkhoff's theorem ([2], Theorem 37), $G_{\varepsilon[\gamma]}$ is a free abelian group with the irreducibles of $\varepsilon[\gamma]$ as basis. □

Corollary 3. *Let $\mathbb{F}_1(\gamma)$ be a simply transcendental modular q-field. For every irreducible π of $\varepsilon[\gamma]$, the localization $\varepsilon[\gamma]_{(\pi)}$ is a prime order in $\mathbb{F}_1(\gamma)$. The map $\pi \mapsto \varepsilon[\gamma]_{(\pi)}$ is a bijection between the irreducibles π of $\varepsilon[\gamma]$ and the prime orders $e < \mathbb{F}_1(\gamma)$ with $\gamma \leqslant e$.*

Proof. By Corollary 2, every unit in $\mathbb{F}_1(\gamma)$ is of the form $\alpha\beta^{-1}$ with coprime $\alpha, \beta \leqslant \varepsilon[\gamma]$. Hence $\alpha\beta^{-1} \leqslant \varepsilon[\gamma]_{(\pi)}$ if and only if $\pi \nmid \beta$. Thus $\varepsilon[\gamma]_{(\pi)}$ is a prime order in $\mathbb{F}_1(\gamma)$.

Conversely, let $e < \varepsilon(\gamma)$ be a prime order in $\mathbb{F}_1(\gamma)$ with radical p and $\gamma \leqslant e$. Then $p \wedge \varepsilon[\gamma]$ is a prime ideal of $\varepsilon[\gamma]$. If $p \wedge \varepsilon[\gamma] = 0$, then $\varepsilon(\gamma) \leqslant e$, which is impossible. Hence $p \wedge \varepsilon[\gamma]$ is maximal by Corollary 1. So there is an irreducible π of $\varepsilon[\gamma]$ with $p \wedge \varepsilon[\gamma] = (\pi)$. If $\alpha \in [\varepsilon[\gamma]] \smallsetminus [(\pi)]$, then $\alpha \nleqslant p$, which yields $\alpha^{-1} \leqslant e$. Thus $\varepsilon[\gamma]_{(\pi)} \leqslant e$. Now let $\alpha\beta^{-1}$ be a unit in $\mathbb{F}_1(\gamma)$ with coprime $\alpha, \beta \leqslant \varepsilon[\gamma]$. Assume that $\alpha\beta^{-1} \leqslant e$. Then $\alpha = \alpha\beta^{-1} \cdot \beta$. Hence, $\beta \leqslant (\pi)$ would imply that $\alpha \in (\pi)$, which contradicts the coprimeness of α and β. Thus $\alpha\beta^{-1} \in \varepsilon[\gamma]_{(\pi)}$. □

Corollary 4. *Let $\mathbb{F}_1(\gamma)$ be a simply transcendental modular q-field. Then $\overline{\varepsilon} = \varepsilon$ and $\varepsilon[\gamma] = \bigwedge\{e \in V(\mathbb{F}_1(\gamma)) \mid \gamma \leqslant e\}$. Each prime order $e \geqslant \varepsilon[\gamma]$ in $\mathbb{F}_1(\gamma)$ with radical p satisfies $\varepsilon[\gamma] \vee p = e$.*

Proof. By Corollary 2, every unit in $\mathbb{F}_1(\gamma)^\times$ is of the form $\alpha\beta^{-1}$ with coprime units $\alpha, \beta \leqslant \varepsilon[\gamma]$. If $\alpha\beta^{-1}$ is integral over $\varepsilon[\gamma]$, then $\alpha^{n+1}\beta^{-n-1} \leqslant \varepsilon[\gamma](\varepsilon \vee \alpha\beta^{-1} \vee \cdots \vee \alpha^n\beta^{-n})$ for some $n \in \mathbb{N}$. Hence $\alpha^{n+1} \leqslant \varepsilon[\gamma]\beta$, and thus $\beta = \varepsilon$. By Proposition 17, this proves that $\varepsilon[\gamma] = \bigwedge\{e \in V(\mathbb{F}_1(\gamma)) \mid \gamma \leqslant e\}$. Now let $\alpha \in \mathbb{F}_1(\gamma)^\times$ be integral over ε. Then $\alpha \leqslant \varepsilon[\gamma]$, and $\alpha^{m+1} \leqslant \varepsilon \vee \alpha \vee \cdots \vee \alpha^m$ with $m \in \mathbb{N}$ minimal. So ε cannot be removed. By Lemma 1, this gives $\varepsilon \leqslant \alpha \vee \cdots \vee \alpha^{m+1}$. Hence $\alpha|\varepsilon$, and thus $\alpha = \varepsilon$.

Finally, let $e \geqslant \varepsilon[\gamma]$ be a prime order in $\mathbb{F}_1(\gamma)$ with radical p. By Corollary 3, $\varepsilon[\gamma] \wedge p = (\pi)$ with an irreducible π of $\varepsilon[\gamma]$, and $e = \varepsilon[\gamma]_{(\pi)}$. Assume that $\alpha\beta^{-1} \leqslant e$ with coprime $\alpha, \beta \leqslant \varepsilon[\gamma]$ in $\mathbb{F}_1(\gamma)^\times$. Then $\pi \nmid \beta$, which yields $\alpha \leqslant \varepsilon[\gamma]\beta \vee (\pi) \leqslant \varepsilon[\gamma]\beta \vee p$. Hence $\alpha\beta^{-1} \leqslant \varepsilon[\gamma] \vee p\beta^{-1} = \varepsilon[\gamma] \vee p$. □

Corollary 5. *Let $\mathbb{F}_1(\gamma)$ be a simply transcendental modular q-field. The degree function in $\varepsilon[\gamma]$ extends uniquely to a valuation*

$$\deg\colon \mathbb{F}_1(\gamma)^\times \twoheadrightarrow \mathbb{Z}$$

with corresponding prime order $\varepsilon[\gamma^{-1}]_{(\gamma^{-1})}$, the only prime order e with $\gamma \nleqslant e$. All prime orders of $\mathbb{F}_1(\gamma)$ are discrete.

Proof. Let e be a prime order in $\mathbb{F}_1(\gamma)$ with $\gamma \nleqslant e$. Let p be the radical of e. Then $\gamma^{-1} \leqslant p$ and $p \wedge \varepsilon[\gamma^{-1}] = (\gamma^{-1})$. By Corollary 3, $e = \varepsilon[\gamma^{-1}]_{(\gamma^{-1})}$. For a unit $\alpha \in \mathbb{F}_1(\gamma)^\times$ with $\alpha \leqslant \varepsilon[\gamma]$, we have $\deg \alpha = n$ if and only if $\alpha \leqslant \varepsilon \vee \gamma \vee \cdots \vee \gamma^n$ with n minimal, that is, $\alpha\gamma^{-n} \leqslant \gamma^{-n} \vee \cdots \vee \gamma^{-1} \vee \varepsilon$, where ε cannot be removed. Hence $\deg \alpha = n \Longrightarrow \alpha\gamma^{-n} \in \varepsilon[\gamma^{-1}]^\times_{(\gamma^{-1})} = e^\times$, and thus $\deg = \varphi_e$. So the valuation corresponding to the prime order $\varepsilon[\gamma^{-1}]_{(\gamma^{-1})}$ extends the degree function with respect to $\varepsilon[\gamma]$. □

Remark. In Example 12, we considered the special case $\mathfrak{P}(C) = \mathbb{F}_1(\gamma)$ with an infinite cyclic group $C = \langle\gamma\rangle$, where $\varepsilon[\gamma]$ is a prime order with radical (γ). The degree function $\deg\gamma^n = n$ then gives the other prime order $\varepsilon[\gamma^{-1}]$ in $\mathbb{F}_1(\gamma)$.

For a modular q-field F, we slightly extend Definition 10 by saying that a family $(a_i)_{i\in I}$ of elements is *independent* if each family $(\alpha_i)_{i\in I}$ of units with $\alpha_i \leqslant a_i$ is independent. To indicate this, we write $\bigvee_{i\in I} a_i = \bigoplus_{i\in I} a_i$ or $a_1 \oplus \cdots \oplus a_n$ if $I = \{1, \ldots, n\}$.

Theorem 9. *Let $\mathbb{F}_1(\gamma)$ be a simply transcendental modular q-field, and let π be an irreducible of $\varepsilon[\gamma]$ with corresponding prime order $e := \varepsilon[\gamma]_{(\pi)}$ of $\mathbb{F}_1(\gamma)$. Then*

$$\ell(\varepsilon(\gamma)/\varepsilon(\pi)) = \deg\pi = \ell(\varepsilon[\gamma]/(\pi)) = d_e. \tag{37}$$

Proof. The third equation follows by Corollary 4 of Proposition 46. With $m := \deg\pi$, we have $\pi \leqslant \varepsilon \vee \gamma \vee \cdots \vee \gamma^m$. Hence $\gamma^m \leqslant \varepsilon \vee \cdots \vee \gamma^{m-1} \vee \pi$. In terms of sup-lattices, this means that we have a decomposition

$$\varepsilon \oplus \gamma \oplus \cdots \oplus \gamma^m = \varepsilon \oplus \gamma \oplus \cdots \oplus \gamma^{m-1} \oplus \pi.$$

This equation remains valid under multiplication with powers of γ. Thus, by induction, we obtain

$$\varepsilon \oplus \gamma \oplus \cdots \oplus \gamma^n = \varepsilon \oplus \gamma \oplus \cdots \oplus \gamma^{m-1} \oplus \pi \oplus \pi\gamma \oplus \cdots \oplus \pi\gamma^{n-m}.$$

Hence $\varepsilon[\gamma] = \varepsilon \oplus \gamma \oplus \cdots \oplus \gamma^{m-1} \oplus \pi\varepsilon[\gamma]$, which yields the second equation of (37).

With $a := \varepsilon \vee \gamma \vee \cdots \vee \gamma^{m-1}$, we have $\varepsilon[\gamma] = a \oplus \pi\varepsilon[\gamma]$. By induction, this gives $\varepsilon[\gamma] = a \oplus \pi a \oplus \cdots \oplus \pi^n a \oplus \pi^{n+1}\varepsilon[\gamma]$ for all $n \in \mathbb{N}$. Hence $\varepsilon[\gamma] = \bigoplus_{i\in\mathbb{N}} a\pi^i$, and thus

$$\varepsilon[\gamma] = \varepsilon[\pi] \oplus \varepsilon[\pi]\gamma \oplus \cdots \oplus \varepsilon[\pi]\gamma^{m-1}. \tag{38}$$

In particular, $\ell(\varepsilon(\gamma)/\varepsilon(\pi)) \leqslant m$. Suppose that $n := \ell(\varepsilon(\gamma)/\varepsilon(\pi)) < m$. Then $\gamma^n \leqslant \varepsilon(\pi)(\varepsilon\vee\gamma\vee\cdots\vee\gamma^{n-1})$. Hence $\gamma^n \leqslant \alpha_0\alpha_n^{-1}\vee\alpha_1\alpha_n^{-1}\gamma\vee\cdots\vee\alpha_{n-1}\alpha_n^{-1}\gamma^{n-1}$ with $\alpha_i \leqslant \varepsilon[\pi]$ in $\varepsilon(\pi)^\times$. Now Eq. (38) implies that the $\pi^i\gamma^j$ with $i \in \mathbb{N}$ and $0 \leqslant j < m$ are independent, contrary to the relation

$$\alpha_n\gamma^n \leqslant \alpha_0 \vee \alpha_1\gamma \vee \cdots \vee \alpha_{n-1}\gamma^{n-1}.$$

Thus $n = m$, which completes the proof of Eqs. (37). □

Remarks. 1. As in Section 4, we can consider the group $J_F := \coprod_{e\in X(\mathbb{F}_1(\gamma))} G_e$ of replete ideals in $\mathbb{F}_1(\gamma)$ with unit element $e_{\mathbb{F}_1(\gamma)} := (e)_{e\in X(F)}$. By Corollary 4 of Proposition 46, the map $\alpha \mapsto e_{\mathbb{F}_1(\gamma)}\alpha$ gives an embedding $\mathbb{F}_1(\gamma)^\times \hookrightarrow J_F$. For an irreducible π of $\varepsilon[\gamma]$, let $p \in P(\mathbb{F}_1(\gamma))$ be the corresponding prime with $\pi \leqslant p$, and let $q \in P(\mathbb{F}_1(\gamma))$ be the prime with $\gamma \nleqslant q$. Then $e_{\mathbb{F}_1(\gamma)}\pi = pq^{-1}$, and $\ell(\varepsilon(\gamma)/\varepsilon(\pi)) = \deg\pi = \ell(q^{-1}/e_{\mathbb{F}_1(\gamma)})$, in accordance with Theorem 4 in the case of function q-fields.

2. The irreducibles $\pi \leqslant \varepsilon[\gamma]$ of degree 1 form an affine line $[\varepsilon \vee \gamma] \smallsetminus \{\varepsilon\}$. Compare with with the classical case $k[x]$ where the linear polynomials $x - \lambda$ correspond to the elements $\lambda \in k$. So the "elements" of $\mathbb{F}_1 \subset \mathbb{F}_1(\gamma)$ can be identified with units in $[\varepsilon \vee \gamma]$.

Corollary 1. *Let $\mathbb{F}_1(\gamma)$ be a simply transcendental modular q-field. Then $\pi \in \mathbb{F}_1(\gamma)^\times$ satisfies $\mathbb{F}_1(\pi) = \mathbb{F}_1(\gamma)$ if and only if $e_{\mathbb{F}_1(\gamma)}\pi = pq^{-1}$ with distinct primes $p, q \in P(\mathbb{F}_1(\gamma))$ such that $d_{p\to p} = d_{q\to q} = 1$.*

Proof. The necessity follows since γ has this property. Conversely, assume that $e_{\mathbb{F}_1(\gamma)}\pi = pq^{-1}$. Then either $\gamma \leqslant q \to q$ or $\gamma^{-1} \leqslant q \to q$. By symmetry, we can assume that $\gamma^{-1} \leqslant q \to q$. So $\varepsilon[\gamma^{-1}] \wedge q = (\sigma)$ for some irreducible $\sigma \leqslant \varepsilon[\gamma^{-1}]$ of degree 1. Hence $\varepsilon[\sigma] = \varepsilon[\gamma^{-1}]$. Since $p \neq q$, it follows that $\varepsilon[\sigma^{-1}] \leqslant p \to p$. So there is an irreducible τ of $\varepsilon[\sigma^{-1}]$ of degree 1 with $\varepsilon[\sigma^{-1}] \wedge p = (\tau)$. Thus $\varepsilon[\tau] = \varepsilon[\sigma^{-1}]$. So $\varepsilon(\tau) = \varepsilon(\sigma) = \varepsilon(\gamma)$ and $e_{\mathbb{F}_1(\gamma)}\tau = pq^{-1}$, which yields $e_{\mathbb{F}_1(\gamma)}\pi\tau^{-1} = e_{\mathbb{F}_1(\gamma)}$. By Corollary 4 of Proposition 46, $\tau = \pi$. □

Corollary 2. *A simply transcendental extension of q-fields is split monic in* $\mathbf{Flr}^q$.

Proof. Let $\mathbb{F}_1(\gamma)$ be a simply transcendental modular q-field. Then $e := \varepsilon[\gamma]_{(\gamma)}$ is a prime order in $\mathbb{F}_1(\gamma)$, and Theorem 9 gives an isomorphism of q-fields $\mathbb{F}_1 \cong \varepsilon[\gamma]/(\gamma) \xrightarrow{\sim} \partial e$. Hence $q_e|_{\mathbb{F}_1}$ is invertible. □

15 Exceptional connected modular q-fields

Before we return to categorical questions, we have to deal with q-fields of small length which do not arise from field extensions. By Theorem 3, every connected modular q-field of length $\geqslant 4$ is isomorphic to $F(K|k)$ for a field extension $K|k$. Connected modular q-fields which cannot be represented in this way will be called *exceptional*. The q-field $\mathbb{F}_1$ of length 1 is not exceptional. So the exceptional connected modular q-fields are either *lines* (length 2) or *planes* (length 3).

Since every finite multiplicative subgroup of a field is cyclic (the equation $x^n = 1$ has at most n solutions), the smallest exceptional connected modular q-field is a line with 4 elements, forming a Klein Four group $C_2 \times C_2$. The next smallest one is the cyclic group C_7. As there is no field with 6 elements, this gives a 7-point exceptional projective line.

More interesting are the exceptional connected planes. They are non-Desarguesian planes where the points form an abelian group such that the left multiplications are automorphisms of the plane. Such a structure has been called an *incidence group* [17, 18]. Equivalently, an incidence group is a group which acts freely and transitively by collineations on a projective plane. These groups have also been called *Singer groups* [28]. It is an old open problem whether non-Desarguesian finite Singer groups exist. This question is deeply related to problems in number theory [29] and the notorious "field with one element" [7, 8].

Infinite (abelian) Singer groups abound. Thas ([28], Theorem 5.1) has shown that an infinite abelian group G can be represented as a Singer group of a projective plane (hence as the unit group of a modular q-field of length 3) if and only if it has no elements of order 2. By a result of Brandis [5], the group $K^{\times}/k^{\times}$ of a non-trivial field extension $K|k$ cannot be finitely generated unless the base field k is finite. Therefore, every finitely generated infinite abelian group without elements of order 2 gives rise to an exceptional q-field of length 3.

The first example of a non-Desarguesian infinite cyclic incidence group was given by Hall [15]. The method was generalized by Hughes [16]. We will adapt it to our framework to construct exceptional connected q-fields of length 3. Before we do this, we take the opportunity to give a simple characterization of modular q-fields in general.

Proposition 47. *A q-field F is modular if and only if the following are satisfied for $\alpha, \beta, \gamma \in F^{\times}$ and non-zero $a, b \in F$:*

(a) *If $\alpha \leqslant \beta \vee \gamma$ and $\alpha \neq \beta$, then $\gamma \leqslant \alpha \vee \beta$.*
(b) *If $\gamma \leqslant a \vee b$, there exist $\alpha \in [a]$ and $\beta \in [b]$ with $\gamma \leqslant \alpha \vee \beta$.*

Proof. The necessity follows by Lemma 1 and Lemma 9. Conversely, assume that (a) and (b) are satisfied, and let $a, b, c \in F$ with $a \leqslant c$ be given. We have to verify that $(a \vee b) \wedge c \leqslant a \vee (b \wedge c)$. Without loss of generality, we can assume that $a \wedge b > 0$. Then (b) implies that any $\gamma \in [(a \vee b) \wedge c]$ satisfies $\gamma \leqslant \alpha \vee \beta$ for some $\alpha \in [a]$ and $\beta \in [b]$. If $\gamma = \alpha$, then $\gamma \leqslant a \leqslant a \vee (b \wedge c)$. Otherwise, (a) gives $\beta \leqslant \alpha \vee \gamma \leqslant c$. Thus, in any case, $\gamma \leqslant a \vee (b \wedge c)$. ☐

As every element $a \in F$ is determined by the closed subset $[a] \subset F^{\times}$, condition (b) can be interpreted as a definition of joins in terms of closed sets: For $a, b \in F$,

$$[a \vee b] = \bigcup_{\alpha \in [a],\, \beta \in [b]} [\alpha \vee \beta].$$

Definition 28. Let G be an abelian group. A subset D of G is said to be a *difference set* if for each non-zero $g \in G$ there is a unique pair $(a, b) \in D \times D$ with $a - b = g$.

To any difference set D of a (multiplicative) abelian group G we associate a projective plane as follows. Points are the elements of G. For any $\alpha \in G$ we define a *line*

$$\ell_\alpha := \{\beta \in G \mid \alpha^{-1}\beta \in D\}.$$

Proposition 48. *Any difference set D of an abelian group G determines a connected modular q-field F_D of length 3 with $F_D^{\times} = G$. If G is finitely generated and without elements of order 2, the q-field F_D is exceptional.*

Proof. For distinct points $\alpha, \beta \in G$, there are unique $\gamma, \delta \in D$ with $\alpha^{-1}\beta = \gamma^{-1}\delta$. Hence $\gamma^{-1}\alpha = \delta^{-1}\beta$. Thus $\alpha, \beta \in \ell_{\gamma^{-1}\alpha} = \ell_{\delta^{-1}\beta}$. Conversely, $\alpha, \beta \in \ell_\zeta$ implies that $\zeta^{-1}\alpha = \gamma$ and $\zeta^{-1}\beta = \delta$ with $\gamma, \delta \in D$. Hence $\alpha^{-1}\beta = \gamma^{-1}\delta$, and thus $\zeta = \gamma^{-1}\alpha = \delta^{-1}\beta$. So there is a unique line $\alpha \vee \beta$ which connects α and β. By symmetry, the same argument shows that two distinct lines ℓ_α and ℓ_β intersect in a singleton of a point in G.

Now we define a subset $A \subset G$ to be *closed* if for all pairs of distinct $\alpha, \beta \in A$, the line $[\alpha \vee \beta]$ belongs to A. So the closed subsets form a complete lattice F_D. For a line ℓ_α and a point $\beta \notin \ell_\alpha$, every point $\gamma \neq \beta$ determines a line $\beta \vee \gamma$ which intersects ℓ_α in a single point δ. Thus $\gamma \in \beta \vee \delta$, which shows that $\ell_\alpha \vee \beta = G$. This proves that F_D is of length 3. Moreover, it follows that F_D satisfies condition (b) of Proposition 47. Condition (a) holds since any pair of distinct points determines a unique connecting line.

For $\alpha, \beta, \gamma, \delta \in G$, we have $\gamma \in \beta \cdot \ell_\alpha \Leftrightarrow \gamma\beta^{-1} \in \ell_\alpha \Leftrightarrow \alpha^{-1}\gamma\beta^{-1} \in D \Leftrightarrow \gamma \in \ell_{\alpha\beta}$. Hence $\beta \cdot \ell_\alpha = \ell_{\alpha\beta}$. Assume that $\ell_\alpha = \gamma \vee \delta$. Then $\beta\gamma \in \ell_{\alpha\beta}$ and $\beta\delta \in \ell_{\alpha\beta}$. Thus $\beta(\gamma \vee \delta) = \beta\gamma \vee \beta\delta$, which shows that F_D is a q-field. By Proposition 47, F_D is modular, and $F_D^\times = G$. The remaining statement has been verified already. $\square$

Example 13. To construct a difference set D on the additive group of $\mathbb{Z}$, we represent D as $\{d_i \mid i \in \mathbb{N}\}$ with a strictly increasing sequence (d_i). We start with $d_0 = 1$ and $d_1 = 2$. This gives a single positive difference $2 - 1 = 1$. With $d_3 = 4$, we get two further (positive) differences 2 and 3. Of course, each difference $d > 0$ comes with its negative counterpart $-d$. Suppose that $a_0, \ldots, a_n$ have been constructed, and let d be the smallest difference that has not yet been obtained. Then we may check whether $a_{n+1} := a_n + d$ is possible. This means that the differences $a_{n+1} - a_i$ with $i < n$ have not been obtained. Otherwise, we try $a_{n+1} := a_n + d + 1$, and so on. So we get new differences which are all distinct. The procedure leads to the so-called Mian-Chowla sequence

$$1, 2, 4, 8, 13, 21, 31, 45, 66, 81, 97, 123, 148, 182, 204, 252, 290, 361, 401, 475, 565, 593, \ldots$$

However, it seems that the difference 33 never occurs. So we have to modify the construction. If d is the smallest difference that has not yet occurred, we add two big numbers a_{n+1} and a_{n+2} with difference d. If a_{n+1} is big enough, the new differences $a_{n+1} - a_i$ and $a_{n+2} - a_i$ are all distinct. So we obtain infinitely many difference sets on $\mathbb{Z}$. By Proposition 48, they all lead to exceptional q-fields.

For finite field extensions, we have a degree formula. The next result shows that this remains true for arbitrary modular q-fields.

Proposition 49. *Let F be a modular q-field of finite length with a rational order e. Then*

$$\ell(F) = \ell(\partial e) \cdot \ell(F/e).$$

Proof. Let $e\alpha_1, \ldots, e\alpha_m$ be a maximal independent set of units in F/e, where $\alpha_1, \ldots, \alpha_m \in F^\times$. Then $1 = e\alpha_1 \oplus \cdots \oplus e\alpha_m$. Hence $\ell(F/e) = m$, and thus $\ell(F) = \ell(e\alpha_1/0) + \cdots + \ell(e\alpha_m/0) = m \cdot \ell(\partial e)$. $\square$

As an immediate consequence, we get

Corollary. *The rational orders of an exceptional connected modular q-field are 1 and ε.*

16 The connected part of a modular q-field

Let us call a modular q-field F *totally disconnected* if the connected components of $F^\times$ are trivial.

Lemma 10. *Let F be a modular q-field. Assume that $\alpha \leqslant \alpha_1 \vee \cdots \vee \alpha_n$, with units $\alpha, \alpha_i \in F^\times$. Then α is connected to one of the α_i.*

Proof. We can assume that $n > 1$. Then Lemma 9 implies that $\alpha \leqslant \beta \vee \alpha_n$ with $\beta \in [\alpha_1 \vee \cdots \vee \alpha_{n-1}]$. Thus α is connected to β or to α_n, and the result follows by induction. □

Proposition 50. *A modular q-field F is totally disconnected if and only if $F \cong \mathfrak{P}(F^\times)$. If F is totally disconnected, then $[\bar{\varepsilon}]$ is the torsion part of $F^\times$.*

Proof. Let F be totally disconnected, and $\alpha \leqslant \alpha_1 \vee \cdots \vee \alpha_n$ with $\alpha, \alpha_i \in F^\times$ and $n > 1$. By Lemma 10, $\alpha = \alpha_i$ for some i. So the natural epimorphism $\mathfrak{P}(F^\times) \twoheadrightarrow F$ is invertible. In particular, $\alpha^{n+1} \leqslant \varepsilon \vee \alpha \vee \cdots \vee \alpha^n$ implies that $\alpha^{n+1} = \alpha^i$ with $i \leqslant n$. Thus $\alpha^{n+1-i} = \varepsilon$, that is, α belongs to the torsion part of $F^\times$. So $[\bar{\varepsilon}] = T(F^\times)$. Conversely, $\mathfrak{P}(F^\times)$ is totally disconnected. □

Proposition 51. *Let F be a modular q-field. There is a rational order e in F such that ∂e is the largest connected sub-q-field of F. The q-field F/e is totally disconnected.*

Proof. Let $C \subset F^\times$ be the connected component which contains ε. Since multiplication with a unit $\alpha \in F^\times$ is an automorphism of F, each connected pair ε, β is mapped to a connected pair $\alpha, \alpha\beta$. Hence $e := \bigvee C$ is an order. By Lemma 10, any $\alpha \in [e]$ is connected to an element of C. Hence $\alpha \in C$, which yields $C = [e]$. For any $\alpha \in C$, multiplication with α^{-1} gives $\{\alpha^{-1}, \varepsilon\} = \{\varepsilon, \alpha\}\alpha^{-1}$. Hence e is a rational order. The connected components of $F^\times$ are $[e\alpha]$ with $\alpha \in F^\times$. Thus F/e is totally disconnected. □

With the notation of Proposition 51, we call e the *connected component of* ε. The $e\alpha$ with $\alpha \in F^\times$ will be called the *connected components* of F. Furthermore, we call $F_c := \partial e$ the *connected part* of F. Accordingly, we call a modular q-field F *exceptional* if F_c is exceptional. Note that $F \mapsto F_c$ is functorial.

By Proposition 51, every embedding $i \colon \partial f \hookrightarrow F$ with a rational order f admits a unique factorization $i \colon \partial f \hookrightarrow \partial e \hookrightarrow F$ such that e is the connected component of the unit element in F/f, and F/e is totally disconnected. Accordingly, we call i *connected* if $\partial e = F$ and a *trivial embedding* if $e = f$.

Recall that a q-group can be regarded as a group G with a nucleus on $\mathfrak{P}(G)$ which is *finitary* in the sense that any element α in the closure $\overline{A}$ of a subset $A \subset G$ already belongs to the closure of a finite subset of A (see the remarks after Proposition 3). The finitary property says that the units are compact. Then $\widehat{G}$ is a q-field if and only if G is abelian with the "Hausdorff" property that $\{\varepsilon\}$ is closed.

Corollary 1. *Let F be a modular q-field, and let e be the connected component of ε. With respect to the nucleus given by the natural morphism $\mathfrak{P}(F^\times) \twoheadrightarrow F$, a subset A of $F^\times$ is closed if and only if $A\alpha \cap [e]$ is closed in ∂e for all $\alpha \in F^\times$.*

Proof. The necessity is obvious. Since every circuit in $F^\times$ is contained in a single connected component of $F^\times$, the sufficiency follows. □

Corollary 1 leads to a construction of arbitrary modular q-fields with prescribed connected component of ε. Let F be a modular q-field, and let $h\colon F^\times \hookrightarrow H$ be a monomorphism of abelian groups. We are looking for a trivial embedding $i\colon F \hookrightarrow G$ into a modular q-field G such that $i^\times = h$. We call such a q-field G a *trivial extension* of F along h.

$$\begin{array}{ccc} F & \overset{i}{\hookrightarrow} & G \\ \uparrow & & \uparrow \\ F^\times & \overset{h}{\hookrightarrow} & H \end{array}$$

As a lattice, $G \cong F^{H/F^\times}$. Thus G is modular. The elements of G can be represented as closed subsets of H, where $A \subset H$ is closed if and only if its intersection with each residue class in $H/F^\times$ is closed. As the $F\alpha \cong F$ with $\alpha \in H$ are all isomorphic, G is unique up to isomorphism. So we obtain

Corollary 2. *Let F be a modular q-field, and let $F^\times$ be a subgroup of an abelian group H. The trivial extension of F along $F^\times \hookrightarrow H$ extsts and is modular. Every modular q-field is a trivial extension of its connected part.*

The special case where the short exact sequence $F^\times \hookrightarrow H \twoheadrightarrow F^\times/H$ splits is given by the following

Corollary 3. *Let F be a modular q-field, and let G be a totally disconnected q-field. Then $F \otimes G$ is a trivial extension of F along $F^\times \hookrightarrow F^\times \times G^\times$.*

Proof. Since $G \otimes F \cong Q(G, F^\circ)^\circ \cong Q(\mathfrak{P}(G^\times), F^\circ)^\circ$, the elements of $F \otimes G \cong G \otimes F$ can be represented as maps $G^\times \to F$. By Eq. (29), a simple tensor $\beta \otimes a \in G \otimes F$ with $\beta \in G^\times$ maps β to $a \in F$, and all other units $\beta' \in G^\times$ to 0. Thus, as a lattice, $G \otimes F$ coincides with the trivial extension. Furthermore, the multiplication in $F \otimes G$ coincides with that of the trivial extension. □

Note that for a tensor product $F \otimes G$ of q-fields (30), the embedding $i\colon F \hookrightarrow F \otimes G$ has a retraction $p\colon F \otimes G \twoheadrightarrow F$, induced by the identity $1_F\colon F \to F$ and the zero morphism $0\colon G \to F$, which gives a short exact sequence

$$G \hookrightarrow F \otimes G \twoheadrightarrow F.$$

Proposition 52. *Let F be a modular q-field with a rational order f, and let e be the connected component of ε. Then $(F/f)_c = \partial(ef)$ and $(\partial f)_c = \partial(e \wedge f)$.*

Proof. Let $\{f, f\alpha, f\beta\}$ be a circuit in F/f with $\alpha, \beta \in F^\times$. Then $\varepsilon \leqslant f\alpha \vee f\beta$. By Lemma 9, $\varepsilon \leqslant \gamma\alpha \vee \delta\beta$ for some $\gamma, \delta \in [f]$. Since $\varepsilon \neq \delta\beta$, this implies that $\gamma\alpha \in [e]$. Hence $\alpha \leqslant ef$, and thus $(F/f)_c = \partial(ef)$. For any $\alpha \in [e \wedge f]$ with $\alpha \neq \varepsilon$, there is a circuit $\{\varepsilon, \alpha, \beta\}$ in F. Hence $\beta \leqslant \varepsilon \vee \alpha \leqslant f$, and thus $\alpha \in (\partial f)_c$. $\square$

As an immediate consequence, we get

Corollary 1. *Let F be a modular q-field with a rational order f. Then F is connected if and only if ∂f and F/f are connected.*

Corollary 2. *A rational embedding $i\colon F \hookrightarrow G$ is trivial if and only if i_c is invertible.*

Proof. Let f be the rational order in G with $\partial f = F$, and let e be the connected component of ε in G. Then: "i is trivial" $\Leftrightarrow ef = f \Leftrightarrow e \leqslant f \Leftrightarrow e = e \wedge f$. $\square$

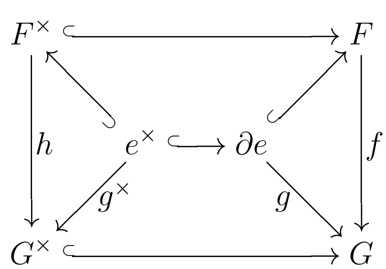
Proposition 53. *A morphism $f\colon F \to G$ of modular q-fields is a rational embedding if and only if f_c is a rational embedding and $(f^\times)^{-1}(G_c^\times) = F_c^\times$.*

Proof. The necessity follows by Proposition 52. So let us assume that f_c is a rational embedding with $(f^\times)^{-1}(G_c^\times) = F_c^\times$. For $\alpha \in F^\times$ with $f(\alpha) = \varepsilon$, this gives $\alpha \in F_c^\times$. Since f_c is a rational embedding, it follows that $\alpha = \varepsilon$. Thus $f^\times$ is injective. Now assume that $\beta \leqslant f(\alpha_1) \vee \cdots \vee f(\alpha_n)$ with n minimal, $\alpha_i \in F^\times$, and $\beta \in G^\times$. Assume first that $\alpha_1 = \varepsilon$. Then $f(\alpha_i) \in G_c^\times$ for all i. Hence $\alpha_i \in (f^\times)^{-1}(G_c^\times) = F_c^\times$. Since $f|_{F_c}$ is a rational embedding, $\beta = f(\alpha)$ for some $\alpha \in F_c^\times$ with $\alpha \leqslant \alpha_1 \vee \cdots \vee \alpha_n$. Now let α_1 be arbitrary. Then $\beta f(\alpha_1)^{-1} \leqslant \varepsilon \vee f(\alpha_2\alpha_1^{-1}) \vee \cdots \vee f(\alpha_n\alpha_1^{-1})$, which yields $\beta f(\alpha_1^{-1}) = f(\alpha)$ for some $\alpha \in F_c^\times$ with $\alpha \leqslant \varepsilon \vee \alpha_2\alpha_1^{-1} \vee \cdots \vee \alpha_n\alpha_1^{-1}$. Thus $\beta = f(\alpha\alpha_1)$ and $\alpha\alpha_1 \leqslant \alpha_1 \vee \cdots \vee \alpha_n$. $\square$

By Proposition 51, every modular q-field F is a trivial extension of a connected modular q-field. Accordingly, the morphisms in $\mathbf{Fld}^q$ are characterized by the following

Proposition 54. *Let F, G be modular q-fields, and let e be a rational order in F with F/e totally disconnected. For any morphism $g\colon \partial e \to G$ in $\mathbf{Fld}^q$ and a group homomorphism $h\colon F^\times \to G^\times$ with $h|_{e^\times} = g^\times$, there is a unique morphism $f\colon F \to G$ in $\mathbf{Fld}^q$ with $f|_{\partial e} = g$ and $f^\times = h$.*

Proof. By assumption, the left-hand triangle and the two squares in the diagram



commute. We have to complete the diagram by a commuting right-hand triangle. Every closed subset A of $F^\times$ satisfies

$$A = \bigcup_{\alpha \in F^\times} (A \cap \alpha e^\times) = \bigcup_{\alpha \in F^\times} \alpha(\alpha^{-1} A \cap e^\times).$$

For a morphism $f\colon F \to G$, this gives $f(A) = \bigvee_{\alpha \in F^\times} f(\alpha) f(\alpha^{-1} A \cap e^\times)$. So f is uniquely determined by g and h, which leads us to define

$$f(A) := \bigvee_{\alpha \in F^\times} h(\alpha) g(\alpha^{-1} A \cap e^\times). \tag{39}$$

For closed $A, B \subset F^\times$, this gives $f(\overline{AB}) = f\big(\bigcup_{\alpha,\beta \in F^\times} \alpha\beta\overline{(\alpha^{-1} A \cap e^\times)(\beta^{-1} B \cap e^\times)}\big) = \bigvee_{\alpha,\beta \in F^\times} h(\alpha)h(\beta) g(\alpha^{-1} A \cap e^\times) g(\beta^{-1} B \cap e^\times) = f(A)f(B)$. If $(A_i)_{i \in I}$ is a family of closed subsets of $F^\times$,

$$f(\bigvee_{i \in I} A_i) = \bigvee_{\alpha \in F^\times} h(\alpha) g(\bigvee_{i \in I} \alpha^{-1} A_i \cap e^\times) = \bigvee_{\alpha \in F^\times} \bigvee_{i \in I} h(\alpha) g(\alpha^{-1} A_i \cap e^\times) = \bigvee_{i \in I} f(A_i).$$

Thus f is a morphism of q-fields. For $A = \{\beta\}$, Eq. (39) gives $f(A) = h(\beta)$, while for $A \subset e^\times$, we obtain $f(A) = g(A)$. □

Corollary 1. *A morphism $f\colon F \to G$ of modular q-fields is equivalent to a morphism $g\colon F_c \to G_c$ together with a group homomorphism $h\colon F^\times \to G^\times$ satisfying $g^\times = h_c$:*

$$\begin{array}{ccc} F_c & \xrightarrow{\ g\ } & G_c \\ \uparrow & & \uparrow \\ F_c^\times & \xrightarrow{\ g^\times\ } & G_c^\times \end{array} \qquad \begin{array}{ccc} F^\times & \xrightarrow{\ h\ } & G^\times \\ \uparrow & & \uparrow \\ F_c^\times & \xrightarrow{\ h_c\ } & G_c^\times . \end{array}$$

The morphism f is given by $f_c = g$ and $f^\times = h$.

Proof. This follows since g factors through $G_c \hookrightarrow G$. □

Corollary 2. *Let F be a modular q-field with a rational order f, and let e be the connected component of ε in F. Then ef is a rational order with $(ef)^\times = e^\times f^\times$, and*

$$\begin{array}{ccc} \partial e & \hookrightarrow & \partial(ef) \\ \uparrow & & \uparrow \\ \partial(e \wedge f) & \hookrightarrow & \partial f \end{array} \tag{40}$$

is a pushout diagram in $\mathbf{Fld}^q$.

Proof. Every $\gamma \in [ef]$ satisfies $\gamma \leqslant e\beta_1 \vee \cdots \vee e\beta_n$ with $\beta_i \in [f]$. By Proposition 51, $\gamma \leqslant e\beta_i$ for some i. Hence ef is rational, with $(ef)^\times = e^\times f^\times$. Now let $i\colon \partial e \to G$ and $j\colon \partial f \to G$ be morphisms in $\mathbf{Fld}^q$ with $i|_{\partial(e\wedge f)} = j|_{\partial(e \wedge f)}$. Then $i^\times|_{(e\wedge f)^\times} = j^\times|_{(e\wedge f)^\times}$.

So there is a unique group homomorphism $h\colon e^\times f^\times \to G^\times$ with $h|_{e^\times} = i^\times$ and $h|_{f^\times} = j^\times$. By Proposition 52, e is the connected component of ε in $\partial(ef)$. So there is a unique morphism $g\colon \partial(ef) \to G$ with $g|_{\partial e} = i$ and $g^\times = h$. Since $g|_{f^\times} = j^\times$, we have $g|_{\partial f} = j$. Thus (40) is a pushout diagram in $\mathbf{Fld}^q$. $\square$

Example 14. Let $G = \langle \alpha, \beta \rangle$ be a free abelian group with two generators. We call a subset A of G *closed* if $\alpha^i\beta^j, \alpha^{i+n}\beta^{j-n} \in A$ with a non-zero integer n implies that $\alpha^{i+n}\beta^{j-n} \in A$ for all $n \in \mathbb{Z}$. It is easily checked that the corresponding closure operation is a nucleus in $\mathfrak{P}(G)$ which defines a modular q-field F with $F^\times = G$. The connected component of ε in F is $\{\alpha^n\beta^{-n} \mid n \in \mathbb{Z}\}$. On the other hand, $e := \langle \alpha \rangle$ and $f := \langle \beta \rangle$ are rational orders with $e \wedge f = \varepsilon$ and $ef = 1$ such that ∂e and ∂f are trivial. The example shows that this does not imply that $\partial(ef)$ is trivial.

The next corollary gives an enhanced version of Corollary 2.

Corollary 3. *Let F be a modular q-field with rational orders e and f such that $\partial(e \wedge f) \hookrightarrow \partial e$ is connected and $\partial e \hookrightarrow F$ is trivial. Then ef is a rational order with $(ef)^\times = e^\times f^\times$, and the commutative diagram* (40) *is a pushout.*

Proof. Let e' be the connected component of ε in F. Since $\partial e \hookrightarrow F$ is trivial, $e' \leqslant e$. By Proposition 52, we have a commutative diagram

$$\begin{array}{ccccc} \partial e' & \hookrightarrow & \partial e & \hookrightarrow & \partial(ef) \\ \uparrow & \text{PO} & \uparrow & & \uparrow \\ \partial(e' \wedge f) & \hookrightarrow & \partial(e \wedge f) & \hookrightarrow & \partial f \end{array} \tag{41}$$

with $e = e'(e \wedge f)$ and $\partial(e' \wedge f) = (\partial f)_c$. Hence $ef = e'f$. So Corollary 2 implies that the left square PO and the whole diagram (41) is a pushout. Therefore, the right-hand square in (41) is a pushout. Furthermore, $ef = e'f$ is a rational order, and $(ef)^\times = (e')^\times f^\times \subset e^\times f^\times \subset (ef)^\times$. $\square$

Now we prove an exterior version of Corollary 3.

Corollary 4. *Let F be a modular q-field with a rational order f such that $i\colon \partial f \hookrightarrow F$ is connected, and let $j\colon \partial f \hookrightarrow G$ be a trivial embedding of modular q-fields. There is a pushout*

$$\begin{array}{ccc} F & \overset{h}{\hookrightarrow} & H \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle g} \\ \partial f & \overset{j}{\hookrightarrow} & G \end{array} \tag{42}$$

in $\mathbf{Fld}^q$ with a trivial embedding h and a connected rational embedding g.

Proof. Consider the pushout of abelian groups

$$\begin{array}{ccc} F^\times & \overset{s}{\hookrightarrow} & C \\ \uparrow{\scriptstyle i^\times} & & \uparrow{\scriptstyle r} \\ f^\times & \overset{j^\times}{\hookrightarrow} & G^\times, \end{array}$$

and let $h\colon F \hookrightarrow H$ be the trivial embedding of F along s. Then $rj^\times = si^\times = (hi)^\times$. Hence there is a unique morphism $g\colon G \to H$ with $hi = gj$ and $g^\times = r$. Proposition 53 implies that g is a rational embedding. So (42) is a pushout by Corollary 3. By Proposition 52, g is connected. □

Proposition 55. *Let F be a modular q-field with rational orders e and f such that F/e is totally disconnected. Then ef is a rational order with $(ef)^\times = e^\times f^\times$, and*

$$\partial(ef)/f \cong \partial e/(e \wedge f).$$

Proof. Since F/e is totally disconnected, ef is a rational order with $(ef)^\times = e^\times f^\times$ (see the proof of Corollary 2 of Proposition 54). The map $(e \wedge f)\alpha \mapsto f\alpha$ induces a morphism $g\colon \partial e/(e \wedge f) \to \partial(ef)/f$ of q-fields. It is easily checked that $g^\times$ is bijective. Let $f\alpha \leqslant f\alpha_1 \vee \cdots \vee f\alpha_n$ hold in $\partial(ef)/f$ for some $\alpha, \alpha_i \in e^\times$, and let n be minimal. Then $\alpha \leqslant \alpha_1\beta_1 \vee \cdots \vee \alpha_n\beta_n$ with $\beta_i \in f^\times$. Since F/e is totally disconnected, $\beta_i \in e^\times$ for all i. Hence $\beta_i \in (e \wedge f)^\times$. Thus g is an isomorphism of q-fields. □

17 Stable lax morphisms and projective q-fields

We have seen that every modular q-field is a trivial extension of its connected part which is either exceptional or equivalent to a field extension. The class of rigid morphisms is well suited for the study of connected modular q-fields, but it is too narrow for a proper treatment of the non-connected case. On the other hand, the class of all lax morphisms is too wide. In this section we introduce an intermediate class of morphisms.

Definition 29. We call a lax morphism $f\colon F \to G$ of modular q-fields *stable* if the restriction $f|_{F_c}$ to the connected part of F is rigid.

Proposition 56. *Every prime surjection $q_e\colon F \twoheadrightarrow \partial e$ is stable. If f denotes the rational order in F with $F_c = \partial f$, there is a commutative diagram*

$$\begin{array}{ccc} F & \overset{q_e}{\twoheadrightarrow} & \partial e \\ \uparrow & & \uparrow \\ F_c & \overset{q_{e\wedge f}}{\twoheadrightarrow} & \partial(e \wedge f) \end{array}$$

where $\partial(e \wedge f)$ is the connected part of ∂e.

Proof. The commutative diagram is given by Proposition 44. So q_e is stable. Let p be the radical of e. Since every unit in $\partial(e \wedge f)$ is of the form $\alpha \vee (p \wedge f)$ with $\alpha \in (e \wedge f)^\times$, the units of $\partial(e \wedge f)$ are connected. Thus $\partial(e \wedge f)$ is connected.

Conversely, let $\alpha \in e^\times$ be a unit such that $\alpha \vee p$ is connected to $\varepsilon \vee p$ in ∂e and $\alpha \vee p \neq \varepsilon \vee p$. Then $\alpha \leqslant \varepsilon \vee \beta \vee p$ for a unit $\beta \in e^\times$ with $\alpha \nleqslant \beta \vee p$. By Lemma 9, this implies that $\alpha \leqslant \varepsilon \vee \gamma$ with a unit $\gamma \in [\beta \vee p] \smallsetminus \{\alpha\}$. Hence $\alpha \in (e \wedge f)^\times$. Thus $\partial(e \wedge f) = (\partial e)_c$. □

Corollary 1. *A lax morphism $f\colon F \to G$ of modular q-fields is stable if and only if $f = gq$ with a prime surjection q and a stable morphism $g \in \mathbf{Fld}^q$.*

Proof. Let f be stable. By Proposition 43, there is a factorization $f = gq$ with a prime surjection q and a morphism $g \in \mathbf{Fld}^q$. The induced morphism $f_c = g_c q_c$ is rigid. Hence g_c is rigid, and thus g is stable. The converse follows by Proposition 56. □

Corollary 2. *Modular q-fields with stable lax morphisms form a subcategory of $\mathbf{Fl}^q$.*

Proof. By Corollary 1, a lax morphism $f\colon F \to G$ is stable if and only if there is a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\ f\ } & G \\ \uparrow & & \uparrow \\ F_c & \xrightarrow{\ f_c\ } & G_c \end{array}$$

with a rigid morphism f_c. So the corollary follows since rigid morphisms are closed under composition. □

We will write $\mathbf{Fls}^q$ for the subcategory of $\mathbf{Fl}^q$ with stable lax morphisms. By Corollary 1, the analysis of $\mathbf{Fls}^q$ reduces to the study of stable morphisms in $\mathbf{Fld}^q$. They form a full subcategory of $\mathbf{Fls}^q$ which we denote by $\mathbf{Flds}^q$. For example, every rational embedding is stable, and for any modular q-field F, the natural morphism $\mathfrak{P}(F^\times) \twoheadrightarrow F$ is stable. The category $\mathbf{Flds}^q$ will be investigated in Sections 19.

Recall that a subcategory $\mathscr{C}$ of a category $\mathscr{D}$ is said to be *co-reflective* if the inclusion $I\colon \mathscr{C} \hookrightarrow \mathscr{D}$ admits a right adjoint $R\colon \mathscr{D} \to \mathscr{C}$, the *co-reflection*. Equivalently, this means that any object D in $\mathscr{D}$ admits a morphism $\varepsilon_D\colon C \to D$ with $C \in \mathscr{C}$ such that every morphism $C' \to D$ with $C' \in \mathscr{C}$ factors uniquely through ε_D:

$$\begin{array}{ccc} C & \xrightarrow{\ \varepsilon_D\ } & D \\ \uparrow{\scriptstyle \exists!} & \nearrow & \\ C' & & \end{array}$$

Then the object map of R is given by $RD := C$, and $\varepsilon_D\colon RD \to D$ is the D-component of the co-unit $\varepsilon\colon IR \to 1$ of the adjunction $I \dashv R$ (see [20], Chapter IV). For example, the connected modular q-fields form a co-reflective full subcategory of $\mathbf{Fld}^q$. Here ε_F is the narural embedding $F_c \hookrightarrow F$.

Proposition 57. *The totally disconnected q-fields form a co-reflective full subcategory of* $\mathbf{Fld}^q$ *which is equivalent to the category* $\mathbf{Ab}$ *of abelian groups.*

Proof. The functor $A \mapsto \mathfrak{P}(A)$ gives an equivalence between $\mathbf{Ab}$ and the category of totally disconnected q-fields. For a modular q-field F, consider the natural morphism $\varepsilon_F \colon \mathfrak{P}(F^\times) \twoheadrightarrow F$. For any $f \colon G \to F$ in $\mathbf{Fld}^q$ with G totally disconnected, $f^\times$ induces a unique morphism $g \colon G = \mathfrak{P}(G^\times) \to \mathfrak{P}(F^\times)$ with $\varepsilon_F g = f$. □

In other words, Proposition 57 says that the power set functor $\mathfrak{P} \colon \mathbf{Ab} \hookrightarrow \mathbf{Fld}^q$ is a full embedding with right adjoint $F \mapsto F^\times$. Now we turn our attention to modular q-fields P in $\mathbf{Fls}^q$ which are *projective* with respect to surjective lax morphisms, that is, every stable lax morphism $P \to G$ factors through each surjective $F \twoheadrightarrow G$ in $\mathbf{Fls}^q$.

Corollary 1. *A modular q-field F is projective in* $\mathbf{Fls}^q$ *if and only if F is totally disconnected and* $F^\times$ *is a free abelian group.*

Proof. Assume that F is projective. So the natural morphism $\varepsilon_F \colon \mathfrak{P}(F^\times) \twoheadrightarrow F$ admits a section, a morphism $s \colon F \to \mathfrak{P}(F^\times)$ with $\varepsilon_F s = 1_F$. Thus s is injective on $F^\times$ and $\mathrm{rad}(s) = 0$. For $\alpha, \alpha_i \in F^\times$ with $\alpha \leqslant \alpha_1 \vee \cdots \vee \alpha_n$, we have $s(\alpha) \leqslant s(\alpha_1) \vee \cdots \vee s(\alpha_n)$. Hence $s(\alpha) = s(\alpha_i)$ for some i, and thus $\alpha = \alpha_i$. So F is totally disconnected. Choose an epimorphism $p \colon H \twoheadrightarrow F^\times$ in $\mathbf{Ab}$ with a free abelian group H. Then $\mathfrak{P}(p) \colon \mathfrak{P}(H) \twoheadrightarrow F$ is a split epimorphism, which shows that p is split epic. Whence $F^\times$ is free.

Conversely, let $p \colon F \twoheadrightarrow G$ be a surjective morphism in $\mathbf{Fls}^q$. So there is a prime order e in F with $p = gq_e$ with a morphism $g \colon \partial e \twoheadrightarrow G$ in $\mathbf{Fld}^q$. Any morphism $f \colon \mathfrak{P}(H) \to G$ with a free abelian group H is left adjoint to a morphism $f' \colon H \to G^\times$. So there is a homomorphism $h \colon H \to e^\times \hookrightarrow F^\times$ in $\mathbf{Ab}$ with $ph = f'$. Therefore, h induces a morphism $h' \colon \mathfrak{P}(H) \to F$ with $ph' = f$. Whence $\mathfrak{P}(H)$ is projective. □

Corollary 2. *A morphism* $f \in \mathbf{Fls}^q$ *is monic if and only if* $f \in \mathbf{Fld}^q$ *with* $f^\times$ *injective.*

Proof. Let $f \colon F \to G$ be a monomorphism in $\mathbf{Fls}^q$. Suppose that $p := \mathrm{rad}(f) > 0$. Then there is a unit $\alpha \in [p]$. Hence $\alpha^2 \neq \alpha \leqslant p$. Let $C = \langle \gamma \rangle$ be an infinite cyclic group. By Proposition 57, the morphism $C \to F^\times$ with $\gamma \mapsto \alpha$ induces a morphism $g \colon \mathfrak{P}(C) \to F$ with $g(\gamma) = \alpha$. Hence $fg(\gamma) = 0$, which implies that $fg(\gamma^n) = 0$ for $n > 0$ and $fg(\gamma^n) = \infty$ for $n < 0$. Similarly, $\gamma \mapsto \alpha^2$ induces a morphism $\mathfrak{P}(C) \to F$ with this property, a contradiction. Thus $\mathrm{rad}(f) = 0$. With a similar argument, it follows that $f^\times$ must be injective.

Conversely, let $f \colon F \to G$ be a stable morphism in $\mathbf{Fld}^q$ with $f^\times$ injective. Let $g, h \colon H \to G$ be morphisms in $\mathbf{Fls}^q$ with $fg = fh$. Then $\mathrm{rad}(g) = \mathrm{rad}(h)$. By Proposition 43, we can assume without loss of generality that $g, h \in \mathbf{Fld}^q$. Hence $g^\times = h^\times$, and thus $g = h$. □

18 Injective modular q-fields

In this section, we study the modular q-fields I which are *injective* in the full subcategory $\mathbf{Fls}_0^q$ of non-exceptional modular q-fields in $\mathbf{Fls}^q$, that is, each $f\colon F \to I$ in $\mathbf{Fls}_0^q$ factors through all rational embeddings $F \hookrightarrow G$ of q-fields in $\mathbf{Fls}_0^q$. The restriction to non-exceptional q-fields is dictated by the fact that there are no non-trivial morphisms between exceptional and non-exceptional q-fields. For example, if I is exceptional, the morphism $\mathbb{F}_1 \to I$ cannot factor through the embedding $\mathbb{F}_1 \hookrightarrow F(\mathbb{C}|\mathbb{R})$. If I is non-exceptional, $\mathbb{F}_1 \to I$ does not factor through any embedding $\mathbb{F}_1 \hookrightarrow G$ with G exceptional. So there are no injectives at all in $\mathbf{Fls}^q$.

First, we deal with the pushforward of rational embeddings along prime surjections.

Proposition 58. *Let F' be a modular q-field with a rational sub-q-field F. Every prime surjection $q\colon F \twoheadrightarrow G$ extends to a prime surjection $q'\colon F' \twoheadrightarrow G'$ such that G is a rational sub-q-field of G':*

$$\begin{array}{ccc} F' & \xrightarrow{\;q'\;} & G' \\ \uparrow & & \uparrow \\ F & \xrightarrow{\;q\;} & G. \end{array}$$

Proof. There is a prime order e in F with $q = q_e$. By Proposition 11, there exists a prime order $e' \geqslant e$ in F' with radical p' such that $e \wedge p'$ is the radical of e. Thus $\partial e_0 = F$ and $e_0 \wedge e' = e$. By Proposition 44, $G = \partial e$ is isomorphic to a rational sub-q-field of $G' := \partial e'$. □

In accordance with category theory, we define an *essential embedding* in $\mathbf{Fls}_0^q$ to be a rational embedding $i\colon F \hookrightarrow G$ such that a morphism $f\colon G \to H$ in $\mathbf{Fls}_0^q$ is a rational embedding whenever fi is so.

Proposition 59. *Every essential embedding $i\colon F \hookrightarrow G$ in $\mathbf{Fls}_0^q$ is algebraic.*

Proof. Let f be the rational order in G with $F = \partial f$. Suppose that i is not algebraic. Then there exists a unit $\gamma \in G^\times$ which is transcendental over f. By Corollary 2 of Theorem 9, the embedding $j\colon F \hookrightarrow F(\gamma)$ is split monic with a retraction $q\colon F(\gamma) \twoheadrightarrow F$ in $\mathbf{Flr}^q$. By Proposition 58, q extends to G, which gives a commutative diagram

$$\begin{array}{ccccc} G & = & G & \xrightarrow{\;q'\;} & F' \\ \uparrow & & \uparrow{\scriptstyle i'} & & \uparrow{\scriptstyle j'} \\ F & \xrightarrow{\;j\;} & F(\gamma) & \xrightarrow{\;q\;} & F \end{array}$$

with $q'i'j = j'qj = j'$. Since $i'j$ is essential, q' is a rational embedding. Thus q' and q are invertible, a contradiction. So i is algebraic. □

Proposition 60. *Let $ji\colon F \hookrightarrow G$ be rational embedding of modular q-fields with $i\colon F \hookrightarrow H$ connected and $j\colon H \hookrightarrow G$ trivial. Then ji is essential in $\mathbf{Fls}_0^q$ if and only if i is algebraic and j essential in $\mathbf{Fls}_0^q$.*

Proof. Assume that ji is essential. Then $j\colon H \hookrightarrow G$ is essential. By Proposition 59, ji is algebraic. Hence i is algebraic.

To prove the converse, we have to show that every connected algebraic embedding $i\colon F \hookrightarrow G$ is essential in $\mathbf{Fls}_0^q$. Let $pq_e\colon G \to G'$ be a morphism with a prime order e in G and a stable morphism $p\colon \partial e \to G'$ in $\mathbf{Fld}^q$ such that $pq_e i$ is a rational embedding. If f denotes the rational order in G with $\partial f = F$, then $f \leqslant e$, which implies that $q_e i$ is a rational embedding. Since i is algebraic, Corollary 2 of Proposition 17 implies that $\dim G/f = 0$, which yields $e = 1$. So we can assume without loss of generality that $q_e = 1_G$.

By Proposition 52, and Corollary 2 of Proposition 54, we have a pushout

$$\begin{array}{ccc} G_c & \overset{j}{\hookrightarrow} & G \\ \uparrow & & \uparrow i \\ F_c & \hookrightarrow & F. \end{array}$$

Since p is stable, pj is a rational embedding. Let e and f be the rational orders in G' such that $\partial e = G_c$ and $\partial f = F$ with respect to the rational embeddings pj and pi. By Corollary 3 of Proposition 54, we can identify G with $\partial(ef)$ in G'. Whence $p\colon G \to G'$ is a rational embedding. $\square$

Corollary. *Let $F \hookrightarrow G$ be a connected rational embedding of modular q-fields. There exists a prime surjection $q\colon G \twoheadrightarrow H$ such that $q|_F$ is an essential embedding in $\mathbf{Fls}_0^q$.*

Proof. Let f be the rational order in G with $\partial f = F$. By Proposition 16, there is a minimal prime order $e \in V(G/f)$. If p denotes the radical of e, then $f \leqslant e$ implies that $p \wedge f = 0$. So there is a rational embedding $\partial f \hookrightarrow \partial e$ of q-fields. By the corollary of Proposition 15, $\dim \partial e/f = 0$. So Corollary 2 of Proposition 17 implies that the rational embedding $q_e|_F\colon F \hookrightarrow \partial e$ is algebraic, hence essential. $\square$

Recall that an abelian group G is said to be *divisible* if the maps $a \mapsto na$ with non-zero $n \in \mathbb{N}$ and $a \in G$ are surjective. In the category $\mathbf{Ab}$ of abelian groups, the injective objects are exactly the divisible groups (see [12], Section 21). Each injective group splits into a direct sum of indecomposables, and the indecomposable injective groups are the additive group of rationals $\mathbb{Q}$, and the *Prüfer groups* $\mathbb{Z}_{p^\infty} = \varinjlim \mathbb{Z}/(p^n)$.

Theorem 10. *A modular q-field F is injective in $\mathbf{Fls}_0^q$ if and only if F is of the form $F \cong F(K|k) \otimes \mathfrak{P}(G)$ with a field extension $K|k$ and a divisible abelian group G such that K is algebraically closed.*

Proof. Assume that F is injective. Let $e \in F$ be the connected component of ε. Then $\partial e = F(K|k)$ with a field extension $K|k$. Let $\overline{K}$ be the algebraic closure of K.

So the embedding $\partial e \hookrightarrow F$ extends to $F(\overline{K}|k)$, which gives a commutative diagram

$$\begin{array}{ccc} F(\overline{K}|k) & \xrightarrow{g} & F \\ \Big\uparrow i & & \Big\uparrow \\ F(K|k) & = & \partial e. \end{array}$$

By Proposition 60, i is essential. Hence g is a rational embedding. Since $F(\overline{K}|k)$ is connected, i is invertible.

As K contains all roots of unity, the group $K^\times$ is divisible. So the factor group $e^\times = K^\times/k^\times$ is divisible, too. Since $e^\times$ is injective in $\mathbf{Ab}$, this implies that $F^\times \cong e^\times \times G$ for some abelian group G. By Corollary 3 of Proposition 51, this yields $F \cong F(K|k) \otimes \mathfrak{P}(G)$. Let G^d be the divisible hull of G. Then we have a rational embedding $j\colon F(K|k) \otimes \mathfrak{P}(G) \hookrightarrow F(K|k) \otimes \mathfrak{P}(G^d)$. Since F is injective, j admits a retraction r. As every lax morphism respects connected components, j and r induce a factorization $1\colon \mathfrak{P}(G) \hookrightarrow \mathfrak{P}(G^d) \hookrightarrow \mathfrak{P}(G)$ of the identity. Thus G is a direct summand of G^d, which shows that G is divisible.

Conversely, assume that $F = F(K|k) \otimes \mathfrak{P}(G)$ with K algebraically closed and G divisible. Let F' be a non-exceptional modular q-field with a rational order f, and let $g\colon \partial f \to F$ be a morphism in $\mathbf{Fls}^q$. So there is a factorization $g = g'q$ with a prime surjection q and a stable morphism $g' \in \mathbf{Fld}^q$. We have to show that g factorizes through $j\colon \partial f \hookrightarrow F'$. By Proposition 58, we can assume that $g \in \mathbf{Fld}^q$.

Consider a factorization of $\partial f \hookrightarrow F'$ into a connected embedding $i\colon \partial f \hookrightarrow H$ and a trivial embedding $j\colon H \hookrightarrow F'$. By Proposition 59 and the corollary of Proposition 60, there is a prime surjection $q\colon H \twoheadrightarrow H'$ such that qi is an algebraic embedding. Thus, using again Proposition 58, we can assume without loss of generality that $\partial f \hookrightarrow F'$ factorizes into ji with i algebraic and j trivial.

Let e be the connected component of ε in F'. By Proposition 55, $\partial(ef)/f \cong \partial e/(e \wedge f)$. So the embedding $\partial(e \wedge f) \hookrightarrow \partial e$ is algebraic. By Proposition 52, $e \wedge f$ is the connected component of ε in ∂f. Since g is stable, this implies that $g|_{\partial(e\wedge f)}\colon \partial(e \wedge f) \hookrightarrow F$ is a rational embedding. So $e \wedge f$ can be regarded as a rational order in the connected part $F(K|k)$ of F. Since K is algebraically closed, $g|_{\partial(e\wedge f)}$ extends to ∂e. By Corollary 3 of Proposition 54, we have a pushout:

$$\begin{array}{ccc} \partial e & \hookrightarrow & \partial(ef) \\ \Big\uparrow & & \Big\uparrow \\ \partial(e \wedge f) & \hookrightarrow & \partial f. \end{array}$$

Therefore, g extends to a stable morphism $h\colon \partial(ef) \to F$.

Since $F^\times \cong K^\times/k^\times \times G$ is a divisible group, $h^\times$ extends to a group homomorphism $(F')^\times \to F^\times$. By Proposition 54, this implies that h extends to a morphism in $\mathbf{Flds}^q$. Whence F is injective. □

Corollary 1. *A modular q-field F is injective in $\mathbf{Fls}_0^q$ if and only if $F^\times$ is divisible and $F_c \cong F(K|k)$ with K algebraically closed.*

Proof. The necessity follows since $F(K|k)^\times \cong K^\times/k^\times$ is divisible. The converse follows by Corollary 3 of Proposition 51. $\square$

Corollary 2. *Every non-exceptional modular q-field F admits an injective envelope in $\mathbf{Fls}_0^q$, that is, an essential embedding $F \hookrightarrow \overline{F}$ with $\overline{F}$ injective.*

Proof. Let e be the connected component of ε in F. Then $\partial e = F(K|k)$ with a field extension $K|k$. So there is an algebraic closure $\overline{K}$ of K. Hence $F(K|k) \hookrightarrow F(\overline{K}|k)$ is algebraic. By Corollary 4 of Proposition 54, there is a pushout

$$\begin{array}{ccc} F & \overset{i}{\hookrightarrow} & G \\ \uparrow & & \uparrow j \\ \partial e & \hookrightarrow & F(\overline{K}|k) \end{array}$$

with i connected and j trivial. Hence $F(\overline{K}|k) = G_c$. By Corollary 3 of Proposition 54, $G^\times = F(\overline{K}|k)^\times F^\times$. Since $\partial e \hookrightarrow F(\overline{K}|k)$ is algebraic, Proposition 55 implies that i is algebraic. By Proposition 60, i is an essential embedding.

Let $h\colon G^\times \hookrightarrow H$ be the divisible hull of $G^\times$, and let $j'\colon G \hookrightarrow G'$ be the trivial extension of G along h. Let $p\colon G' \to G''$ be a morphism in $\mathbf{Fls}_0^q$ such that pj' is a rational embedding. Then $p^\times$ is injective. Let α be a unit in G' with $p(\alpha) \in G''_c$. Suppose that $\alpha \notin G^\times$. Since h is an essential embedding in $\mathbf{Ab}$, there is a smallest integer $n > 1$ with $\alpha^n \in G^\times$. Thus $\alpha^n \in G \cap G''_c = G_c = F(\overline{K}|k)$. Since $\overline{K}$ is algebraically closed, there is a unit $\beta \in G_c$ with $\beta^n = \alpha^n$. Hence $(\alpha\beta^{-1})^n = \varepsilon$. Since $\alpha\beta^{-1} \in (G')^\times \smallsetminus G^\times$, there is an integer $m > 1$ with $m < n$ and $(\alpha\beta^{-1})^m \in G^\times$. Thus $\alpha^m \in G^\times$, a contradiction to the minimality of n.

So we obtain $\alpha \in G^\times$, which yields $\alpha \in G_c^\times$. By Proposition 53, it follows that p is a rational embedding. Hence j' is essential, and thus Proposition 60 implies that $j'i\colon F \hookrightarrow G'$ is essential. By Corollary 1, G' is injective. $\square$

19 Stable morphisms and fibred categories

In this section, we provide a relative version of the fact that any modular q-field can be represented as a trivial extension of its connected part. "Relative" means that we give an analogue on the level of morphisms. The master of relativization is Grothendieck who introduced such an approach into modern algebraic geometry. The nature of stable morphisms will be clarified by using Grothendieck's concept of fibred category [14].

Let $p\colon \mathscr{F} \twoheadrightarrow \mathscr{E}$ be a functor. Regarding $\mathscr{F}$ as a category "over $\mathscr{E}$", we say that $f \in \mathscr{F}$ *lies over* $e \in \mathscr{E}$ if $p(f) = e$. For an object x of $\mathscr{E}$, the morphisms $f \in \mathscr{F}$ over 1_x form a subcategory $\mathscr{F}_x$ of $\mathscr{F}$, the *fiber* of x. Now a morphism $f\colon a \to b$ in $\mathscr{F}$ is said to be *cartesian* if for each $g\colon a' \to b$ in $\mathscr{F}$ with $p(g) = p(f)$ there is a unique morphism $h\colon a' \to a$ with $g = fh$. Dually, f is said to be *co-cartesian* if for each $g\colon a \to b'$ with

$p(g) = p(f)$ in $\mathscr{F}$ there is a unique $h\colon b \to b'$ in $\mathscr{F}$ with $g = hf$. Note that up to isomorphism, a cartesian morphism $f\colon a \to b$ is uniquely determined by its target b and its projection $p(f)$.

A category $\mathscr{F}$ is said to be *fibred over* $\mathscr{E}$ if for any morphism $\varphi\colon x \to p(b)$ in $\mathscr{E}$ there is a cartesian morphism $f\colon a \to b$ with $p(f) = \varphi$, and cartesian morphisms are closed under composition. Typical examples are sheaves or vector bundles over topological spaces. *Co-fibred* categories are defined dually. We apply these concepts to the category $\mathbf{Flds}^q$ of modular q-fields with stable morphisms. By $\mathbf{Flds}^q_c$ we denote the full subcategory of connected modular q-fields. Note that the morphisms in $\mathbf{Flds}^q_c$ are rational embeddings.

Proposition 61. *The category $\mathbf{Flds}^q$ is co-fibred via $F \mapsto F_c$ over the full subcategory $\mathbf{Flds}^q_c$. Co-cartesian morphisms are exactly the connected embeddings.*

Proof. Let F be a modular q-field, and let $h\colon F_c \hookrightarrow H$ be a morphism in $\mathbf{Flds}^q_c$. By Corollary 4 of Proposition 54, there is a pushout

$$\begin{array}{ccc} F & \overset{i}{\hookrightarrow} & G \\ \uparrow & & \uparrow j \\ F_c & \overset{h}{\hookrightarrow} & H \end{array}$$

with i connected and j trivial. Hence $h = i_c$. Now let $f\colon F \hookrightarrow G'$ be a morphism in $\mathbf{Flds}^q$ with $f_c = i_c$. By the pushout property, there is a unique morphism $g\colon G \to G'$ with $gi = f$ and $g_c = 1_H$. In particular, g is stable.

By Proposition 52, i is a connected embedding. Conversely, every connected embedding $i\colon F \hookrightarrow G$ satisfies $G^\times = F^\times G^\times_c$. By Corollary 2 of Proposition 54, this implies that G is a pushout of $F_c \hookrightarrow G_c$ along $F_c \hookrightarrow F$. $\square$

Definition 30. We say that a morphism $f \in \mathbf{Flds}^q$ is *disconnected* if f_c is invertible.

Thus every trivial embedding is disconnected. As a counterpart to Proposition 61, we have

Proposition 62. *The category $\mathbf{Flds}^q$ is fibred via $F \mapsto F^\times/F^\times_c$ over the category $\mathbf{Ab}$ of abelian groups. Cartesian morphisms are the disconnected morphisms in $\mathbf{Flds}^q$.*

Proof. Let F be a modular q-field, and let $h\colon H \to F^\times/F^\times_c$ be a group homomorphism in $\mathbf{Ab}$. Consider the embedding $i\colon F_c \hookrightarrow F$ and the pullback

$$\begin{array}{ccccc} F^\times_c & \hookrightarrow & H' & \overset{q'}{\twoheadrightarrow} & H \\ \| & & \downarrow h' & \text{PB} & \downarrow h \\ F^\times_c & \overset{i^\times}{\hookrightarrow} & F^\times & \overset{q}{\twoheadrightarrow} & F^\times/F^\times_c \end{array} \tag{43}$$

of abelian groups. We define F' to be the trivial extension of F_c along $F_c^\times \hookrightarrow H'$. By Proposition 54, there is a unique morphism $f\colon F' \to F$ with $f|_{F_c} = i$ and $f^\times = h'$. Thus $F'_c = F_c$, which yields $f_c = 1$. So f is a disconnected stable morphism of q-fields. Let $p\colon \mathbf{Flds}^q \to \mathbf{Ab}$ be the functor $F \mapsto F^\times/F_c^\times$. Then the commutative diagram (43) shows that $p(f) = h$.

Now let $g\colon G \to F$ be any stable morphism with $p(g) = h$. By the pullback property of (43), there is a unique group homomorphism $s\colon G^\times \to H'$ with $g^\times = h's$ which extends (43) to a commutative diagram

$$\begin{array}{ccccc}
G_c^\times & \hookrightarrow & G^\times & \twoheadrightarrow & G^\times/G_c^\times \\
\downarrow{\scriptstyle g_c^\times} & & \downarrow{\scriptstyle s} & & \| \\
F_c^\times & \hookrightarrow & H' & \overset{q'}{\twoheadrightarrow} & H \\
\| & & \downarrow{\scriptstyle h'} & \text{PB} & \downarrow{\scriptstyle h} \\
F_c^\times & \overset{i^\times}{\hookrightarrow} & F^\times & \overset{q}{\twoheadrightarrow} & F^\times/F_c^\times .
\end{array}$$

Since $s_c = (g_c)^\times$, Corollary 1 of Proposition 54 gives a unique morphism $j\colon G \to F'$ of q-fields with $j_c = g_c$ and $j^\times = s$. Hence j is stable. Since $(fj)_c = j_c = g_c$ and $(fj)^\times = h'j^\times = h's = g^\times$, we have $fj = g$. Conversely, let $j\colon G \to F'$ be a morphism with $fj = g$ and $p(j) = 1$. Then $f^\times j^\times = g^\times = f^\times s$ and $q'j^\times = q's$. Hence $j^\times = s$ and $j_c = g_c$. Thus j is unique. If g is disconnected, then s is invertible. So g is cartesian. □

Let us call a category $\mathscr{F}$ with a pair of functors $p\colon \mathscr{F} \to \mathscr{E}$ and $q\colon \mathscr{F} \to \mathscr{E}'$ *bi-fibred* if $\mathscr{F}$ is fibred over $\mathscr{E}$ and co-fibred over $\mathscr{E}'$ such that every morphism $f \in \mathscr{F}$ admits a factorization $f = pq$ into a cartesian morphism p over $\mathscr{E}$ and a co-cartesian morphism q over $\mathscr{E}'$. Note that up to isomorphism, such a factorization must be unique.

Corollary 1. *The category* $\mathbf{Flds}^q$ *is bi-fibred over* $\mathbf{Ab}$ *and* $\mathbf{Flds}_c^q$.

Proof. Let $f\colon F \to G$ be a stable morphism of modular q-fields. By Corollary 4 of Proposition 54, there exists a pushout of f_c along $F_c \hookrightarrow F$ which gives a commutative diagram

$$\begin{array}{ccccc}
F & \overset{i}{\hookrightarrow} & H & \overset{g}{\longrightarrow} & G \\
\uparrow & & \uparrow & & \uparrow \\
F_c & \overset{f_c}{\hookrightarrow} & H_c & = & G_c
\end{array}$$

with $gi = f$. Moreover, i is a connected embedding, and g is disconnected. □

In particular, we obtain

Corollary 2. *Every stable morphism of modular q-fields factors into a connected embedding followed by a disconnected morphism.*

20 Afterword and brief outlook

Where are we now? We have shown that classical field extensions and abelian groups fit into the more general framework of modular q-fields, a special class of quantales, where a rich interplay between connected and trivial extensions takes place. Algebraic and transcendental field extensions, the arithmetic of function fields, valuation theory, the Riemann-Roch theorem, and the ring of adèles, appear in a new guise, in the extended view of quantales where the additive structure is converted to pure geometry. In fact, every modular q-field is a projective space with a group structure on its points, so that multiplication by a unit is a collineation of the space.

The initial q-field $\mathbb{F}_1$ is omnipresent in the new setup: Like an empty box, it contains nothing, but looking at it from outside, it is contained in everything! The relevance of this point of view became explicit in the study of $\mathbb{F}_1(\gamma)$ with a transcendental unit γ. Unique prime factorization in $\mathbb{F}_1[\gamma]$ and its many consequences like in the classical case are present in this purified structure where coefficients have disapperaed. Explicitly, we found that the points on the affine line $[\varepsilon \vee \gamma] \smallsetminus \{\varepsilon\}$ stand for the units in $\mathbb{F}_1$ though there are none except ε.

Pursuing this line of thought, it is natural to proceed with the really big challenge, to reveal the hidden function field structure of $\mathbb{Q}$, the field of rationals. In classical terms, $\mathbb{Q}$ is a prime field, an empty box with no subfields. There is a smallest subring $\mathbb{Z}$, a principal ideal domain like the polynomial ring $\mathbb{C}[x]$, the reason why $\mathbb{Q}$ behaves very similar to the function field $\mathbb{C}(x)$ of the Riemann sphere. The main difference is that $\mathbb{Q}$ has a *continuous* prime, distinguished from all others by the range of its valuation, a dense subgroup of $\mathbb{R}$. The other primes are *discrete*, like all primes of the function field $\mathbb{C}(x)$. In a finite extension of $\mathbb{Q}$, the continuous primes split into finitely many continuous primes, while the discrete primes ramify into discrete ones.

Despite of this big chasm between $\mathbb{Q}$ and $\mathbb{C}(x)$, there is reason to believe in an overarching connexion. In many cases, discrete and continuous structures are just two sides of one medal. The prototypical situation appears in harmonic analysis where a continuous function is transformed into a sequence of amplitudes. Similarly, analytic functions are represented by coefficients of a power series. The overall pattern is given by Pontrjagin duality in the category **LCA** of locally compact abelian groups: Every such group has a dual and is isomorphic to the dual of its dual. So the discrete abelian groups are dual to the compact groups, and vice versa.

There are exactly two indecomposable projective objects in **LCA**: the infinite cyclic group $\mathbb{Z}$, and the additive group of $\mathbb{R}$ with its natural topology. These are exactly the groups which figure as codomain for the two types of valuations on $\mathbb{Q}$. Discrete valuations go into $\mathbb{Z}$, while continuous ones have their range in $\mathbb{R}$. Both types of prime orders appeared in the theory of q-fields. To uncover the geometry of $\mathbb{Q}$, more has to be done. The approximation theorem, as stated in the lecture, does not hold in the presence of continuous primes. It does hold if the topology of $\mathbb{R}$ is taken into account. Another piece of evidence that $\mathbb{Q}$ can be represented as a kind of "function q-field" over $\mathbb{F}_1$, with the speciality that one of the prime orders is a continuous one.

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