Set-theoretic solutions to the YBE. $S : X \times X \to X \times X$ 

$$(S\times 1)(1\times S)(S\times 1) = (1\times S)(S\times 1)(1\times S) : X\times X\times X \to X\times X\times X.$$

$S(x,y) = ({}^x y, x^y)$ is non-degenerate if the maps $y \mapsto {}^x y$ and $x \mapsto x^y$ are bijective.

Prop C. Every cycle set X gives a solution with ${}^x y := x^y \cdot y$. It is non-degenerate iff X is non-degenerate.

Pf. $S(x,y) = S(x^y\cdot y, x^y) \rightsquigarrow S(y\cdot x, y) = (x\cdot y, x)$. $(\Rightarrow S^2 = 1_X)$ Note first that $(z,y,x) \longmapsto (x\cdot z, x\cdot y, x) \longmapsto ((x\cdot y)\cdot(x\cdot z), x\cdot y, x)$ is bijective. Now

$$\begin{tikzcd}
((x\cdot y)\cdot(x\cdot z), x\cdot y, x) \arrow[r, mapsto, "S\times 1"] \arrow[d, mapsto, "1\times S"'] & ((x\cdot z)\cdot(x\cdot y), x\cdot z, x) \arrow[r, mapsto, "1\times S"] & ((z\cdot x)\cdot(z\cdot y), z\cdot x, z) \arrow[d, mapsto, "S\times 1"] \\
((y\cdot x)\cdot(y\cdot z), y\cdot x, y) \arrow[r, mapsto, "S\times 1"] & ((y\cdot z)\cdot(y\cdot x), y\cdot z, y) \arrow[r, mapsto, "1\times S"] & ((z\cdot y)\cdot(z\cdot x), z\cdot y, z)
\end{tikzcd}$$

We have ${}^{x\cdot y}x = (x\cdot y)^x \cdot x = y\cdot x$. $\rightsquigarrow {}^{x\cdot x}x = x\cdot x$. So if $y \mapsto {}^x y$ is bijective, x can be retrieved from $x\cdot x$. Furthermore, ${}^x y \cdot {}^x y = (x^y\cdot y)\cdot(x^y\cdot y) = (y\cdot x^y)\cdot(y\cdot y) = x\cdot(y\cdot y)$. So if $x \mapsto x\cdot x$ is bijective, y can be retrieved from ${}^x y$. $\square$

Prop. D (cf. Venkov's example). Let X be a Hilbert algebra. Then $S(x,y) := (x\cdot y, x)$ is a solution of the YBE.

Pf.
$$\begin{tikzcd}
(x,y,z) \arrow[r, mapsto, "S\times 1"] \arrow[d, mapsto, "1\times S"'] & (x\cdot y, x, z) \arrow[r, mapsto, "1\times S"] & (x\cdot y, x\cdot z, x) \arrow[r, mapsto, "S\times 1"] & ((x\cdot y)\cdot(x\cdot z), x\cdot y, x) \arrow[d, equal] \\
(x, y\cdot z, y) \arrow[r, mapsto, "S\times 1"] & (x\cdot(y\cdot z), x, y) \arrow[r, mapsto, "1\times S"] & (x\cdot(y\cdot z), x\cdot y, x) &
\end{tikzcd}$$
$\square$

Thus every top. space gives rise to a solution S. After discussing the logic of Hilbert algebras, we will show that Hilbert algebras are „very non-comm.", that is, only very trivial Hilbert algebras X are commutative (in the sense that $S(X)$ is commutative). The above proof also yields

Corollary. Every (conjugacy class of a) group G gives a solution $x\cdot y := xyx^{-1}$.

Pf. $(x\cdot y)\cdot(x\cdot z) = xyx^{-1}\cdot xzx^{-1} = xyx^{-1}xzx^{-1}xy^{-1}x^{-1} = xyzy^{-1}x^{-1} = x\cdot(y\cdot z)$. $\square$

Def. We call a cycle set X <u>cocyclic</u> if $G(X)$ is cyclic, $G(X) = \langle\gamma\rangle$.

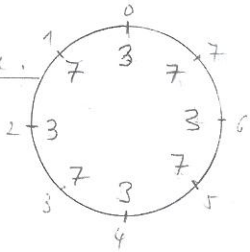
Then γ splits X into orbits $X = \coprod_{i \in I} X_i$, and the X_i are indec. sub-cycle sets. For any $x \in X$, we have $\sigma(x) = \gamma^{e(x)}$ with $e(x) \in \mathbb{Z}$, and the function $e : X \to \mathbb{Z}$ determines X. We fix an elt $0_i \in X_i$ for any $i \in I$. Then X_i can be regarded as a cyclic group, and for $y \in X_i$ we have

$$x \cdot y = x \cdot 0_i + y.$$

Now assume that <u>X is indecomposable</u>. Then X can be regarded as a cyclic group, with $x \cdot y = x \cdot 0 + y$. Since $(x \cdot y) \cdot (x \cdot z) = (x \cdot 0 + y) \cdot (x \cdot 0 + z) = (x \cdot 0 + y) \cdot 0 + x \cdot 0 + z$, the cycle set condition is

$$(x \cdot 0 + y) \cdot 0 + x \cdot 0 = (y \cdot 0 + x) \cdot 0 + y \cdot 0. \qquad (L)$$

<u>Example.</u>



$$x \cdot 0 = \begin{cases} 3 & \text{for } x \text{ even} \\ 7 & \text{for } x \text{ odd.} \end{cases}$$
(L) becomes either $3 + 7 = 3 + 7$ or $3 + 3 = 7 + 7$.

Square map: $0 \mapsto 3 \mapsto 2 \mapsto 5 \mapsto 4 \mapsto 7 \mapsto 6 \mapsto 1 \mapsto 0$.

P: always 1 cycle?

So this cycle set is non-degenerate. – Now assume that <u>X is finite.</u>

Def. We define the <u>socle</u> of X to be $\mathrm{Soc}(X) := \{x \in X \mid x \cdot 0 = 0 \cdot 0\}$.

Prop. E. $\mathrm{Soc}(X)$ is a subgroup with $x \cdot 0 = y \cdot 0 \iff x - y \in \mathrm{Soc}(X)$, and $\mathrm{Soc}(X) \neq 0$ if $|X| > 1$.

<u>Pf.</u> $x \cdot 0 = y \cdot 0 \iff \sigma(x) = \sigma(y) \implies \sigma(z \cdot x) = \sigma(z \cdot y)$, $\forall z \in X$. Since $G(X)$ is generated by the $\sigma(x)$, $\rightsquigarrow \sigma(x+1) = \sigma(y+1)$. $\rightsquigarrow (x+z) \cdot 0 = (y+z) \cdot 0$, $\forall z \in X$. $\rightsquigarrow$ $x \cdot 0 = y \cdot 0 \iff (x-y) \cdot 0 = 0 \cdot 0 \iff x - y \in \mathrm{Soc}(X)$. $\rightsquigarrow \mathrm{Soc}(X)$ subgrp. If $\mathrm{Soc}(X) = 0$, then $\exists x \in X$: $x \cdot 0 = 0 \overset{(L)}{\rightsquigarrow} y \cdot 0 = (y \cdot 0 + x) \cdot 0 + y \cdot 0 \rightsquigarrow (y \cdot 0 + x) \cdot 0 = 0 = x \cdot 0 \rightsquigarrow y \cdot 0 + x = x$. $\rightsquigarrow y \cdot 0 = 0, \forall y$. $\square$

Thus $X / \mathrm{Soc}(X) \cong \sigma(X)$, the retraction. This can be iterated:

$$X \twoheadrightarrow \sigma X \twoheadrightarrow \sigma^2 X \twoheadrightarrow \sigma^3 X \twoheadrightarrow \cdots \twoheadrightarrow \{1\}.$$

Cor. X is non-degenerate.

<u>Pf.</u> Induction: If $x \cdot x = y \cdot y$, then $\sigma(x) = \sigma(y)$, since $\mathrm{Soc}(X) \neq 0$. $\rightsquigarrow y \cdot x = x \cdot x = y \cdot y$, $\rightsquigarrow x = y$. $\square$

X can be constructed from σX as follows. Let $S \subset X$ be a representative system for the cosets of $\mathrm{Soc}(X) \subset X$. Then $x \cdot 0 = x * 0 + f(x)$ with $x * 0 \in S$ and $f(x) \in \mathrm{Soc}(X)$. The group $X / \mathrm{Soc}(X)$ induces a group structure on S. The elements x, y in (L) can be taken in S. So (L) becomes $(x * 0 + y) * 0 + f(x * 0 + y) + x * 0 + f(x) = (y * 0 + x) * 0 + f(y * 0 + x) + y * 0 + f(y)$. So the extension $X \twoheadrightarrow \sigma X$ is given by a map $f : \sigma X \to \mathrm{Soc}(X)$, with

$$f(x \cdot 0 + y) + f(x) = f(y \cdot 0 + x) + f(y). \qquad (L')$$

For given X and a map $f : X \to C$ into a cyclic group C, we obtain an extension

$$C \hookrightarrow Y \twoheadrightarrow X$$

with $\mathrm{Soc}(Y) = C$, similar to an abelian extension of groups.

Thus every Hilbert-algebra X embeds into a Brouwerian semilattice $C(X)$.

Def. A BCK-algebra is a set X with a binary operation $\cdot$ and a logical unit 1 satisfying

$$(x\cdot y)\cdot((z\cdot x)\cdot(z\cdot y)) = 1 \qquad (B)$$

$$x\cdot(y\cdot z) = y\cdot(x\cdot z) \qquad (C)$$

$$x\cdot(y\cdot x) = 1 \qquad (K)$$

$$x\cdot y = y\cdot x = 1 \Rightarrow x = y. \qquad (E)$$

Every BCK-algebra has a partial order $x\le y :\Longleftrightarrow x\cdot y = 1$. But it need not be an L-algebra.

Prop. 12. Every Hilbert algebra X is a BCK-algebra.

Pf. $x\cdot(y\cdot z) = (x\cdot y)\cdot(x\cdot z) = (y\cdot x)\cdot(y\cdot z) = y\cdot(x\cdot z)$. $\rightsquigarrow$ (C). In particular,

$$x\le y\cdot z \Longleftrightarrow y \le x\cdot z.$$

For $x = z$, this gives (K). $\rightsquigarrow$ $x\cdot y \le z\cdot(x\cdot y) = (z\cdot x)\cdot(z\cdot y)$. $\rightsquigarrow$ (B). □

Def. An ideal (or a deductive filter) of a Hilbert algebra X is a subset I with

$$1\in I$$

$$x, x\cdot y\in I \Longrightarrow y\in I.$$

Recall that an equivalence relation $\equiv$ is a congruence if it satisfies

$$x\equiv y \Longrightarrow x\cdot z\equiv y\cdot z \text{ and } z\cdot x\equiv z\cdot y.$$

Then the set $X/\equiv$ of equivalence classes satisfies the same equations. For an L-algebra X, this does not imply that $X/\equiv$ is an L-algebra, because $x\cdot y\equiv y\cdot x\equiv 1$ need not imply that $x\equiv y$. Therefore, we have to consider the stronger relation $x\sim y :\Longleftrightarrow x\cdot y = y\cdot x = 1$ which is still a congruence. This leads to the concept of ideal:

Def. An ideal of an L-algebra X is a subset $I\subset X$ with

(a) $1\in I$

(b) $x, x\cdot y\in I \Longrightarrow y\in I$

(c) $x\in I \Longrightarrow (x\cdot y)\cdot y,\ y\cdot x,\ y\cdot(x\cdot y)\in I$.

Then every surjective morphism $X \xrightarrow{f} Y$ of L-algebras is determined by the ideal $I := f^{-1}(1)$, and $f(x) = f(y) \Longleftrightarrow x\cdot y, y\cdot x\in I$. Conversely, every ideal $I \triangleleft X$ yields a surjective morphism $X \twoheadrightarrow X/I$ of L-algebras.

While (a) and (b) are intelligible, (c) is less obvious. For a Hilbert algebra X, condition (c) is redundant. Indeed, (a) and (b) implies

$$y\ge x\in I \Longrightarrow y\in I,$$

because $y\ge x\in I$ yields $x, x\cdot y = 1\in I$. Furthermore, Prop. 12 yields

$$x\overset{(C)}{\le}(x\cdot y)\cdot y,\quad x\overset{(K)}{\le} y\cdot x,\quad x\overset{(K)}{\le} y\cdot(x\cdot y) = 1.$$

<u>Prop. 13.</u> Let A be a Brouwerian semilattice. Then $I \lhd A$ iff $I \neq \emptyset$ and

(a) $b \geqslant a \in I \Longrightarrow b \in I$

(b) $a, b \in I \Longrightarrow a \wedge b \in I$.

<u>Pf.</u> Assume $I \lhd A$ and $a, b \in I$. $\rightsquigarrow$ $a \cdot (a \wedge b) = a \cdot b \geqslant b \in I$. $\rightsquigarrow$ $a \wedge b \in I$. $\rightsquigarrow$ (b).
Conversely, assume (a) and (b). If $a, a \cdot b \in I$, then $a \cdot b \wedge a \leqslant b$. $\rightsquigarrow$ $b \in I$. □

<u>Def.</u> Let X be an L-algebra. We call $x \in X$ <u>dense</u> if $x \cdot y = y$ for some $y \leqslant x$.

<u>Prop. 14.</u> If $x \cdot y = y$ and $x \geqslant y \geqslant z$, then $x \cdot z = z$.

<u>Pf.</u> $y \cdot z = (y \cdot x) \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z) = y \cdot (x \cdot z)$, and $x \cdot z \leqslant x \cdot y = y$: $\rightsquigarrow$ $x \cdot z = x \cdot z \wedge y =$
$(y \cdot (x \cdot z))\, y = (y \cdot z)\, y = (z \cdot y)\, z = z$. □

<u>Cor.</u> If X is <u>bounded</u> (i.e. $\exists$ smallest element 0), then $x \in X$ is dense iff $x \cdot 0 = 0$.

<u>Example.</u> $X \in \mathrm{Top}$. Then $\mathcal{U} \in \mathcal{O}(X)$ is dense iff $\mathcal{U} \cdot \emptyset = \emptyset$, where $\mathcal{U} \cdot \emptyset = X \setminus \overline{\mathcal{U} \setminus \emptyset} =$
$X \setminus \overline{\mathcal{U}}$. So $\mathcal{U}$ is dense iff $\overline{\mathcal{U}} = X$.

<u>Def.</u> Let X be an L-algebra. We call $x \in X$ <u>regular</u> iff $z \cdot x = x$ for all dense $z \in X$.

<u>Prop. 15.</u> Let A be a Brouwerian semilattice. For $x, y \in A$,

$$x = (y \cdot x) \wedge (y \cdot x) \cdot x.$$

<u>Pf.</u> "$\leqslant$" is trivial. $(y \cdot x) \cdot x \leqslant (y \cdot x) \cdot x$ gives $(y \cdot x) \cdot x \wedge (y \cdot x) \leqslant x$. □

<u>Prop. 16.</u> A Hilbert algebra X is a Boolean algebra iff X is bounded, with
$x'' = x$ $(x' := x \cdot 0)$ for all $x \in X$.

<u>Pf.</u> Let X be bounded with $x'' = x$, $\forall x \in X$. $\rightsquigarrow$ If $x \leqslant y$, then $x \leqslant y \cdot \leqslant (y \cdot 0) \cdot 0$ $\rightsquigarrow$
$y' \leqslant x'$. $\therefore C(X)$ is also bounded, and $x \wedge y \leqslant (x \wedge y)'' \leqslant x'' = x$. $\rightsquigarrow$ $(x \wedge y)'' = x \wedge y$,
$(x \wedge y)'' = ((x \wedge y) \cdot 0)' = (x \cdot (y \cdot 0))' = (x \cdot y')' \in X$. $\rightsquigarrow$ $X = C(X)$ with $x \wedge y = (x \cdot y')'$.
So X is a lattice:

$$x \vee y = (x' \wedge y')' = x' \cdot y.$$

$\rightsquigarrow$ $x' \cdot (y \wedge z) = (x' \cdot y) \wedge (x' \cdot z)$. $\rightsquigarrow$ $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. $\rightsquigarrow$ X distributive.
Furthermore $x' \vee x = x \cdot x = 1$. $\rightsquigarrow$ $x \wedge x' = 1' = 0$. The converse is trivial. □

<u>Thm. 4 (Glivenko).</u> Let X be a bounded Hilbert algebra. Then

$$\overline{X} := \{x' \mid x \in X\} = \{x \in X \mid x'' = x\}$$

is a Boolean algebra which consists of the regular elements of X, and $x \mapsto x''$
is a homomorphism $X \twoheadrightarrow \overline{X}$. The kernel is the set $D(X)$ of dense elements.

<u>Pf</u>. For $x, y \in X \subset S(X)$, we have $y \le x\cdot y$. $\rightsquigarrow (x\cdot y)' \le y'$ and $x' = x\cdot 0 \le x\cdot y$. $\rightsquigarrow$ $(x\cdot y)' \le x''$. $\rightsquigarrow \underline{(x\cdot y)' \le x'' \wedge y'}$. Furthermore, $y' \le (x\wedge y)' = (x\cdot y)x\cdot 0 = (x\cdot y)\cdot x' \overset{(c)}{\rightsquigarrow}$ $x\cdot y \le y'\cdot x' \rightsquigarrow y' \le x''\cdot(x\cdot y)' \rightsquigarrow \underline{y'\wedge x'' \le (x\cdot y)'}$. $\rightsquigarrow$

$$(x\cdot y)' = x''\wedge y'.$$

$\rightsquigarrow (x\cdot y)'' = (x''\wedge y')\cdot 0 = x''\cdot y''$. $\rightsquigarrow x \mapsto x''$ is a homomorphism. $\rightsquigarrow \overline{X}$ is a Hilbert algebra. $\overset{\text{Prop. 16}}{\rightsquigarrow}$ $\overline{X}$ is Boolean. Note that $x \le x'' \rightsquigarrow x''' \le x' \le x'''$! $\rightsquigarrow$ $x = x'' \iff \exists y \in X: x = y'$.

If x is dense, then $x' = 0 \rightsquigarrow x\cdot y' = x\cdot(y\cdot 0) \overset{(c)}{=} y\cdot x' = y'$. $\rightsquigarrow y'$ regular. Conversely, let x be regular. $\rightsquigarrow (x'\cdot x)' = x'''\wedge x' = x'$. $\rightsquigarrow ((x'\cdot x)\cdot x)' =$ $(x'\cdot x)''\wedge x' = x''\wedge x' = 0$, since $\overline{X}$ is Boolean. $\rightsquigarrow (x'\cdot x)\cdot x$ is dense. $\rightsquigarrow$ $x'\cdot x = ((x'\cdot x)\cdot x)\cdot x = x \rightsquigarrow x'' = x'\cdot 0 \le x'\cdot x = x$. $\rightsquigarrow x \in \overline{X}$. $\square$

<u>Example</u>. $X \in \mathbf{Top}$. Then $U \in \mathcal{O}(X)$ is <u>regular</u> iff $\exists V \in \mathcal{O}(X): X = V\cdot\emptyset = X\setminus\overline{V}$, i.e. $U = \overset{\circ}{\overline{U}}$. These U have an involutive complement $X\setminus\overline{U}$. They form a Boolean algebra.

<u>Def</u>. Let X be an L-algebra. We call $p \in X$ <u>prime</u> if $p < 1$ and for all $x \in X$,

$$x \le p \quad\text{or}\quad x\cdot p = p.$$

<u>Prop. 17</u>. Let A be a Brouwerian semilattice. Then $p < 1$ is prime iff

$$p = a\wedge b \implies p = a \text{ or } p = b.$$

<u>Pf</u>. Let p be prime, and $p = a\wedge b$. $\rightsquigarrow a \le b\cdot p$. If $b \not\le p$, then $b\cdot p = p \rightsquigarrow a \le p$. So $a = p$ or $b = p$. Conversely, let the implication be satisfied. $\overset{\text{Prop. 15}}{\rightsquigarrow} \forall x \in X$:

$$p = x\cdot p \wedge (x\cdot p)\cdot p.$$

So either $x\cdot p = p$ or $(x\cdot p)\cdot p = p$. In the second case, $x\cdot p = ((x\cdot p)\cdot p)\cdot p = p\cdot p = 1$. $\rightsquigarrow x \le p$. $\square$

<u>Prop. 18</u>. Let A be a Brouwerian semilattice. Every prime $p \in A$ is either dense or regular.

<u>Pf</u>. If p is not dense, every dense $x \in X$ satisfies $x \not\le p$. $\rightsquigarrow x\cdot p = p$. $\square$

Let X be a Hilbert algebra. The set $\hat{X}$ of ideals is $\bigcap$-closed, hence a complete lattice. If X is a Brouwerian semilattice, the join $\bigvee \mathcal{J}$ of $\mathcal{J} \subset \hat{X}$ is the ideal generated by $\mathcal{J}$:

$$a \in \bigvee\mathcal{J} \iff \exists\, a_1, \dots, a_n \in \bigcup\mathcal{J}: a \ge a_1\wedge\dots\wedge a_n.$$

<u>Def</u>. A <u>Heyting algebra</u> is a bounded Brouwerian (semilattice which is a) lattice. A <u>locale</u> is a complete lattice which satisfies $a \wedge \bigvee a_i = \bigvee (a\wedge a_i)$.

Prop. 19. A locale is equivalent to a complete Heyting algebra.

Pf. Let A be a locale. Define
$$a \cdot b := \bigvee \{c \in A \mid c \wedge a \leq b\}.$$
Then $x \wedge a \leq b \Rightarrow x \leq a \cdot b \Rightarrow x \wedge a \leq a \cdot b \wedge a = \bigvee\{c \wedge a \mid c \wedge a \leq b\} \leq b$. $\rightsquigarrow$
A is a complete Heyting algebra. Conversely, let A be a complete Heyting algebra.
Then $c \leq a \cdot b \Leftrightarrow c \wedge a \leq b$ gives the above formula for $a \cdot b$. Verify
$$\left(\bigvee a_i\right) \wedge a \overset{!}{=} \bigvee (a_i \wedge a).$$
"$\geq$" is obvious, and "$\leq$" $\Leftrightarrow \bigvee a_i \leq a \cdot \bigvee(a_i \wedge a) \Leftrightarrow \forall i : a_i \leq a \cdot \bigvee(a_i \wedge a) \Leftrightarrow$
$\forall i : a_i \wedge a \leq \bigvee(a_i \wedge a)$. □

Examples. 1. $X \in \mathbf{Top}$. Then $\mathcal{O}(X)$ is a locale: $U \cap \bigcup U_i = \bigcup (U \cap U_i)$.

2. If A is a Brouwerian semilattice, $\hat{A}$ is a locale: For $I, J \in \hat{A}$,
$$I \cdot J := \{a \in A \mid \uparrow a \cap I \subset J\}$$
is an upper set. If $a, b \in I \cdot J$ and $a \wedge b \leq c \in I$, then $a \leq b \cdot c$ and $b \leq (b \cdot c) \cdot c$.
$\xrightarrow{\text{Prop. 15}}$ $c = b \cdot c \wedge (b \cdot c) \cdot c \in J$. $\rightsquigarrow$ $I \cdot J \in \hat{A}$. By definition,
$$I \subset J_1 \cdot J_2 \Longleftrightarrow \forall a \in I : \uparrow a \cap J_1 \subset J_2 \Longleftrightarrow I \cap J_1 \subset J_2.$$
So $\hat{A}$ is a complete Heyting algebra. $\xrightarrow{\text{Prop. 19}}$ $\hat{A}$ is a locale.

Prop. 20. Let X be a Hilbert algebra. Then $I \mapsto C(I)$ gives an isomorphism
$$C : \hat{X} \xrightarrow{\sim} \widehat{C(X)}.$$

Pf. For $I \in \hat{X}$, assume that $b \geq a \in C(I)$. Then $a = x_1 \wedge x_2 \wedge \ldots \wedge x_n$ with $x_i \in I$.
We show that $b \in C(I)$. By Prop. 15, $b = x_1 \cdot b \wedge (x_1 \cdot b) \cdot b$ with $x_2 \wedge \ldots \wedge x_n \leq x_1 \cdot b$
and $x_1 \leq (x_1 \cdot b) \cdot b$. Inductively, we are reduced to the case $b \geq x \in I \overset{!}{\Rightarrow} b \in C(I)$.
So $b = y_1 \wedge \ldots \wedge y_n$ with $y_i \in X$. $\rightsquigarrow$ $y_i \in I$. $\rightsquigarrow$ $b \in C(I)$. Thus $I \mapsto C(I)$ is well
defined. Consider the inverse map $J \mapsto J \cap X$ for $J \lhd C(X)$. If $x, y \in X$
with $x, x \cdot y \in J \cap X$, then $y \in J \cap X$. So $J \cap X \in \hat{X}$.
For $I \in \hat{X}$ and $J \in \widehat{C(X)}$, we have $I \subset C(I) \cap X$ and $C(J \cap X) \subset J$. For $a \in J$,
$a = x_1 \wedge \ldots \wedge x_n$ with $a \leq x_i \in X$. $\rightsquigarrow$ $x_i \in J \cap X$ $\rightsquigarrow$ $J = C(J \cap X)$. If $x \in C(I) \cap X$, then
$x = x_1 \wedge \ldots \wedge x_n$ with $x_i \in I$. By induction, we can assume that $x_2 \wedge \ldots \wedge x_n =: y \in I$.
$\rightsquigarrow$ $x_1 \cdot x = x_1 \cdot (x_1 \wedge y) = x_1 \cdot y \in I$. $\xrightarrow{x_1 \in I}$ $x \in I$. So $C(I) \cap X = I$. □

Def. Let X be a Hilbert algebra. Let $\operatorname{Spec} X$ be the set of <u>prime ideals</u> $P \in \hat{X}$, that is,
the prime elements in the locale $\hat{X}$. We call $\operatorname{Spec} X$ the <u>spectrum</u> of X.

By Prop. 20, we can restrict ourselves to a Brouwerian semilattice A. For $a \in A$,
we set
$$\mathcal{U}(a) := \{P \in \operatorname{Spec} A \mid a \notin P\}.$$
Then $P \notin \mathcal{U}(a \wedge b) \Leftrightarrow a \wedge b \in P \Leftrightarrow a, b \in P \Leftrightarrow P \notin \mathcal{U}(a) \cup \mathcal{U}(b)$. Hence

$$\mathcal{U}(a \wedge b) = \mathcal{U}(a) \vee \mathcal{U}(b).$$

More generally, we define

$$\mathcal{U}(I) := \{P \in \operatorname{Spec} A \mid I \not\subset P\}, \quad I \in \hat{A}.$$

Thus $\mathcal{U}(a) = \mathcal{U}(\uparrow a)$.

<u>Prop. 21.</u> Let A be a Brouwerian semilattice. Then $P \in \hat{A}$ is prime iff $P \neq A$ and $A \setminus P$ is upper directed: $a, b \notin P \Rightarrow \exists c \notin P : c \geq a, b$.

<u>Pf.</u> For $a, b, c \in A$,

$$a \cdot b = b, \; a \leq c \Rightarrow c \cdot b = b.$$

Indeed, $c \cdot b \leq a \cdot b = b$. Now $P \neq A$ prime $\Leftrightarrow \forall I \in \hat{A} : I \subset P \vee I \cdot P = P$. So we can replace I by a principal ideal: $\forall a \in A : \uparrow a \subset P$ or $\uparrow a \cdot P = P$. Now $b \in \uparrow a \cdot P \Leftrightarrow \uparrow b \cap \uparrow a \subset P$. So P prime $\Leftrightarrow \forall a \notin P : b \notin P \Rightarrow \uparrow b \cap \uparrow a \not\subset P \Leftrightarrow \forall a, b \notin P \, \exists c \in \uparrow b \cap \uparrow a \setminus P$. $\square$

<u>Cor. 1.</u> Let A be a Brouwerian semilattice. For $I \in \hat{A}$,

$$I = \bigcap \{P \in \operatorname{Spec} A \mid I \subset P\}.$$

<u>Pf.</u> Let $a \in A \setminus I$. $\overset{\text{Zorn}}{\rightsquigarrow} \exists P \supset I$ in $\hat{A}$ maximal with $a \notin P$. For ideals $I_1, I_2 \supsetneq P$, this gives $a \in I_1 \cap I_2$, hence $I_1 \cap I_2 \supsetneq P$. So P is prime. $\rightsquigarrow a \notin \bigcap\{P \in \operatorname{Spec} A \mid I \subset P\}$. $\square$

With $V(I) := \operatorname{Spec} A \setminus \mathcal{U}(I) = \{P \in \operatorname{Spec} A \mid I \subset P\}$, Cor. 1 yields

$$I = \bigcap V(I),$$

$$I \subset J \Leftrightarrow V(I) \supset V(J) \Leftrightarrow \mathcal{U}(I) \subset \mathcal{U}(J) \quad \text{for } I, J \in \hat{A},$$

$$V(\bigvee I_i) = \bigcap V(I_i) \quad \text{for } I_i \in \hat{A}, \text{ hence}$$

$$\mathcal{U}(\bigvee I_i) = \bigcup \mathcal{U}(I_i).$$

<u>Cor. 2.</u> Let A be a Brouwerian semilattice. The $\mathcal{U}(I)$ with $I \in \hat{A}$ are the open sets of a topology on $\operatorname{Spec} A$, and the $\mathcal{U}(a)$, $a \in A$, form a basis of open sets.

<u>Pf.</u> For $a, b \in A$ and $P \in \mathcal{U}(a) \cap \mathcal{U}(b)$, we have $a, b \notin P$. $\rightsquigarrow \exists c \notin P : c \geq a, b$. $\rightsquigarrow$ $P \in \mathcal{U}(c) \subset \mathcal{U}(a) \cap \mathcal{U}(b)$. So the $\mathcal{U}(a)$ form a basis of a topology. If $I \in \hat{A}$ is generated by $S \subset A$, then $\bigcup_{a \in S} \mathcal{U}(a) = \mathcal{U}(I)$. $\square$

We endow $\operatorname{Spec} A$ with this topology. So Cor. 2 gives

$$\boxed{\hat{A} \cong \mathcal{O}(\operatorname{Spec} A)}.$$

<u>Cor. 3.</u> Let A be a Heyting algebra. Then $P \in \hat{A}$ is prime iff $P \neq A$ and $\forall a, b \in A$,

$$a \vee b \in P \Rightarrow a \in P \text{ or } b \in P.$$

<u>Pf.</u> $a, b \notin P \Rightarrow \exists c \notin P : c \geq a \vee b \Leftrightarrow a \vee b \notin P$. $\square$

Now we intensify the connection between Hilbert algebras and spaces. For a Hilbert algebra X, the open sets of $\operatorname{Spec} X$ can be identified with the ideals of X. Therefore, we regard the open sets of a topological space X as <u>ideals</u> of X. Accordingly, we write $\operatorname{Spec} X$ for the set of "prime ideals" of X, the prime $P \in \mathcal{O}(X)$. By Prop. 17, $P \neq X$ in $\mathcal{O}(X)$ satisfies:

$$P \in \operatorname{Spec} X \iff \left(\forall\, \mathcal{U}, V \in \mathcal{O}(X) : \mathcal{U} \cap V = P \Rightarrow \mathcal{U} = P \text{ or } V = P\right).$$

So the closed sets $A = X \setminus P$ are <u>irreducible</u>, i.e. $A = B \cup C$ with B, C closed implies $B = A$ or $C = A$. Equivalently, a space X is irreducible iff all non-empty open sets are dense. As before, $\operatorname{Spec} X$ has a topology $\mathcal{O}(\operatorname{Spec} X)$ which can be identified with $\mathcal{O}(X)$, so that for $P \in \operatorname{Spec} X$ and $\mathcal{U} \in \mathcal{O}(X)$

$$P \in \mathcal{U} \iff \mathcal{U} \not\subset P \iff \mathcal{U} \cap (X \setminus P) \neq \emptyset .$$

So from now on, we define

$$\operatorname{Spec} X := \{A \subset X \mid A \neq \emptyset \text{ closed irreducible}\}.$$

For all $x \in X$, the set $\bar{x} := \overline{\{x\}} \in \operatorname{Spec} X$. So we have a continuous map

$$i : X \longrightarrow \operatorname{Spec} X, \quad i(x) := \bar{x}, \quad \mathcal{O}(\operatorname{Spec} X) = \mathcal{O}(X).$$

X is not a Hilbert algebra, but close to a poset:

$$x \le y :\iff \bar{x} \subset \bar{y} \iff x \in \bar{y} \iff \forall\, \mathcal{U} \in \mathcal{O}(X) : x \in \mathcal{U} \Rightarrow y \in \mathcal{U}.$$

X is a <u>T_0-space</u> if the open sets separate points, i.e. $(X; \le)$ is a poset. So

$$X \text{ is a } T_0\text{-space} \iff i \text{ injective}.$$

Def. Let X be a T_0-space. A point $x \in X$ is called <u>generic</u> if $\bar{x} = X$, i.e. x is the greatest element w.r.t. "$\le$". If every $A \in \operatorname{Spec} X$ has a generic point, X is called <u>sober</u>.

Thus X is sober iff i is a homeomorphism:

$$X \text{ sober} \iff X \cong \operatorname{Spec} X.$$

For example, every Hausdorff space is sober. By $\mathcal{S}(X)$ we denote the set of <u>quasicompact</u> open sets. So $\mathcal{U} \in \mathcal{O}(X)$ is in $\mathcal{S}(X)$ if for $\mathcal{U}_i \in \mathcal{O}(X)$

$$\mathcal{U} \subset \bigcup_{i \in I} \mathcal{U}_i \Rightarrow \exists\, J \subset I \text{ finite} : \mathcal{U} \subset \bigcup_{j \in J} \mathcal{U}_j .$$

Def. We call $X \in \mathbf{Top}$ <u>prespectral</u> if X is sober, and $\mathcal{S}(X)$ is a basis.

Prop. 22. Let A be a Brouwerian semilattice. Then $\operatorname{Spec} A$ is prespectral, and $A^{op} \cong \mathcal{S}(\operatorname{Spec} A)$ as a poset.

Pf. For $a, b \in A$, $a \le b \iff \uparrow a \supset \uparrow b \iff \mathcal{U}(a) \supset \mathcal{U}(b)$. So

$$\mathcal{U} : A^{op} \hookrightarrow \mathcal{U}(\operatorname{Spec} A)$$

is an embedding of posets, and the $\mathcal{U}(a)$ form a basis of $\operatorname{Spec} A$ (Cor. 2 of Prop 21).

An inclusion $\mathcal{U}(a) \subset \bigcup \mathcal{U}(I_i)$ with $I_i \in \hat{A}$ is equivalent to $\uparrow a \subset \bigvee I_i$. So $\exists a_1, \dots, a_n \in \bigcup I_i$ with $a \geq a_1 \wedge \dots \wedge a_n$. $\rightsquigarrow$ $\mathcal{U}(a) \in \mathcal{D}(X)$. Conversely, let $I \in \hat{A}$ with $\mathcal{U}(I) \in \mathcal{D}(X)$. $\rightsquigarrow$ $I = \bigcup_{a \in I} \uparrow a$ $\rightsquigarrow$ $\mathcal{U}(I) = \bigcup_{a \in I} \mathcal{U}(a)$ $\rightsquigarrow$ $\exists a_1, \dots, a_n \in I : \mathcal{U}(I) = \mathcal{U}(a_1) \cup \dots \cup \mathcal{U}(a_n) = \mathcal{U}(a_1 \wedge \dots \wedge a_n)$. □

So A is completely determined by $\operatorname{Spec} A$.

<u>Def.</u> A <u>spectral space</u> is a prespectral space X with

$$\mathcal{U}_1, \dots, \mathcal{U}_n \in \mathcal{D}(X) \Longrightarrow \mathcal{U}_1 \cap \dots \cap \mathcal{U}_n \in \mathcal{D}(X).$$

$n = 0$: $(\Rightarrow X \in \mathcal{D}(X))$

A Hausdorff spectral space is said to be a <u>Stone space</u>.

<u>Cor. (Stone 1937).</u> There is a 1-1 correspondence between Stone spaces X and Boolean algebras A: $\quad A^{op} \cong \mathcal{D}(\operatorname{Spec} A), \quad X \cong \operatorname{Spec} \mathcal{D}(X)^{op}$.

<u>Pf.</u> Since $A^{op} \cong \mathcal{D}(\operatorname{Spec} A)$ is a bounded lattice, $\operatorname{Spec} A$ is a spectral space. For $a \in A$, we have $\mathcal{U}(a) \sqcup \mathcal{U}(a') = \operatorname{Spec} A$. So the $\mathcal{U}(a) \in \mathcal{D}(\operatorname{Spec} A)$ are open and closed. $\rightsquigarrow$ $\operatorname{Spec} A$ Hausdorff. Converse:
Let X be a Stone space. $\rightsquigarrow$ X compact $\rightsquigarrow$ $\forall\, \mathcal{U} \in \mathcal{D}(X)$ closed $\rightsquigarrow$ $X \setminus \mathcal{U} \in \mathcal{D}(X)$.
$\rightsquigarrow$ $\mathcal{D}(X)$ Boolean. Let $P \in \operatorname{Spec} \mathcal{D}(X)^{op}$. $\rightsquigarrow$ (P down-closed) and $X \notin P$.
X sp. P is no covering of X. $\rightsquigarrow$ $X \setminus \bigcup P$ irreducible $\rightsquigarrow$ $\exists x \in X : \{x\} = X \setminus \bigcup P$. $\rightsquigarrow$
$P = \{\mathcal{U} \in \mathcal{D}(X) \mid x \notin \mathcal{U}\}$. $\rightsquigarrow$ $X \cong \operatorname{Spec} \mathcal{D}(X)^{op}$. □

Stone's thm. has been the starting point of topological algebra.

<u>Example.</u> Let L be a distributive lattice. Then $S(a,b) := (a \vee b, a \wedge b)$ solves the YBE.

The structure group of a Boolean algebra A can be expressed by the Stone space $X := \operatorname{Spec} A$. As $G(X)$ is abelian, we take $+$ as its operation. For $\mathcal{U}, V \in \mathcal{D}(X)$ we define $\mathcal{U} + V$ by counting $\mathcal{U} \cap V$ twice:

$$\mathcal{U} + V = \mathcal{U} \setminus V + 2(\mathcal{U} \cap V) + (V \setminus \mathcal{U}).$$

So $S(A) = \mathbb{N}^{(A)} / \langle \mathcal{U} + V - \mathcal{U} \cup V - \mathcal{U} \cap V \rangle$ consists of the formal linear combinations

$$n_1 \mathcal{U}_1 + \dots + n_r \mathcal{U}_r, \quad n_i \in \mathbb{N}, \ \mathcal{U}_i \in \mathcal{D}(X)$$

modulo the relation $\quad \mathcal{U} + V = \mathcal{U} \cup V + \mathcal{U} \cap V$.
For $G(A)$ the n_i have to be taken in $\mathbb{Z}$. So $G(A)$ consists of the step functions (over $\mathbb{Z}$) on the measurable space $(X; \mathcal{D}(X))$. Here $G(A)$ is even a ring with multiplication $\cap$: $\quad \mathcal{U} \cap (V + W) = \mathcal{U} \cap V + \mathcal{U} \cap W$. The operation $\mathcal{U} \cap -$ is the restriction to the support $\mathcal{U}$. Therefore, we write $R(A) := (G(A); +, \cap)$.
We regard the elements of $R(A)$ as step functions $f : A \to \mathbb{Z}$. So $a \in A$ is represented by the characteristic function $\chi_{\mathcal{U}(a)}$.

Thus $a \le b \iff \mathcal{U}(a) \supset \mathcal{U}(b) \iff \chi_{\mathcal{U}(a)} \geq \chi_{\mathcal{U}(b)}$. In particular, $a \le 1$ corresponds to $\chi_{\mathcal{U}(a)} \geq \chi_{\emptyset} = 0$: The $\chi_{\mathcal{U}(a)}$ are in the <u>negative cone</u> $S(A) = G(A)^-$.

The $\chi_{\mathcal{U}(a)}$ are exactly the idempotents of $R(A)$. They represent the residue classes modulo $2\,R(A)$. So A can be equipped with a <u>ring structure</u>:

$$2\,R(A) \hookrightarrow R(A) \twoheadrightarrow A,$$

a <u>Boolean ring</u> (cf. the analogy: $B_n \twoheadrightarrow S_n$!), an algebra over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$, for which every element is idempotent. (Compare $a \vee b = a + b + ab$ with Jacobson's $\circ$!)

B.W. Now let X be a spectral space. Every $\mathcal{U} \in \mathcal{O}(X)$ is an upper set: $x \geq y \in \mathcal{U} \Rightarrow x \in \mathcal{U}$, but not vice versa. $\mathcal{D}(X)$ is a bounded distributive lattice. It generates a Boolean algebra $\mathcal{B}(X)$: Every element $B \in \mathcal{B}(X)$ is of the form

$$B = (\mathcal{U}_1 \setminus V_1) \cup \dots \cup (\mathcal{U}_n \setminus V_n)$$

with $\mathcal{U}_i, V_i \in \mathcal{D}(X)$. Indeed, $\mathcal{U}_i \setminus V_i = \mathcal{U}_i \cap V_i^c$ with $V_i^c := X \setminus V_i$. $\rightsquigarrow (\mathcal{U} \cap V^c)^c = \mathcal{U}^c \cup V$. So $\mathcal{B}(X)$ is a basis of a topology on X, the <u>patch topology</u>. We write $\tilde{X}$ for this space, which gives a continuous bijection

$$\tilde{X} \longrightarrow X.$$

<u>Prop. 23.</u> Let X be a spectral space. Then $\tilde{X} \cong \operatorname{Spec} \mathcal{B}(X)$.

<u>Pf.</u> Let $P \in \operatorname{Spec} \mathcal{B}(X)$. $\rightsquigarrow$ Every $B \in P$ contains an element $\mathcal{U} \setminus V$ with $\mathcal{U}, V \in \mathcal{D}(X)$. Consider the elements $\mathcal{U}_i \setminus V_i = \mathcal{U}_i \cap V_i^c \in P$ with $\mathcal{U}_i, V_i \in \mathcal{D}(X)$, so $A \doteq \bigcap V_i^c$ is closed in X. Since $\mathcal{U}_i \in \mathcal{D}(X)$, the intersection property for closed sets in a quasicompact space gives $A \cap \mathcal{U}_i \neq \emptyset$ for all i. $\xrightarrow{\text{Zorn}} \exists$ minimal closed $A_0 \neq 0$ in X with $A_0 \cap \mathcal{U}_i \neq \emptyset$ for all i. Suppose that $A_0 = A_1 \cup A_2$ with $A_i \neq A$ closed in X. Then $\exists\, i, j : \mathcal{U}_i \cap A_1 = \mathcal{U}_j \cap A_2 = \emptyset \rightsquigarrow \mathcal{U}_i \cap \mathcal{U}_j \cap A = \emptyset$, $\lightning$. $\rightsquigarrow A_0$ irreducible. $\rightsquigarrow \exists x \in X : A_0 = \overline{x}$. $\rightsquigarrow x \in \bigcap P$. For $x \in \mathcal{U} \in \mathcal{B}(X)$, $\mathcal{U} \sqcup \mathcal{U}^c = X \in P$ and $\mathcal{U}^c \notin P$. $\rightsquigarrow \mathcal{U} \in P$. $\rightsquigarrow$ $P = \{\mathcal{U} \in \mathcal{B}(X) \mid x \in \mathcal{U}\}$. Since X is T_0, x is unique. For $\mathcal{U} \in \mathcal{B}(X)$, $x \in \mathcal{U} \iff \mathcal{U} \in P$. Thus $\tilde{X} \cong \operatorname{Spec} \mathcal{B}(X)$. $\square$

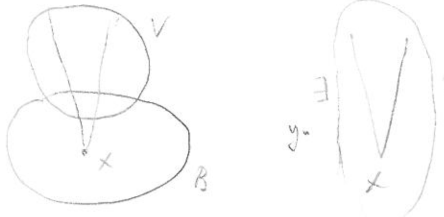
In particular, $\tilde{X}$ is a Stone space. Now X is determined by $\tilde{X}$ and the poset X:

<u>Prop. 24.</u> Let X be a spectral space. Then $\mathcal{U} \in \mathcal{B}(X)$ belongs to $\mathcal{D}(X)$ iff $\mathcal{U}$ is an upper set.

<u>Pf.</u> Let $\mathcal{U} \in \mathcal{B}(X)$ be an upper set. For $x \in \mathcal{U}$ and $y \notin \mathcal{U}$, we have $x \not\leq y$. $\rightsquigarrow$ $\exists\, V \in \mathcal{D}(X) : x \in V,\ y \notin V$. $\rightsquigarrow V^c \in \mathcal{B}(X)$. Since $\mathcal{U}^c$ is closed, hence compact in $\tilde{X}$, this gives $V_1, \dots, V_n \in \mathcal{D}(X)$ with $x \in V_i$ and $\mathcal{U}^c \subset V_1^c \cup \dots \cup V_n^c$. $\xrightarrow{x \in} V := V_1 \cap \dots \cap V_n \subset \mathcal{U}$. $\rightsquigarrow \mathcal{U} \in \mathcal{O}(X)$. Since $\mathcal{U} \in \mathcal{B}(X)$, we get $\mathcal{U} \in \mathcal{D}(X)$. The converse is obvious. $\square$

Thm. 5 (Priestley 1970). A spectral space is equivalent to a Stone space X with a partial order s.th. $\forall\, x \not\leq y\ \exists\, U \in \mathcal{D}(X): x \in U,\ y \notin U$.

Pf. Let X be such a Priestley space. We define $\mathscr{D} := \{U \in \mathcal{D}(X) \mid U = \uparrow U\}$. $\rightsquigarrow$ $\mathscr{D}$ is a sublattice of $\mathfrak{B}(X)$. We take $\mathscr{D}$ as a basis for a coarser topology τ of X so the $U \in \mathscr{D}$ are quasicompact open in τ. By the compatibility of $\leq$, $(X;\tau) \rhd T_0$. All $U \in \mathcal{O}(X;\tau)$ are upper sets. So $x \leq y \iff (\forall\, U \in \mathcal{O}(X): x \in U \Rightarrow y \in U) \iff \overline{x} \leq \overline{y}$.
Let $A \subset X$ be non-empty, closed, and irreducible in $(X;\tau)$. $\rightsquigarrow$ A compact in X. The $U \in \mathscr{D}$ are closed in X, and $U_1 \cap \dots \cap U_n \cap A \neq \emptyset$ for $U_i \in \mathscr{D}$. $\rightsquigarrow$ $\exists\, x \in X: x \in A \cap U$ for all $U \in \mathscr{D}$ with $U \cap A \neq \emptyset$, and x is unique. $\rightsquigarrow$ $\overline{x} = A$. $\rightsquigarrow$ $(X;\tau)$ spectral. It remains to show that the Boolean algebra $\mathcal{D}(X)$ is generated by $\mathscr{D}$, i.e. $\mathcal{D}(X) = \mathcal{D}(X;\tau)$. Let $B \in \mathcal{D}(X)$ and $x \in B$. $\rightsquigarrow$ $\forall\, y \not\geq x\ \exists\, U \in \mathscr{D}: x \in U,\ y \notin U$. $\rightsquigarrow$



$\uparrow x = \overline{x}$ kompakt in X. $\rightsquigarrow$ $\overline{x} \setminus B$ compact. $\forall\, z \in \overline{x} \setminus B: z > x$ $\rightsquigarrow$ $\exists\, U \in \mathscr{D}: z \in U,\ x \notin U$. $\rightsquigarrow$ $\exists\, V \in \mathscr{D}: \overline{x} \setminus B \subset V,\ x \notin V$. Now $X \setminus (B \cup V)$ compact. For $y \notin B \cup V$: $x \not\leq y$ $\rightsquigarrow$ $\exists\, U \in \mathscr{D}: x \in U,\ y \notin U$. $\rightsquigarrow$ $\exists\, U \in \mathscr{D}: x \in U \subset B \cup V$. $\rightsquigarrow$ $x \in U \setminus V \subset B$. So the $U \setminus V$ with $U, V \in \mathscr{D}$ form a basis for $\mathcal{D}(X)$. $\rightsquigarrow$ $\mathcal{D}(X) = \mathcal{D}(X;\tau)$. □

Next we show that a spectral space X is completely given by the distributive lattice $\mathscr{D}(X)$.

Prop. 25. Let A be a Boolean algebra and $P \in \operatorname{Spec} A$. Then $A/P \cong \mathbb{B} = \{0,1\}$.

Pf. For $a \in A$, we have $a \vee a' = 1 \in P$. $\rightsquigarrow$ $a \in P$ or $a' \in P$ (not both). So $A/P = \mathbb{B}$. □

$\mathbb{B}$ is a Boolean algebra and a spectral space with $\mathscr{D}(\mathbb{B}) = \{\emptyset, \{1\}, \mathbb{B}\}$. 

More generally, let D be a bounded distributive lattice. By $X(D)$ we denote the set of lattice homomorphisms $x: D \to \mathbb{B}$ with $x(0) = 0$ and $x(1) = 1$. For $a \in D$, let $\hat{a}: X(D) \to \mathbb{B}$ be given by

$$\hat{a}(x) = x(a).$$

They form a sublattice of $\mathbb{B}^{X(D)}$ (pointwise lattice operations), e.g. $\hat{a}(x) \vee \hat{b}(x) = x(a) \vee x(b) = x(a \vee b) = \widehat{a \vee b}(x)$. So $\widehat{a \vee b} = \hat{a} \vee \hat{b}$. Also $\hat{1}(x) = x(1) = 1$, and $\hat{0} = 0 \in \mathbb{B}^{X(D)}$.

Prop. 26 Let D be a bounded distributive lattice. Then $a \mapsto \hat{a}$ is a lattice embedding

$$D \hookrightarrow \mathbb{B}^{X(D)}.$$

Pf. Let $a \not\leq b$ in D be given. We call $I \subset D$ an ideal if

$$c, d \in I \iff c \wedge d \in I$$

So $\uparrow a \lhd D$ and $b \notin \uparrow a$. $\overset{\text{Zorn}}{\rightsquigarrow}$ $\exists$ ideal $P \supset \uparrow a$, maximal with $b \notin P$.