Flat covers: The direct method

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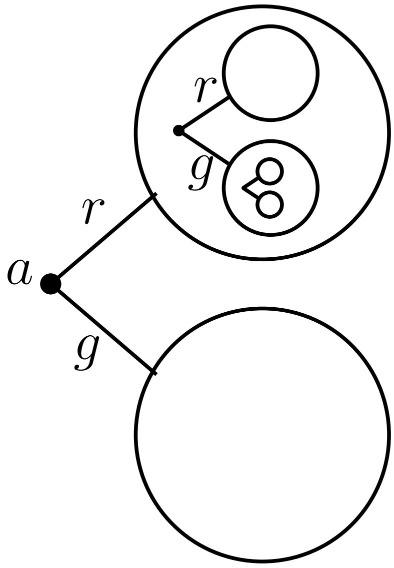
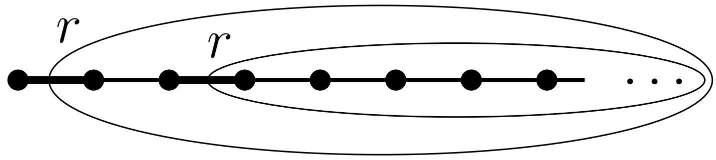
1. An additive Ramsey theorem

Let K be an infinite complete graph with red and green edges. Ramsey's theorem states that there exists an infinite monochromatic full subgraph.

Proof. Choose a vertex a. Then $K \setminus \{a\}$ splits into two parts. Continue with each part ...



So K becomes an infinite binary tree. Hence K has an infinite branch:



Without loss of generality, there are infinitely many red edges in the branch. □

Definition. A *tree* is a poset T such that

$$\widetilde{a} := \{b \in T \mid b \leqslant a\}$$

is a well-ordered chain for every $a \in T$.

- Special case: A tree with one branch is an *ordinal*. Normalized: $\lambda = \{\alpha \in \mathbf{Ord} \mid \alpha < \lambda\}$.

- Topology on λ: Set of open sets $\mathcal{O}(\lambda) := \lambda + 1 = \{\alpha \in \mathbf{Ord} \mid \alpha \leqslant \lambda\}$.

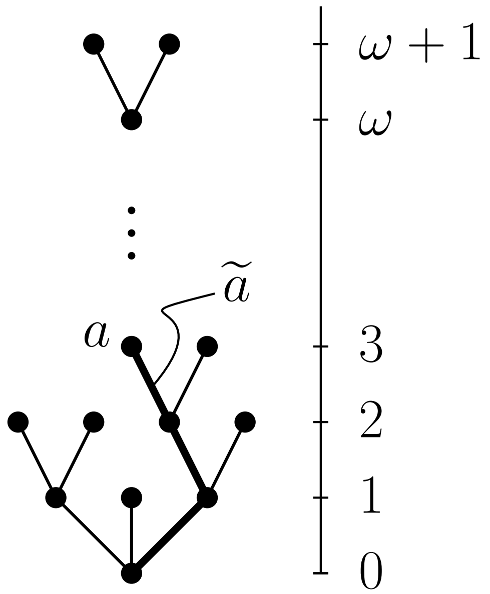
New definition:

A *tree* is a sheaf of sets on some $\lambda \in \mathbf{Ord}$.

Stalks: $T_\alpha = T(\alpha + 1)$, $\alpha < \lambda$ (levels)

Etale space: $T = \bigsqcup_{\alpha<\lambda} T_\alpha$

Basis of $\mathcal{O}(T)$: $\{\widetilde{a} \,|\, a \in T\}$



Height function: $T \xrightarrow{h} \lambda$

In what follows, we identify a tree T with its étale space. So the height function h can be regarded as the natural projection to the base space λ.

Every tree T is contained in a bigger tree:

$$T \hookrightarrow \widetilde{T} := \{\text{open chains in } T, \subset\}.$$
$$a \mapsto \widetilde{a}$$

Identification with its étale space: $\widetilde{T} = \bigsqcup_{\alpha \leqslant \lambda} T(\alpha)$.

$B(T) := \{\text{maximal elements of } \widetilde{T}\} =$ *branches* of T

Every functor $F\colon T^{\mathrm{op}} \to \mathbf{Ab}$ extends to the open sets U of T via

$$F(U) := \mathrm{Lim}_{a\in U} F(a).$$

Thus F becomes a sheaf: For $U, V \in \mathcal{O}(T)$, we have a pullback

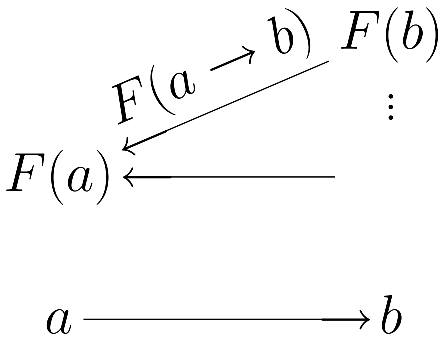
$$\begin{array}{ccccc} & & F(U) & & \\ & \nearrow & & \searrow & \\ F(U\cup V) & & & & F(U\cap V) \\ & \searrow & & \nearrow & \\ & & F(V) & & \end{array}$$

The étale space $F_{\text{ét}}$ of the sheaf F gives rise to a local homeomorphism

$$F_{\text{ét}} \longrightarrow T \xrightarrow{\;h\;} \lambda.$$

Thus F is again a tree!

We define the *purity* of $\mathfrak{p}F$ of F as follows:



Choose subgroups $U_a \subset F(a)$ with $U_a \subset \bigcup_{a<b} \mathrm{Im}\, F(a\to b)$ for all $a \in T$.

The purity of F is given by

$$\boxed{\mathfrak{p}F := \inf\{\aleph \mid \forall a \in \widetilde{T} \smallsetminus B(T)\colon |F(a)/U_a| \leqslant \aleph\}}$$

Additive Ramsey Theorem: Let F be an abelian sheaf on a tree T. For every root $e \in T_0$:

$$\boxed{|F(e)| \leqslant \sum_{e \in b \in B(T)} (\mathfrak{p}F)^{|b|} \cdot |F(b)|}$$

2. What is a module?

An R-module is an additive functor $R \to \mathbf{Ab}$. But R has only one object. Enlarge R:

$$R\text{-}\mathbf{mod} := \{M \in R\text{-}\mathbf{Mod} \mid \exists R^m \to R^n \twoheadrightarrow M\}.$$

This category need not be abelian. We introduce the category of *small* modules:

$$R\text{-}\mathbf{mos} := \{M \in R\text{-}\mathbf{Mod} \mid |M| \leqslant w(R)\},$$

where $w(R) := \max\{|R|, \aleph_0\}$ is the *weight* of R. This category is abelian, skeletally small and $w(R)$-cocomplete: Every colimit in R-**mos** with index set of cardinality $\leqslant w(R)$ exists in R-**mos**.

Proposition 1. *R-**Mod** is equivalent to the category of $w(R)$-continuous additive functors $(R\text{-}\mathbf{mos})^{\mathrm{op}} \to \mathbf{Ab}$.*

Proof. Consider the embedding $R \hookrightarrow R$-**mos**. Every $M \in R$-**Mod** has a natural $w(R)$-continuous extension: $M = \mathrm{Hom}_R(-, M)$ to R-**mos**. □

Note: Proposition 1 depends on the cardinal $w(R)$.

To get rid of this dependence, endow R-**mos** with the *canonical* topology (=the finest Grothendieck topology such that $\mathrm{Hom}_R(-, A)$ is a sheaf for all $A \in R$-**mos**). Then we can prove:

$$\boxed{R\text{-}\mathbf{Mod} \approx \mathrm{Sh}(R\text{-}\mathbf{mos})}$$

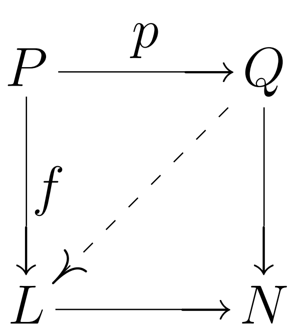
Thus R-modules are abelian sheaves on R-**mos**.

This formulation is suitable for a study of purity. Recall: A short exact sequence

$$0 \to L \xrightarrow{a} M \xrightarrow{b} N \to 0$$

is *pure exact* if it stays exact under $X \otimes_R -$ for any right R-module X. Equivalently: It stays exact under $\mathrm{Hom}_R(E, -)$ for $E \in R$-**mod** (Stenström). Accordingly, a is called a *pure monomorphism* and b is called a *pure epimorphism*.

Consequently: $L \to M$ is a pure monomorphism if and only if for every commutative diagram



with $P, Q \in R$-**proj** (finitely generated projective), the morphism f factors through p.

Recall: For $M \in R$-**Mod** and $A \in R$-**mos**

$$M(A) = \mathrm{Hom}_R(A, M).$$

Define:

$SM(A) := \{f \in M(A) | f \text{ not pure monomorphism}\}$

R-**nos** $:= \{f \in R\text{-}\mathbf{mos} | f \text{ not pure monomorphism}\}$

Note: $f, g \in R$-**nos** $\Rightarrow$ $gf \in R$-**nos**,
but $1_A \notin R$-**nos**.

In general, SM is *not* a subfunctor of M. Every R-linear map $f\colon L \to M$ induces a homomorphism $f_A\colon L(A) \to M(A)$ for $A \in R$-**mos**.

Proposition 2.
(a) *$f\colon L \to M$ is a pure monomorphism $\Longleftrightarrow$ $f_A^{-1}(SM(A)) \subset SL(A)$ for all $A \in R$-**nos**.*
(b) *$SM(A) = \bigcup\{\mathrm{Im}\, M(g) \,|\, g\colon A \to B$ in R-**nos**$\}$.*

Proof. a)
$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ \uparrow{\scriptstyle a} & & \\ A & & \end{array}$$
f pure monomorphism $\overset{!}{\Leftrightarrow}$
(a pure m. $\Rightarrow fa$ pure m.)
"$\Rightarrow$" is trivial.

b)
$$\begin{array}{ccc} M & & \\ \uparrow{\scriptstyle f} & \nwarrow & \\ A & \xrightarrow{g} & B \end{array}$$
$f \in \mathrm{Im}\, M(g)$, $g \in R$-**nos**
$\Rightarrow f \in SM(A)$.
This proves "$\supset$".

Proposition 2 reduces purity in R-**Mod** to R-**mos**.

Now we come back to trees. First we prove

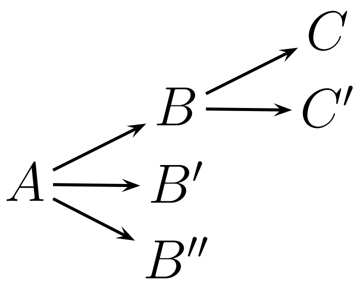
Proposition 3. *For all $A \in R$-**mos**, there exists a set $I(A)$ of morphisms $f\colon A \to B$ in R-**nos** with $|I(A)| \leqslant w(R)$ such that every $f'\colon A \to B'$ in R-**nos** factors through some $f \in I(A)$.*

Proof. Let $A \to B$ in R-**nos** be given. Consider the pushouts

$$\begin{array}{ccc} R^m & \xrightarrow{e} & R^n \\ \big\downarrow{\scriptstyle g} & & \big\downarrow \\ A & \longrightarrow & B \end{array}$$

The pairs (g, e) form a set of cardinality $\leqslant w(R)$. Collect the pushouts of e along g to get $I(A)$. $\square$

Put the $I(A)$ together to get a tree T in R-**mos**:



To each object A of R-**mos**, attach the arrows of $I(A)$. For limit ordinals, we take the direct limit. Note that $I(0) = \varnothing$.

Proposition 4. *For every (well-ordered) chain $A_0 \to A_1 \to A_2 \to \cdots$ in T, there exists an ordinal λ with $|\lambda| \leqslant w(R)$ and $\varinjlim_{\alpha<\lambda} A_\alpha = 0$.*

Proof. Find λ, $|\lambda| \leqslant w(R)$, such that $A_\lambda \to A_{\lambda+1}$ must be a pure monomorphism. Thus $A_\lambda = 0$. $\square$

In particular, all branches b of T satisfy $|b| \leqslant w(R)$. The tree T encodes the information on purity in R-**mos**, hence also in R-**Mod**.

Restrict $M \in R\text{-}\mathbf{Mod} \approx \mathrm{Sh}(R\text{-}\mathbf{mos})$ to $T \subset R$-**mos**. This makes M into an abelian sheaf on T. Hence: Every R-module is a tree!

In particular, we can associate a *purity* $\mathfrak{p}M$ to every R-module M. It is a cardinal invariant of M. (It does not depend on the choice of $T \subset R$-**mos**).

Now apply the additive Ramsey theorem! This gives:

Theorem 1. *The cardinality of an R-module M can be estimated by means of its purity:*

$$\boxed{|M| \leqslant (\mathfrak{p}M)^{w(R)}}$$

More generally, this theorem holds for abelian sheaves on a suitable site.

Remarks. 1. There seems to be no logical implication between the additive Ramsey theorem and non-additive (classical) Ramsey theory.

2. Theorem 1 can be regarded as a condensed form of El Bashir's (and Bican's) main argument in their proof of the flat cover conjecture.

3. Flat covers

Let $\mathcal{A}$ be an additive category with directed limits. $E \in \operatorname{Ob}\mathcal{A}$ is called *finitely presented* if $\operatorname{Hom}_{\mathcal{A}}(E, -)$ respects direct limits:

$$\varinjlim \operatorname{Hom}_{\mathcal{A}}(E, A_i) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}}(E, \varinjlim A_i).$$

$\mathcal{A}$ is *locally finitely presented* if every $A \in \operatorname{Ob}\mathcal{A}$ is a direct limit of finitely presented objects in $\mathcal{A}$.

Examples. Let R-**Flat** $\subset$ R-**Mod** denote the category of flat R-modules.

1. R-**Mod** is locally f. p.; $\operatorname{fp}(R\text{-}\mathbf{Mod}) = R\text{-}\mathbf{mod}$.

2. R-**Flat** is locally f. p.; $\operatorname{fp}(R\text{-}\mathbf{Flat}) = R\text{-}\mathbf{proj}$.

3. Let X be a ringed space such that the quasi-compact open sets form a basis of X. The category $\operatorname{Sh}(X)$ of abelian sheaves on X is locally f. p.

4. Let X be a noetherian space. The (non-abelian) category of separated presheaves on X is loc. f. p.

Every locally finitely presented category $\mathcal{A}$ with $\mathcal{C} := \operatorname{fp}(\mathcal{A})$ admits a natural full embedding into a generalized module category

$$\mathcal{A} \hookrightarrow \mathbf{Mod}(\mathcal{C})$$

such that $\mathcal{A}$ is equivalent to the full subcategory of flat $\mathcal{C}$-modules. Via this embedding, the concepts of cardinality $|A|$ and purity $\mathfrak{p}A$ apply to objects of $\mathcal{A}$.

It is natural to apply Stenström's definition of pure exact sequences to a locally finitely presented category $\mathcal{A}$. This makes $\mathcal{A}$ into an *exact* category.

More generally, we define a *left exact* structure on $\mathcal{A}$ to be a subcategory $\mathcal{D}$ consisting of cokernels (called *deflations*) such that the following hold:

(D) $\quad gf \in \mathcal{D} \;\Rightarrow\; g \in \mathcal{D}$

(P) $\quad \mathcal{D}$ is stable under pullback.

We indicate deflations by twoheadarrows $\twoheadrightarrow$.

Property (P) implies that every deflation b has a kernel a:

$$\begin{array}{ccc} A & \dashrightarrow & 0 \\ {\scriptstyle a}\,\downarrow & \text{PB} & \downarrow \\ B & \overset{b}{\twoheadrightarrow} & C \end{array}$$

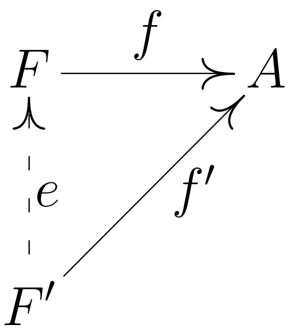
Therefore, a left exact structure is given by a class of short exact sequences.

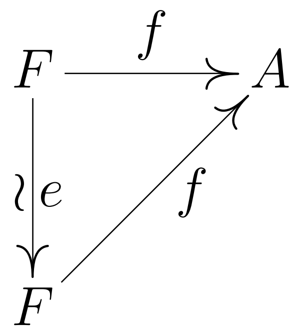
Definition. Let $\mathcal{A}$ be a locally finitely presented left exact category. An object F is *flat* if every deflation $A \twoheadrightarrow F$ is a pure epimorphism.

A morphism $f\colon F \to A$ is a *flat cover* if

(1) F is flat and every $F' \to A$ with F' flat factors through f.

(2) $fe = f \;\Rightarrow\; e$ is invertible.

(1) 

(2) 

Flat covers are unique up to isomorphism.

Theorem 2. *Let $\mathcal{A}$ be a locally finitely presented left exact category. Assume that every pure epimorphism is a deflation. Then every $A \in \operatorname{Ob}\mathcal{A}$ has a flat cover $F \to A$ with*

$$|F| \leqslant |A|^{w(\mathrm{fp}(\mathcal{A}))}.$$

Proof. We start with any $g\colon F \to A$ where F is flat. We define $f' \leqslant f$ by the above diagram (1). Using Enochs' old argument (Isr. J. Math. 1981; to show that precovers (1) are covers), we replace g by a larger f' such that every e in a diagram (1) must be monic. Then we use the estimation of Theorem 1 to show that f' must be a flat cover. □

We list some examples of additive categories $\mathcal{A}$ which satisfy the assumptions of Theorem 2.

Example 1. Let $\mathcal{B}$ be a locally finitely presented Grothendieck category, and let $\mathcal{A}$ be a reflective full subcategory which is closed under direct limits.

Example 2. Let $\mathcal{B}$ be a locally finitely presented Grothendieck category with a torsion theory $(\mathcal{T}, \mathcal{F})$ generated by finitely presented objects. We can take $\mathcal{A} := \mathcal{F}$ or $\mathcal{A} := \mathcal{B}/\mathcal{T}$.

In particular, flat covers exist in any locally finitely presented Grothendieck category. (This answers a question of Cuadra and Simson).

Example 3. Let $\mathcal{A}$ be a locally finitely presented category which is *quasi-abelian*, that is, kernels and cokernels exist and the short exact sequences make $\mathcal{A}$ into an exact category. Such categories arise in algebra, functional analysis, theory of topological groups, and algebraic geometry. Special cases are: categories of topological vector spaces like Fréchet spaces, locally convex, nuclear or bornological spaces; torsion or torsion-free classes in abelian categories, categories of representations of classical orders over a Dedekind domain, reflexive modules over an integrally closed noetherian domain, monopresheaves of abelian groups; categories of abelian varieties ...). Some of them are locally finitely presented.

The examples of locally convex spaces are locally finitely presented, but they have the property that all objects are flat (since n-dimensional separated topological vector spaces have the same topology).

Example 4. $\mathcal{A} := \mathrm{Sh}(X)$, where X is a ringed space with a basis of quasi-compact open sets.

Example 5. Categories $\mathcal{A}$ of continuous separated presheaves on such a space X. Special cases are: $\mathcal{A} = \{$separated presheaves on a noetherian space$\}$ or $\mathcal{A} := \{$filtered objects of $\mathcal{B}\}$, where $\mathcal{B}$ is a locally finitely presented Grothendieck category.
(These categories are non-abelian in general.)

Flat covers have also been shown to exist in some categories which are not locally finitely presented, e. g., sheaves on an arbitrary ringed space, or quasi-coherent sheaves on a scheme. For such categories, however, Stenström's definition of purity does not apply. For example, the line bundles $\mathcal{O}(n)$ on the projective line are finitely presented and flat, but not projective. This happens since flatness of a sheaf is usually defined by the flatness of its stalks. In this case, we first apply Theorem 2 to the affine case and then patch the flat covers of the affine pieces together to get a global flat cover.

4. Small flats

Enochs' proof of the flat cover conjecture uses the fact that pure submodules of flat modules are flat. Therefore, every flat module has a continuous filtration of flat pure submodules of bounded cardinality.

In the situation of Theorem 2, such an argument is no longer feasible.

A locally finitely presented Grothendieck category $\mathcal{A}$ has *enough flat objects* if every $A \in \operatorname{Ob}\mathcal{A}$ admits an epimorphism $F \to A$ with F flat. Cuadra and Simson (2007) ask whether this implies that $\mathcal{A}$ has enough projectives. On the positive side, we have

Theorem 3. *Let $\mathcal{A}$ be a locally finitely presented quasi-abelian category with enough flat objects. (For such $\mathcal{A}$, flat covers exist by Theorem 2!) Then every flat $F \in \operatorname{Ob}\mathcal{A}$ is a direct limit of small pure subobjects which are flat.*

Proof. $F = \varinjlim A_i$ for small pure subojects A_i of F. We blow the A_i up until they become flat. □

If "small" could be replaced by "finitely presented", there would be enough projectives. However, the distance from finitely presented objects to the biggest "small" objects is very far.

Theorem 3 shows that Enochs' existence proof of flat covers also applies to very general categories - provided that there are enough flats (an assumption that is highly desirable in practice). In particular, there is no need to extend "locally finitely presented" to regular cardinals greater than $\aleph_0$.