The mystery of e and π

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Dedicated to B. V. M.

Abstract. Relationships between the Eulerian number e and the Ludolphine number π are recalled, explaining why e is the *natural* basis of logarithm, while π relates to arc-length and area in the plane, distinguished by the imaginary unit i. Old and new conjectures around Euler's identity are discussed, indicating that the mysteries about e and π are far from resolved.

> Gentlemen, that is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means. But we have proved it, and therefore we know it must be the truth.
>
> BENJAMIN PEIRCE

As reported by one of his students, the 19th century mathematician and astronomer Benjamin Peirce (1809-1880) once got amazed by an Eulerian formula

$$e^{\pi/2} = \sqrt[i]{i}$$

and, after a few minutes' contemplation, praised it in the highest terms [20]. With a slight demystification, the formula turns into the well-known $e^{\pi i/2} = i$, so that one might say [20]: "We certainly *can* understand what Peirce always called the 'mysterious formula', and we certainly *do* know what it means." Nahin [20] adds: "But, yes, it *is* still a wonderful, indeed beautiful, expression; no amount of "understanding" can ever diminish its power to awe us ..."

In this note, we start with a brief review of basic facts on Euler's identity, hopefully to revive at least part of the early fascination about this formula. Much has been written about that, but it seems that in the standardised introductory courses of our days, the "well-known" facts stand to loose their attraction if not taught as a coherent unity. Having recalled such basic matters, our second concern is to emphasize that with the elementary facts about e and π, the case is by no means closed. We focus upon the scandal that after a century of hard work, fundamental questions on both transcendentals are widely open, and indicate new developments concerning the heart of our number system which must not be ignored.

Coming back to Nahin's awe-inspiring amazement, are we really sure that "no understanding" can ever distract us from mathematical beauty? We certainly cannot be distracted from something we never had in mind. So the question cannot be answered unless our relation to mathematics is accompanied by a strong estimation for its beauty.

On the other hand, a working mathematician usually struggles with truth or falsity rather than beauty. When a theory is complete, beauty will come as a reward and satisfaction for what has been done. To be sure, this kind of self-involved, accidential beauty, a product of human activity, was certainly *not* what Benjamin Peirce had in mind when he was struck with admiration by that remarkable connection between e and π. Precisely, the purpose of this note is threefold. First, we revisit the question

1. What is natural about the *natural* basis e of the exponential function?

An obvious answer would perhaps refer to the derivative of the exponential function at zero - or to the special form of its series expansion. Both are good explanations, touching the essential point in an *analytic* fashion. Moreover, both explanations belong to *real analysis*, somewhat remote from the geometric nature of π. So we ask for a more intuitive, *geometric* reason why the Eulerian constant e is the *natural* basis of exponentiation. It will turn out that the reason is essentially connected with π. To highlight the geometry of this connection between e and π, our second goal is to

2. Explain the essence of e and π in connection with length and area.

The exercise to prove "analytic" facts geometrically pursues another purpose, namely, to regain the awareness that "analysis" - in its original meaning - refers to a *method* rather than to a genuine part of mathematics itself.

With the increasing demand of rigour, triggered by the foundational crisis, twentieth century mathematics developed a tendency to look upon geometric methods with suspicion. An early advocate of this viewpoint was Edmund Landau, whose definition of $\pi/2$ (the smallest root of the cosine function) has become commonplace (see, e. g., the treatise of Dieudonné [5]). Landau himself [11] speaks of the "universal constant" π, which suggests that his unintuitive definition may well be taken as an amusing gag of a talented teacher. His rigorous telegram style (definition - theorem - proof) was new at that time. Combined with his ability to reduce proofs to a minimum of assumptions, he had a strong impact to the following generations, including the Bourbakians until the seventies. However, mathematics cannot be fully captured in a formal system, just as the physical world could not be encapsulated in a "theory of everything" [7, 14, 15]. This partly explains the persistence of mysteries, problems in mathematics with little hope for a solution in the near future. As an example, we discuss a long-standing conjecture on exponentials in connection with the relationship between e and π:

3. Schanuel's conjecture and Euler's identity

Our exposition is completely elementary, making wide use of geometric arguments in a stringent but not too formal way, assuming only the most evident facts about infinite series.

To set the stage, we briefly recall Euler's limit formula for the exponential function

$$\exp\colon \mathbb{C} \longrightarrow \mathbb{C}, \tag{1}$$

the heart of the whole matter, defined by the infinite series

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{2}$$

In what follows, we usually write z or w for complex, and x or y for real variables. The ratio $|\frac{z^{n+1}}{(n+1)!} : \frac{z^n}{n!}| = \frac{|z|}{n+1}$ tends to zero with increasing n. So the series (2), being majorized by a geometric series, converges for all $z \in \mathbb{C}$. A simple calculation, using the binomial theorem

$$(a+b)^n = \sum_{j=0}^{n} \tbinom{n}{j} a^j b^{n-j}, \tag{3}$$

shows that (1) is a homomorphism from the additive group of the field $\mathbb{C}$ of complex numbers onto its multiplicative group $\mathbb{C}^\times$:

$$\exp(z+w) = \exp(z) \cdot \exp(w). \tag{4}$$

In particular,

$$\exp(nz) = \exp(z)^n \tag{5}$$

for $z \in \mathbb{C}$ and $n \in \mathbb{Z}$. The following well-known representation of the exp-function is due to Euler [6].

Proposition 1. *For arbitrary $z \in \mathbb{C}$,*

$$\exp(z) = \lim_{n\to\infty} \left(1 + \tfrac{z}{n}\right)^n.$$

Proof. By the binomial theorem (3),

$$\left(1+\tfrac{z}{n}\right)^n = \sum_{j=0}^{n} \tbinom{n}{j} \left(\tfrac{z}{n}\right)^j = \sum_{j=0}^{n} \tfrac{n(n-1)\cdots(n-j+1)}{n^j} \cdot \tfrac{z^j}{j!} = \sum_{j=0}^{\infty} \left(1-\tfrac{1}{n}\right)\cdots\left(1-\tfrac{j-1}{n}\right)\tfrac{z^j}{j!}.$$

For fixed z and $n \to \infty$, the sum converges uniformly to $\exp(z)$, whence the result. $\square$

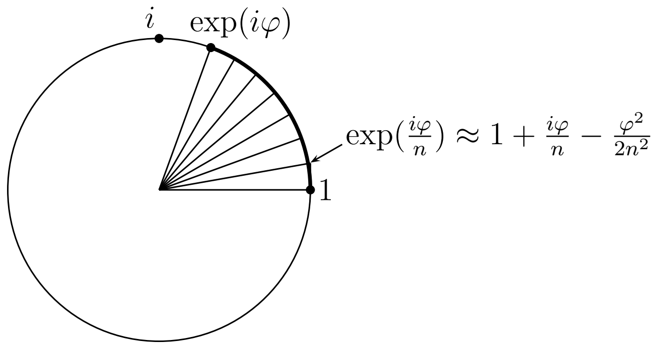
Two aspects of plane geometry arise when the exponential function (1) is restricted either to real or to purely imaginary $z \in \mathbb{C}$. Accordingly, there are two classes of real functions related to area and arc length of circles in the complex plane. We treat the two cases separately.

1 Arc length: The imaginary case

Consider the function $\exp(i\varphi)$, with $\varphi \in \mathbb{R}$. We set

$$\exp(i\varphi) = x + iy \tag{6}$$

with $x, y \in \mathbb{R}$. From the series expansion (2), it is clear that the complex conjugate $x - iy$ of $\exp(i\varphi)$ is $\exp(-i\varphi)$. Hence $|\exp(i\varphi)|^2 = x^2 + y^2 = (x+iy)(x-iy) = \exp(i\varphi)\exp(-i\varphi) = \exp(0) = 1$ by virtue of (4). Thus $\exp(i\varphi)$ is located on the unit circle in the complex plane $\mathbb{C}$. Where on the unit circle?



By Eq. (5), we have $\exp(i\varphi) = \exp(\frac{i\varphi}{n})^n$, and the series expansion (2) yields $\exp(\frac{i\varphi}{n}) \approx 1 + \frac{i\varphi}{n} - \frac{\varphi^2}{2n^2}$, with a second order term in direction of the negative real axis. The first order term $\frac{i\varphi}{n}$ shows that the arc length from 1 to $\exp(\frac{i\varphi}{n})$ is $\frac{\varphi}{n}$ for large n. So the total arc length from 1 to $\exp(i\varphi)$ is exactly φ.

Thus φ can be interpreted as the *argument*, that is, the polar angle of $\exp(i\varphi)$. As an immediate consequence, this yields Euler's formula

$$\exp(i\varphi) = \cos\varphi + i\sin\varphi, \tag{7}$$

with the usual geometric meaning of the sine and cosine function. Accordingly, every complex number $z \in \mathbb{C}^\times$ can be written uniquely as $z = |z| \cdot \exp(i \arg z)$, where $\arg z$ denotes the argument of z.

Now it is time to write $\exp(z)$ in its usual form as a power e^z. With $e := \exp(1)$, Eq. (5) gives $\exp(n) = e^n$ for $n \in \mathbb{N}$, and with $z = \frac{1}{n}$, the same formula gives $e = \exp(\frac{1}{n})^n$, that is, $\exp(\frac{1}{n}) = e^{\frac{1}{n}}$. Furthermore, Eq. (4) with $z = 1$ and $w = -1$ gives $\exp(-1) = e^{-1}$. Thus, $\exp(x) = e^x$ holds for all rational numbers. Thus, it is reasonable to *define* the power e^x for real x to be $\exp(x)$. For imaginary numbers, a similar procedure is suggested by the above discussion of angles. So we can safely write $\exp(z) = e^z$ for all $z \in \mathbb{C}$. With this notation, Euler's remarkable formula

$$e^{i\pi} = -1 \tag{8}$$

follows if, as usual, π is defined to be the length of the half unit circle.

What is more important, the geometric interpretation of $\exp(i\varphi) = e^{i\varphi}$ shows that $e = 2.718281828459045...$ is the *natural* basis of exponentiation: If

$$\ln\colon \mathbb{R}_{>0} \xrightarrow{\sim} \mathbb{R} \tag{9}$$

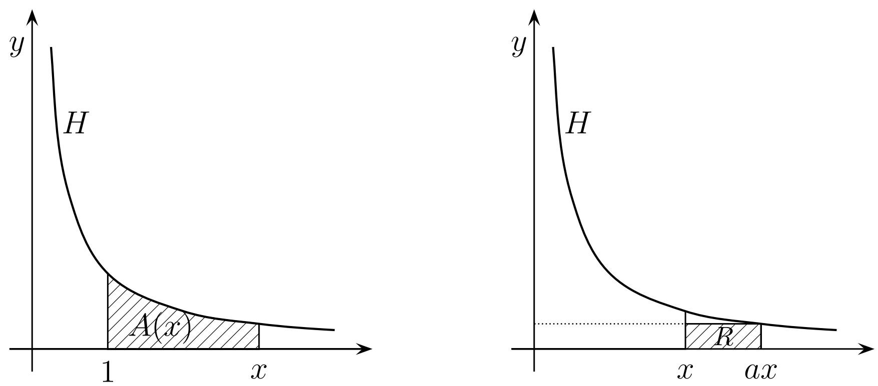
denotes the inverse function to $\exp|_{\mathbb{R}}$, then $a^{i\varphi} = e^{i\varphi \ln a}$ runs along the unit circle. However, e is the unique exponential basis a for which the length from 1 to $a^{i\varphi}$ on the unit circle is exactly φ.

2 Inversion

The natural logarithm (9), inverse to the exp-function, is closely related to the inversion $x \mapsto x^{-1}$, an order-reversing automorphism

$$\mathbb{R}_{>0} \xrightarrow{\sim} \mathbb{R}_{>0}$$

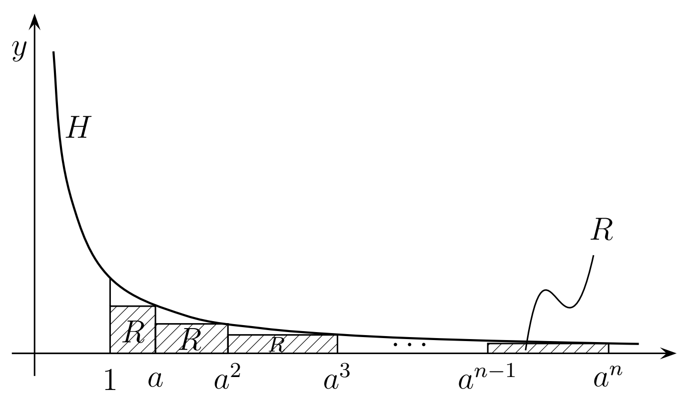
of the multiplicative group of $\mathbb{R}_{>0}$. We would like to determine the



area $A(x)$ between the hyperbola H with equation $xy = 1$, and the x-axis restricted to the interval $[1, x]$. For any $a > 1$, consider the rectangle R between x and ax of height $\frac{1}{ax}$. The area of R does not depend on x:

$$R = (ax - x)\tfrac{1}{ax} = 1 - \tfrac{1}{a} = \tfrac{a-1}{a}.$$

Therefore, the sequence $1, a, a^2, \ldots, a^n$ gives rise to rectangles of equal area:



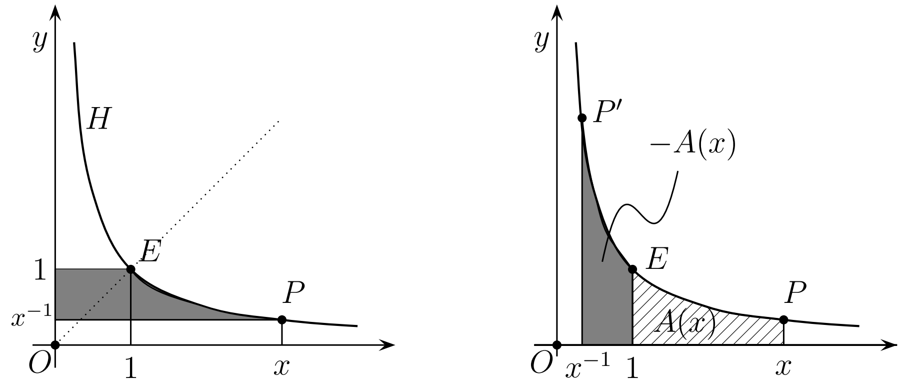
For $n \in \mathbb{N}$ and $a := 1 + \frac{x}{n}$, this yields $R = \frac{x}{n(1+\frac{x}{n})} = \frac{x}{n+x}$ and $a^n = (1 + \frac{x}{n})^n$. Hence Proposition 1 gives

$$A(e^x) = \lim_{n\to\infty} nR = x.$$

This shows that $A(x)$ represents the logarithm $\ln x$. If $0 < x < 1$, the area $A(x)$ has to be taken negative, in accordance with $\ln x < 0$. Precisely, we should have

$$A(x^{-1}) = -A(x). \tag{10}$$

Again, this can be visualized geometrically:



In the first picture, the unit square (between O and E), and the rectangle between O and P are both of area 1. By adding the unit square to $A(x)$ and subtracting the rectangle, we get the shaded area between the horizontal lines of height x^{-1} and 1. Reflecting this area at the dotted line $\overline{OE}$, we obtain the shaded area in the second diagram, which establishes (10).

We started with two manifestations of e in Proposition 1. An analogue for the logarithm can be found in Euler's "introduction" [6].

Proposition 2. *For all* $z \in \mathbb{C}$,

$$z = \lim_{n\to\infty} n(e^{z/n} - 1).$$

Proof. By definition, we have $e^{z/n} = 1 + \frac{z}{n} + \frac{z^2}{2n^2} + \cdots$, which yields $n(e^{z/n} - 1) = z + \frac{z^2}{2n} + \frac{z^3}{3!n^2} + \cdots$. Whence the result. □

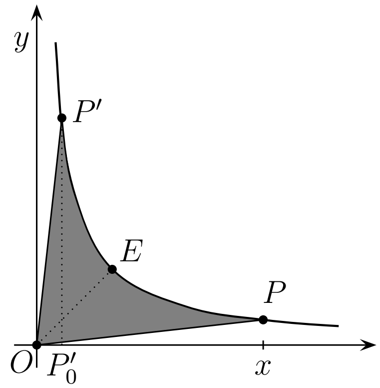
From the polar representation $z = |z| \cdot e^{\arg z}$, it follows that exp maps the strip $-\frac{\pi}{2} < z < \frac{\pi}{2}$ bijectively onto the open half plane $\Re(z) > 0$. Therefore, the inverse $z \mapsto \ln z$ of exp is uniquely defined in this half plane. Replacing z by $\ln z$, Proposition 2 thus gives

$$\ln z = \lim_{n\to\infty} n(z^{1/n} - 1)$$

for $\Re(z) > 0$, where $z^{1/n}$ denotes the n-th root with argument $\frac{1}{n}\arg z$.

3 Area: The real case

Note that the points P and P' are symmetric with respect to $\overline{OE}$. On the x-axis, this symmetry corresponds to the automorphism $x \mapsto x^{-1}$. Now let P_0 be the projection of an arbitrary point P on H to the x-axis. Take the area $A(x)$ and its mirror image $-A(x^{-1})$ together, add the triangle $OP_0'P'$ and subtract the congruent triangle OP_0P to obtain the shaded area in the following picture:



Now we return to the exponential function (1) and restrict it to real numbers. Let us denote the real part x and the imaginary part y of a complex number $z = x + iy$ by $\Re(z)$ and $\Im(z)$, respectively. Representing the hyperbola H in the complex plane $\mathbb{C}$ (like the unit circle in Section 2), its equation becomes

$$H\colon \Im(z^2) = 2.$$

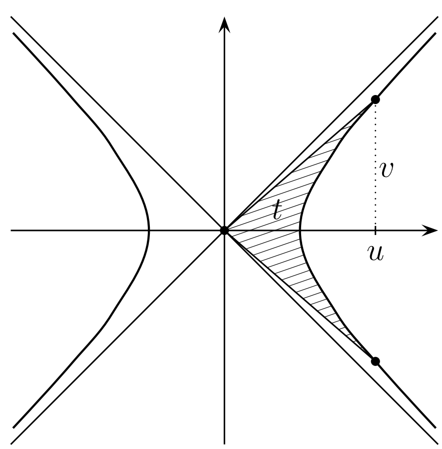
Indeed, $z^2 = (x^2 - y^2) + 2ixy$. So the real part gives another hyperbola H' with equation

$$H'\colon \Re(z^2) = 1,$$

that is, $x^2 - y^2 = 1$. The distance between H' and the origin O is 1, while it is $\sqrt{2}$ for H. Therefore, the hyperbola H' should be regarded as the proper analogue of the circle, rather than H. The map $z \mapsto (i+1)\overline{z}$ transforms H' into H. Indeed, $(i+1)^2\overline{z}^2 = 4xy + 2i(x^2 - y^2)$. Since $|i + 1| = \sqrt{2}$, the size of any area is doubled by this map. Let us denote the coordinates of H' by (u, v). Then $(i + 1)(u - iv) = x + iy$ with $x = u + v$ and $y = u - v$. Hence

$$u = \frac{x + y}{2} \qquad v = \frac{x - y}{2}.$$

The area $A(x)$, that is, half of the shaded area in the previous diagram of H, can thus be represented by t in the graph of the hyperbola H':



Now recall that $t = A(x) = \ln x$. So we obtain the *hyperbolic functions*

$$u = \cosh t = \frac{e^t + e^{-t}}{2}, \qquad v = \sinh t = \frac{e^t - e^{-t}}{2}$$

which give the even and odd part of the series (2):

$$\cosh t = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}, \qquad \sinh t = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}.$$

This completes our sketch of the real and imaginary part of the exponential function. Further details can be gathered from any standard textbook like [1].

4 Relations between e and π

The fundamental importance of π as a "constant of nature" has been an inexhaustible source of inspiration since the ancient days of Archimedes. For a unit circle, π is its area and half of its circumference. Euler's discovery of e after so many centuries where π stood alone as a unique phenomenon adds a new dimension to the perception of numbers. In fact, Euler's invention appears as a last step in the convincement that the habitat of numbers is two-dimensional. Our brief reflection on e and π should leave no doubt that e is more fundamental than π, opening a deep shaft into the rock layers of $\mathbb{C}$. Connecting the additive with the multiplicative group of $\mathbb{C}$, every exponential function maps imaginary numbers onto the unit circle. Only the natural basis e gives the proper arc length in $\mathbb{C}$, which leads to the constant π. If the unit circle is replaced by a hyperbola, the variable x of the function e^x turns out to be an area, which leads to the hyperbolic functions. Providing arc length and area of circles in the plane with an inseparable link to the number system, the constant e has opened a new era after π. Note that from the mere definition of $\mathbb{C}$, the dominance of e is not obvious at first sight. It needed a genius like Euler to bring it to light.

For a non-zero algebraic number $\alpha \in \mathbb{C}$, Lindemann [16] showed that e^α is transcendental. Thus Euler's formula (8) shows that π is transcendental. Furthermore $e = e^1$ is transcendental, a result that was proved earlier by Hermite [8]. More generally, Lindemann [16] and Weierstraß [24] proved that if $\alpha_1, \ldots, \alpha_n$ are $\mathbb{Q}$-linearly independent algebraic numbers, $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent (cf. [9]). More generally, Schanuel's conjecture [12] states that for $\mathbb{Q}$-linearly independent numbers $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, the transcendence degree of $\mathbb{Q}(\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n})$ is at least n. This very general statement implies many known results on transcendental numbers. For example, it implies that e and π are algebraically independent, while it is not even known that $e + \pi$ is irrational. Schanuel's conjecture also implies [17, 22] that in the field $\mathbb{C}$ of complex numbers together with the exponential operation (1), there are no relations between $i = \sqrt{-1}$ and π other than Euler's identity (8).

Although there are not many fields with a natural exponential operation, exponentiation in the field $\mathbb{C}$ of complex numbers is fundamental and should be included as a third operation besides addition and multiplication, justifying the concept of *exponential field* [23], a field K with a function $E\colon K \to K$ such that $E(0) = 1$ and $E(a + b) = E(a)E(b)$. As a consequence, E is a group homomorphism from the additive group of K to $K^\times$. The fact that Schanuel's conjecture is widely open reveals an embarrassing limitation of knowledge concerning most basic properties of our number system. An analogue of Schanuel's conjecture for rings of formal power series is well known [2]: If $f_1, \ldots, f_n \in \mathbb{C}[[x_1, \ldots, x_n]]$ are $\mathbb{Q}$-linearly independent modulo constants, the transcendence degree of $\mathbb{C}(f_1, \ldots, f_n, e^{f_1}, \ldots, e^{f_n})$ is at least n plus the rank of the Jacobian $J(f_1, \ldots, f_n)$.

In 2004, Boris Zilber introduced a class of fields with exponentiation satisfying Schanuel's conjecture. Among Zilber's fields [25, 26] there is, up to isomorphism, a unique field $\mathbb{B}$ of cardinality $2^{\aleph_0}$. Zilber conjectured [26] that as an exponential field, $\mathbb{B}$ is isomorphic to $\mathbb{C}$. A slight evidence for this striking claim, Kirby [10] proved that Zilber's conjecture is true if $\mathbb{B}$ and $(\mathbb{C}; \exp)$ are elementary equivalent as logical systems. Recently, Bays and Kirby [3] improved Zilber's approach and showed that his conjecture is equivalent to the conjunction of Schanuel's conjecture and another unproved condition called "strong exponential-algebraic closedness". Furthermore, Bays and Kirby [3] extend Schanuel's and Zilber's conjecture to a wider context including exponential maps of elliptic curves, which leads to a close relationship to the André-Grothendieck conjecture [4] on periods of 1-motives.

Much more could be said about the influence of Euler's relation and its variants to further developments until recently. For example, Diophantine geometry plays a big role in the study of transcendental numbers [13]. Together with model theory, it applies to the investigation of Zilber's exponential fields [26, 10, 19]. The fundamental role of exponentiation also occurs in the topos-theoretic approach to set theory [21, 18].

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