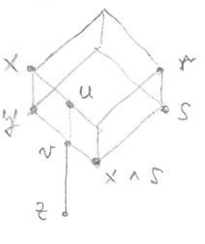


<u>Pf.</u> Choose $r \geq s$ in D with $x/y = r/s$. $\rightsquigarrow$ $x \vee s = y \vee r$ and $x \wedge s = y \wedge r$. $\rightsquigarrow$ $v := z \vee (s \wedge x) = (z \vee s) \wedge x \in D$. $\rightsquigarrow$ $u := (x \wedge r) \vee v \in D$. $\rightsquigarrow$ $z \leq v \leq u$. $\rightsquigarrow$ $y \vee u = y \vee (r \wedge x) \vee v = ((y \vee r) \wedge x) \vee v = ((x \vee s) \wedge x) \vee v = x \vee v = x$, and $v = z \vee (y \wedge r) \leq y$. $\rightsquigarrow$ $y \wedge u = (v \vee (x \wedge r)) \wedge y = v \vee (x \wedge r \wedge y) = v \vee (r \wedge y) = v \vee (x \wedge s) = v$. $\square$

<u>Prop. 41.</u> Let X be an MV-algebra, and $x \geq y \geq z$. Then $x \cdot z = (x \cdot y)(y \cdot z)$.

<u>Pf.</u> $(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z) = y \cdot z$. $\rightsquigarrow$ $(y \cdot z)(x \cdot y) = ((x \cdot y) \cdot (x \cdot z))(x \cdot y) = x \cdot y \wedge x \cdot z = x \cdot z$. $\square$

<u>Thm. 10.</u> Let X be an MV-algebra. There is a unique measure $\mu : B(X) \to X$ with $\mu|_X = 1_X$.

<u>Pf.</u> By the Corollary of Prop. 39, every $a \in B(X)$ gives $a = (x_1 \smallsetminus x_2) + \dots + (x_{2n-1} \smallsetminus x_{2n})$ with $x_1 \geq \dots \geq x_{2n}$ in X. Define

$$\boxed{\mu'(a) := (x_1 \cdot x_2)(x_3 \cdot x_4) \cdots (x_{2n-1} \cdot x_{2n})}.$$

For a second representation of a, we get a common refinement of x_i. If $x \geq y \geq z$ with $x, z \in X$. Prop. 40 yields $x/z = x/u + u/v + v/z$, where $x/y = u/v$ and $y/z = x/u + v/z$, with $u, v \in X$. So μ' is well defined, and

$$\mu'(a+b) = \mu'(a) + \mu'(b)$$

for disjoint $a, b \in B(X)$. In $B(X)$ we have $a \cdot b = 1 \smallsetminus (a \smallsetminus b) = (a \smallsetminus b)^c$. Hence

$$\mu(a) := \mu'(a^c)$$

gives the desired (unique) measure. $\square$

We call μ the <u>canonical measure</u> of X. The group $\pi_1(X) := G(\mu)$ will be called the <u>fundamental group</u> of X.

<u>Prop. 42.</u> Let X be an MV-algebra, and $x \geq y$ in X. There is a μ-invariant lattice isomorphism $B([x \cdot y, 1]) \xrightarrow{\sim} [(x \smallsetminus y)^c, 1]_{B(X)}$.

<u>Pf.</u>

$$\begin{array}{ccccccccc} [x \cdot y, 1]_X & \xrightarrow{\sim} & [y, x]_X & \hookrightarrow & [y, x]_{B(X)} & \xrightarrow{\sim} & [0, x \smallsetminus y]_{B(X)} & \xrightarrow{\sim} & [(x \smallsetminus y)^c, 1]_{B(X)} \\ z & \longmapsto & zx & \longmapsto & zx & \longmapsto & zx \smallsetminus y & \longmapsto & (zx \smallsetminus y) \vee (x \smallsetminus y)^c \end{array}$$

is a lattice embedding. Since $1 \cdot z = z = x \cdot zx$, this is μ-invariant. $\square$

<u>Prop. 43.</u> Let X be an MV-algebra, and $a, b \in B(X)$ with $\mu(a) = \mu(b)$. Then there is a μ-invariant lattice isomorphism $[a, 1]_{B(X)} \xrightarrow{\sim} [b, 1]_{B(X)}$.

<u>Pf.</u> Assume first that $a = (x \smallsetminus y)^c$ and $b = (z \smallsetminus t)^c$ with $x \geq y$ and $z \geq t$ in X. $\rightsquigarrow$ $x \cdot y = \mu(a) = \mu(b) = z \cdot t$. $\overset{\text{Prop. 42}}{\rightsquigarrow}$ $[a, 1]_{B(X)} \cong B([x \cdot y, 1]) = B([z \cdot t, 1]) \cong [b, 1]_{B(X)}$. By symmetry, it is enough to find a μ-invariant lattice isomorphism

$$B([\mu(a), 1]) \xrightarrow{\sim} [a, 1]_{B(X)}.$$

For disjoint $x, y \in X$ we have $0 = (1 \smallsetminus x) \wedge (1 \smallsetminus y) = (x \vee y)^c$, that is $x \vee y = 1$.
So $(x \cdot y) \cdot y = 1$, which gives $x \cdot y \le y \le x \cdot y$. Since $x \wedge y = (y \cdot x) y$, this gives

$$x \vee y = 1 \iff x \cdot y = y \iff y \cdot x = x \iff xy = x \wedge y.$$

$\rightsquigarrow [xy, 1] = [x, 1] \times [y, 1]. \rightsquigarrow \underline{B([xy, 1]) = B([x, 1]) \times B([y, 1])}.$

To establish $B([\mu(a), 1]) \xrightarrow{\sim} [a, 1]_{B(X)}$, we proceed by induction. For $n = 1$, Prop. 42 gives $B([\mu(a), 1]) = B([x_1 \cdot x_2, 1]) \cong [(x \smallsetminus y)^c, 1]_{B(X)} = [a, 1]_{B(X)}$.
Now let the isomorphism be valid for some n, and assume that $b^c = a^c + (x \smallsetminus y)$.
$\rightsquigarrow \mu(b) = \mu(a)(x \cdot y)$. $\rightsquigarrow B([\mu(b), 1]) = B([\mu(a), 1]) \times B([x \cdot y, 1]) \cong [a, 1]_{B(X)} \times [(x \smallsetminus y)^c, 1]_{B(X)} = [b, 1]_{B(X)}$. By construction the lattice isomorphism is μ-invariant. □

<u>Thm. 11.</u> Let X be an MV-algebra. The canonical measure $\mu : B(X) \twoheadrightarrow X$ is a strong covering.

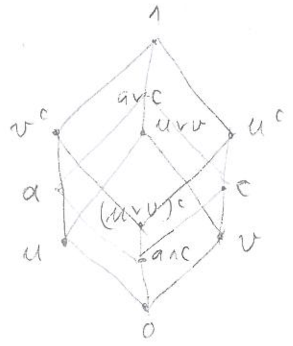
<u>Pf.</u> By construction of μ, the lattice embedding $i : X \hookrightarrow B(X)$ satisfies $\mu i = 1_X$. To verify (G1), let $a \geqslant b$ and $c \geqslant d$ in $B(X)$ with $\mu(a) = \mu(c)$ and $\mu(b) = \mu(d)$ be given. The connecting $\gamma \in G(\mu)$ can be obtained in two steps:

$$\begin{array}{ccccc} a & \xmapsto{\alpha} & \alpha(a) & \xmapsto{\beta} & c \\ \geqslant & & \geqslant & & \geqslant \\ b & \xmapsto{\alpha} & d & \xmapsto{\beta} & d \end{array}$$

So we can assume that $b = d$:
$$\begin{array}{ccc} a & & c \\ & \diagdown \;\; \diagup & \\ & b & \end{array}$$
For $u := a \smallsetminus c$ and $v := c \smallsetminus a$, we have $u \wedge v = 0$. Since $u \wedge a \wedge c = v \wedge a \wedge c = 0$, we have $\mu'(u) + \mu'(a \wedge c) = \mu'(a) = \mu'(b) = \mu'(v) + \mu'(a \wedge c)$. $\rightsquigarrow \mu(u^c) = \mu(v^c)$ and $u^c \vee v^c = 1$. $\xrightarrow{\text{Prop 43}} \exists$ μ-invariant $\alpha : [u^c, 1]_{B(X)} \xrightarrow{\sim} [v^c, 1]_{B(X)}$. Define

$$\begin{tikzcd}[column sep=small] \beta : B(X) = {[u^c, 1]_{B(X)}} \arrow[rrrr, bend right, "\alpha"'] & \times & {[v^c, 1]_{B(X)}} \arrow[rr, bend right, "\alpha^{-1}"'] & \times & {[u \vee v, 1]_{B(X)}} \arrow[rrrr, bend left, "1"] & \xrightarrow{\sim} & {[u^c, 1]_{B(X)}} & \times & {[v^c, 1]_{B(X)}} & \times & {[u \vee v, 1]_{B(X)}} = B(X) \end{tikzcd}$$

$\rightsquigarrow \beta \in G(\mu)$. Since $b \le a \wedge c \le u^c \wedge v^c = (u \vee v)^c$, we have $\beta(b) = b$ and



$\beta(a \wedge c) = a \wedge c$. Furthermore, $a = (a \wedge c) \vee u$, and $c = (a \wedge c) \vee v$. So $\beta(u) = v$ and $\beta(v) = u$ gives $\beta(a) = c$.
$\rightsquigarrow$ (G1). Hence μ is a strong covering. □

Next we describe an MV-algebra as a partial L-algebra.

Prop. 44. An MV-algebra is equivalent to a bounded lattice with a partial operation $x \cdot y$, defined for $x \geq y$, satisfying

(a) $\forall x \geq y \geq z:\ x \cdot z \leq x \cdot y$ and $(x \cdot y) \cdot (x \cdot z) = y \cdot z$

(b) $\forall x \geq y, z:\ x \cdot (y \wedge z) = x \cdot y \wedge x \cdot z$

(c) $(x \vee y) \cdot y = x \cdot (x \wedge y)$

(d) $\forall x \geq y:\ y \leq x \cdot y$ and $(x \cdot y) \cdot y = x$.

Pf. The necessity is obvious. Assume that (a)–(d) holds. Define

$$x \cdot y := x \cdot (x \wedge y).$$

$\rightarrow (x \cdot y) \cdot (x \cdot z) = (x \cdot (x \wedge y)) \cdot (x \cdot (x \wedge z)) \overset{(b)}{=} (x \cdot (x \wedge y)) \cdot (x \cdot (x \wedge y \wedge z)) \overset{(a)}{=} (x \wedge y) \cdot (x \wedge y \wedge z)$.

$\rightarrow$ (L). For $y = 1$, (c) gives $1 \cdot 1 = x \cdot x \xrightarrow{(d)} 1 \leq 1 \cdot 1 = x \cdot x$. $\rightarrow$ $\underline{x \cdot x = 1}$. $\rightarrow$

$x \cdot 1 = x \cdot (x \wedge 1) = 1$. By (d), we have $x \leq 1 \cdot x$ and $(1 \cdot x) \cdot x = 1$. $\rightarrow 1 \cdot x \leq x$.

$\rightarrow$ $\underline{1 \cdot x = x}$. So 1 is a logical unit. Assume that

$x \cdot y = y \cdot x = 1$. $\rightarrow x \cdot (x \wedge y) = 1. \xrightarrow{(d)} x = (x \cdot (x \wedge y)) \cdot (x \wedge y) = 1 \cdot (x \wedge y) = x \wedge y \leq y$.

$\rightarrow x = y$. So X is an L-algebra. $\xrightarrow{(d)} x \leq (x \vee y) \cdot x \overset{(c)}{=} y \cdot (x \wedge y) = y \cdot x$. $\rightarrow$ (K).

Furthermore, $(x \cdot y) \cdot y = (x \cdot (x \wedge y)) \cdot y \overset{(c)}{=} ((x \vee y) \cdot y) \cdot y \overset{(d)}{=} x \vee y$. $\rightarrow$ X is MV-algebra. □

Thm. 12. Let X be an MV-algebra, and let G be a closed subgroup of $G_0(X)$. Then $X \twoheadrightarrow X/G$ is a covering of MV-algebras.

Pf. Let X/G denote the set of orbits of G on X. For $a, b \in X/G$, define

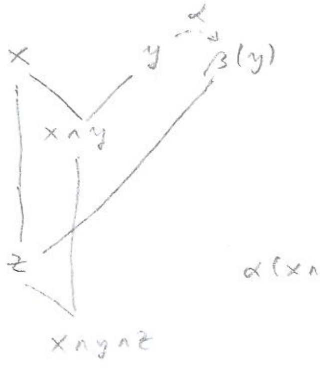
$$a \leq b :\Longleftrightarrow \exists x \in a, y \in b:\ x \leq y.$$

Then (G1) implies that for $u \in a$ and $v \in b$,

$$u \leq v \Longleftrightarrow \exists \alpha \in G: u = \alpha(x),\ v = \alpha(y),$$

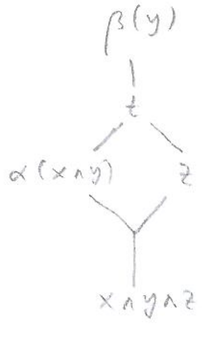
i.e. the pairs $x \in a, y \in b$ with $x \leq y$ form an orbit of G. $\rightarrow$ X/G is a poset.
The antisymmetry follows since all $x \in a$ are pairwise incomparable: If $x \leq y$ in a, then $\exists \alpha \in G:\ \alpha(x) = x,\ \alpha(y) = x \rightarrow x = y$. To show that X/G is a lattice, consider $a, b \in X/G$ with $x \in a$ and $y \in b$. By (G2), y can be chosen so that $x \wedge y$ is maximal. Let $c \in X/G$ satisfy $c \leq a, b$. Choose $z \in c$ with $z \leq x$. $\rightarrow$

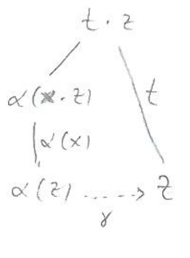
$\exists \beta \in G: z \leq \beta(y)$. $\rightarrow \exists \alpha \in G: \alpha(y) = \beta(y)$ and $\alpha(x \wedge y \wedge z) = x \wedge y \wedge z$.



We set $t := \alpha(x \wedge y) \vee z$. $\rightarrow \alpha(x \cdot z) \leq \alpha((x \wedge y) \cdot z) =$

$\alpha((x \wedge y) \cdot (x \wedge y \wedge z)) = \alpha(x \wedge y) \cdot (x \wedge y \wedge z) \leq t \cdot z$. $\rightarrow$

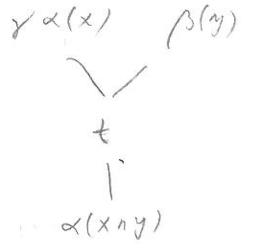




$\exists \gamma \in G: \gamma(t \cdot z) = t \cdot z$ and $\gamma \alpha(z) = z$.

$\rightarrow t \leq \gamma\alpha(x)$. $\rightarrow$

$\alpha(x \wedge y) \leq t \leq \gamma\alpha(x) \wedge \beta(y)$. $\rightarrow$



$\rightsquigarrow \exists \delta \in G$: $\delta\gamma\alpha(x) = x$ and $\delta\alpha(x\wedge y) = x\wedge y$. $\rightsquigarrow x \wedge y \leq \delta(t) \leq$ $x \wedge \delta\beta(y)$. $\overset{x\wedge y \text{ max.}}{\rightsquigarrow} \delta(z) \leq \delta(t) = x\wedge y$. $\rightsquigarrow c = a \wedge b$ in X/G.

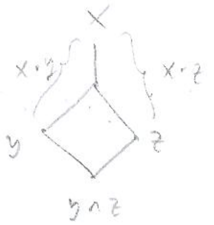
By symmetry, this shows that X/G is a lattice.

For $a \geq b$ in X/G, choose $x \in a$ and $y \in b$ with $x \geq y$. Let $z \in a$, $t \in b$ with $z \geq t$. By $(G\wedge)$ there is an $\alpha \in G$ with $\alpha(x) = z$ and $\alpha(y) = t$. $\rightsquigarrow \alpha(x\cdot y) = z\cdot t$.

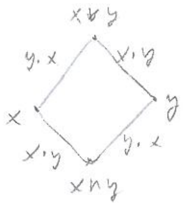
So
$$a\cdot b := x\cdot y$$
is well defined. By Prop. 44, to show that X/G is an MV-algebra, we only have to verify

(b) $a \geq b, c \implies a\cdot(b\wedge c) = a\cdot b \wedge a\cdot c$

(c) $(a\vee b)\cdot b = a\cdot(a\wedge b)$.

b) 

$y\wedge z$ maximal in $y \iff x\cdot y \wedge x\cdot z$ maximal.

c) 

If $x \in a$, $y \in b$, then $x\wedge y \in a\wedge b \iff x\vee y \in a\vee b$.

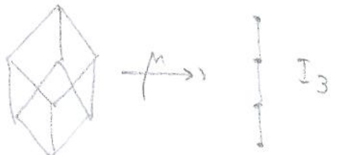
$\rightsquigarrow X/G$ is an MV-algebra.

By definition $X \twoheadrightarrow X/G$ is a covering. $\square$

<u>Problem.</u> When is $X \twoheadrightarrow X/G$ a strong covering?

<u>Corollary.</u> Every MV-algebra admits a canonical representation
$$X \cong B(X)/\pi_1(X).$$

<u>Example.</u> For a MV-chain I_n of length n, $\pi_1(I_n) = S_n$



Here $G_0(B_2^n) = S_n$, which gives a correspondence between finite Boolean algebras and finite chains.

<u>Problems.</u>
- Characterise the MV-algebras with abelian fundamental group.
- Is $X/G_0(X)$ a chain?
- In what sense is the covering $B(X) \twoheadrightarrow X$ universal?
- Which Boolean algebras A can be recovered from $A/G_0(A)$.
- Correspondence between MV-chains and Boolean algebras?
- How do the groups $G_0(A)$ look like?