Similar to Hilbert algebras, commutative L-algebras can be derived from logic. Both are BCK-algebras. Their logic is given by the axioms

$$(B)\quad (p\cdot q)\cdot((r\cdot p)\cdot(r\cdot q))$$
$$(C)\quad (p\cdot(q\cdot r))\cdot(q\cdot(p\cdot r))$$
$$(K)\quad p\cdot(q\cdot p)$$

and the rule (MP) $p,\ p\cdot q \vdash q$.

Using (C), axiom (B) is equivalent to

$$(B')\quad (r\cdot p)\cdot((p\cdot q)\cdot(r\cdot q)).$$

Hence

$$p \le q \;:\Longleftrightarrow\; \vdash p\cdot q$$

is transitive: $r \le p \le q \overset{(B')}{\Longrightarrow} \vdash (p\cdot q)\cdot(r\cdot q) \vdash r\cdot q \Rightarrow r \le q$. Furthermore, (C) with $r=p$ gives $\vdash (p\cdot(q\cdot p))\cdot(q\cdot(p\cdot p)) \overset{(K)}{\vdash} q\cdot(p\cdot p) \vdash p\cdot p$. So $p \le p$, and

$$p \equiv q \;:\Longleftrightarrow\; p \le q \le p$$

is an equivalence relation. By (B) and (B'), it is a congruence relation. So, again, we can form the Lindenbaum algebra over the variable set V:

$$L(V) := P/\equiv$$

with the greatest element 1, the class of true formulae. Now $L(V)$ satisfies

$$(B)\quad p\cdot q \le (r\cdot p)\cdot(r\cdot q)$$
$$(C)\quad p\cdot(q\cdot r) = q\cdot(p\cdot r)$$
$$(K)\quad p \le q\cdot p$$

and $1 = p\cdot p = q\cdot 1$ (since $\vdash q\cdot(p\cdot p)$). By (C), we have

$$p \le q\cdot r \Longleftrightarrow q \le p\cdot r.$$

Hence $1\cdot p \le 1\cdot p$ gives $1 \le (1\cdot p)\cdot p$. Thus $1\cdot p \le p \overset{(K)}{\le} 1\cdot p$. So <u>$L(V)$ is a BCK-algebra</u>, i.e. BCK-algebras are the semantics of BCK-logic.

<u>Prop. 31.</u> An L-algebra X satisfying (C) is a BCK-algebra.

<u>Pf.</u> For $r = p$, (C) gives $p\cdot(q\cdot p) = q\cdot(p\cdot p) = q\cdot 1 = 1$. $\rightsquigarrow$ (K). $\rightsquigarrow$ $p\cdot q \overset{(K)}{\le} (p\cdot r)\cdot(p\cdot q) = (r\cdot p)\cdot(r\cdot q)$. $\rightsquigarrow$ (B). $\square$

So we could speak of a BCKL-algebra or a CL-algebra. We choose a middle course and call these algebras <u>CKL-algebras</u>. They are also known as HBCK-algebras.

It was conjectured by Wroński (1985) that CKL-algebras form a variety (proved by Blok and Ferreirim 2001). Indeed, the axiom $x\cdot y = y\cdot x = 1 \Rightarrow x = y$ can be replaced by Cornish's identity

$$(J) \qquad (((x\cdot y)\cdot y)\cdot x)\cdot x = (((y\cdot x)\cdot x)\cdot y)\cdot y.$$

<u>Prop. 32.</u> Let X be an L-algebra. ◯

(a) X is a CKL-algebra

(b) $x \le y\cdot z \Rightarrow y \le x\cdot z$

(c) X is a KL-algebra with $x \le (x\cdot y)\cdot y$ for all $x, y \in X$.

<u>Pf.</u> (a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (c): $x\cdot y \le x\cdot y$. $\rightsquigarrow$ $x \le (x\cdot y)\cdot y$. For $z = y$, we get (K).

(c) $\Rightarrow$ (a): $a := (x\cdot(y\cdot z))\cdot(x\cdot z) = ((y\cdot z)\cdot x)\cdot((y\cdot z)\cdot z) \overset{(K)}{\geq} (y\cdot z)\cdot z \geq y$. $\rightsquigarrow$

$x\cdot(y\cdot z) \le a\cdot(x\cdot z) \overset{(K)}{\le} (a\cdot y)\cdot(a\cdot(x\cdot z)) = (y\cdot a)\cdot(y\cdot(x\cdot z)) = y\cdot(x\cdot z)$. $\square$

<u>Cor.</u> Every commutative L-algebra X is a CKL-algebra.

<u>Pf.</u> By Prop. 8, X satisfies (K) and (V): $(x\cdot y)\cdot y = (y\cdot x)\cdot x$ $(= x\vee y)$. So $x \overset{(K)}{\le}$ $(y\cdot x)\cdot x = (x\cdot y)\cdot y$. $\square$

Recall that an <u>MV-algebra</u> is a commutative L-algebra with 0. With

$$x' := x\cdot 0$$

every MV-algebra has a multiplication

$$xy := (x\cdot y')', \text{ i.e. } x\cdot y = (xy')'. \quad \left(\text{In } S(X): (x\cdot y')' = xy \vee 0\right)$$

<u>Prop. 33.</u> Every MV-algebra X is a commutative monoid and a lattice with $x'' = x$ and

$$x'\cdot y' = y\cdot x$$
$$xy\cdot z = x\cdot(y\cdot z)$$
$$x\wedge y = (x\cdot y)x \ ; \ x\vee y = (x\cdot y)\cdot y$$

<u>Pf.</u> $x'' = (x\cdot 0)\cdot 0 = (0\cdot x)\cdot x = 1\cdot x = x$. $\rightsquigarrow$ $x'\cdot y' = x'\cdot(y\cdot 0) = y\cdot(x'\cdot 0) = y\cdot x$. $\rightsquigarrow$ $xy = (x\cdot y')' = (x''\cdot y')' = (y\cdot x')' = yx$, and $xy\cdot z = (x\cdot y')'\cdot z = z'\cdot(x\cdot y') = x\cdot(z'\cdot y') = x\cdot(y\cdot z)$. $\rightsquigarrow$ $(xy)z = (xy\cdot z')' = (x\cdot(y\cdot z'))' = (x\cdot(yz)')' = x(yz)$. Furthermore, $1x = (1\cdot x')' = x'' = x$. So X is a monoid. The equation $x'\cdot y' = y\cdot x$ implies that $x' \le y' \Leftrightarrow y \le x$. So $x\wedge y = (x'\vee y')' = ((x'\cdot y')\cdot y')' = (x'\cdot y')y = (y\cdot x)y$. $\square$

Next we show that the elements of an MV-algebra can be regarded as measurable sets. To say that two sets A, B have <u>the same measure</u> means to split A into small pieces, translate them with a group and reassemble them to B.

Def. A <u>measure</u> $\mu : X \to Y$ between L-algebras X, Y is a map with
$$x \geq y \implies \mu(x \cdot y) = \mu(x) \cdot \mu(y),\ \mu(x) \geq \mu(y).$$

Thus $x = y$ gives $\mu(1) = 1$. With $X \subset S(X)$ we get

Prop. 34. $\mu : X \to Y$ is a measure iff
$$\boxed{\exists\, xy \in X \implies \mu(xy) = \mu(x)\mu(y).}$$

Pf. In $S(X)$: $xy \leq z \iff x \leq y \cdot z$, and $z \leq xy \iff ((y \cdot z) \cdot x)(z \cdot y) = 1 \iff y \cdot z \leq x$ and $z \leq y$. So
$$xy = z \iff (x = y \cdot z \text{ and } y \geq z).$$

So the condition of Prop. 34 is equivalent to
$$x = y \cdot z,\ y \geq z \implies \mu(x) = \mu(y) \cdot \mu(z),\ \mu(y) \geq \mu(z). \qquad \square$$

In additive terms: $\exists\, x + y \implies \mu(x+y) = \mu(x) + \mu(y)$. (finite measure)

Now let X be an MV-algebra. By Thm. 3, $G(X)$ is a commutative ℓ-group with $X = [0, 1] \subset S(X) = G(X)^-$, and $x \vee y = (x \cdot y) \cdot y$.

Cor. Let X, Y be MV-algebras. Every measure $\mu : X \to Y$ extends uniquely to a group homomorphism $G(\mu) : G(X) \to G(Y)$, and every group homomorphism $f : G(X) \to G(Y)$ with $f(X) \subset Y$ restricts to a measure $\mu : X \to Y$.

Pf. Let $\mu : X \to Y$ be a measure. Then
$$(1) \qquad \mu(ab) = \mu(a)\mu(b)$$
holds for $0 \leq ab$ in $S(X)$. Every $a \in S(X)$ satisfies $0^n \leq a$ for some $n \in \mathbb{N}$. If $0^{n+1} \leq a$, then $0^n \leq 0 \cdot a$, and $a0 = (a \cdot 0)'(a \cdot 0)a = (a \cdot 0)'(0 \cdot a)0$. so
$$a = (a \cdot 0)'(0 \cdot a) \quad \text{with } (a \cdot 0)' \in X.$$

Thus, recursively, we define
$$(2) \qquad \mu(a) := \mu((a \cdot 0)')\mu(0 \cdot a).$$

Assume that (1) holds for $0^n \leq a, b \leq 1$. We show that for $x, y \in X$ with $xa = yb$,
$$(3) \qquad \mu(x)\mu(a) = \mu(y)\mu(b).$$

Now $xa = yb \iff z := (a \cdot b) \cdot x = (b \cdot a) \cdot y$ with $x \leq a \cdot b$, $y \leq b \cdot a$. So $z(a \cdot b) = a \cdot b \wedge x = x$, and similarly, $z(b \cdot a) = y$. $\rightsquigarrow \mu(z)\mu(a \cdot b) = \mu(x)$, $\mu(z)\mu(b \cdot a) = \mu(y)$. So we have to prove: $\mu(a \cdot b)\mu(a) \overset{!}{=} \mu(b \cdot a)\mu(b)$, which holds by induction.

Now we extend (3) to $0^{n+1} \leq xa$, that is, $\mu(x)\mu(a) \overset{!}{=} \mu(xa)$. By (2), i.e.
$$\mu(x)\mu((a \cdot 0)')\mu(0 \cdot a) \overset{!}{=} \mu((xa \cdot 0)')\mu(0 \cdot xa)$$

Since $y := a\cdot 0 \in X$, $0\cdot xa = (y\cdot x)(0\cdot a) \rightsquigarrow \mu(0\cdot xa) = \mu(y\cdot x)\mu(0\cdot a)$. So

$$\mu(x)\mu(y') \overset{!}{=} \mu((x\cdot y)')\mu(y\cdot x).$$

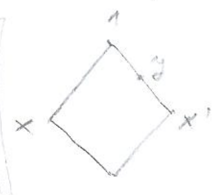
Indeed, $xy'y = x0 = 0x = (x\cdot y)'(x\cdot y)x = (x\cdot y)'(y\cdot x)y \rightsquigarrow xy' = (x\cdot y)'(y\cdot x)$.
The converse follows by Prop. 34. $\square$

For an MV-algebra X, and $x, y \in X \subset S(X)$,

$$xy \in X \iff 0 \le xy \iff 0\cdot xy = 1 \iff 1 = (y'\cdot x)(0\cdot y) = y'\cdot x. \rightsquigarrow$$

$$\boxed{xy \in X \iff y' \le x \iff x' \le y}$$

<u>Example.</u> If X is a Boolean algebra, then $xy \in X \iff x' \le y \iff x \vee y = 1$.



Let $I = [0,1]$, additively, with opposite ordering. $\rightarrow$
$S(I) = \mathbb{R}_{\ge 0}$, $G(I) = \mathbb{R}$, with logical unit 0. For $x, y \in I$,
$x' = 1-x$, and $x\cdot y = y - x$ if $x \le y$. (For $x \geq y$, we have $x\cdot y = 0$)
Let $\mathcal{B}(I)$ be the Boolean algebra generated by $\mathcal{O}(I)$ (<u>Borel sets</u>).
So $x+y \in I$ iff $x+y \le 1$, and in $\mathcal{B}(I)$: $A+B \in \mathcal{B}(I) \iff X \setminus A \supset B \iff A \cap B = \emptyset$.
$\xrightarrow{\text{Prop. 34}}$ A measure $\mu : \mathcal{B}(I) \longrightarrow I$ is a map with

$$\mu(A \sqcup B) = \mu(A) + \mu(B).$$

Note that there are two extreme cases of MV-algebras: Boolean algebras (like $\mathcal{B}(I)$) and <u>MV-chains</u>, i.e. linearly ordered MV-algebras (like I).

<u>Prop. 35.</u> Let X be an MV-algebra, and $I \subset X$. ⊖

(a) I is an (L-algebra) ideal.
(b) $1 \in I$, and $x, x\cdot y \in I \Rightarrow y \in I$.
(c) $I \overset{\neq\emptyset}{}$ is an upper set, and $x, y \in I \Rightarrow xy \in I$
(d) $I \neq \emptyset$ is an upper set, and $x, y \in I$, $x' \le y \Rightarrow xy \in I$.

Pf. (a) $\Rightarrow$ (b) is trivial, (b) $\Rightarrow$ (a) follows since X is a CKL-algebra (Prop. 32).
(b) $\Rightarrow$ (c): If $x, y \in I$, then $y\cdot xy = y\cdot(x\cdot y')' = (x\cdot y')\cdot y' = x \vee y' \geq x$. $\rightsquigarrow y, y\cdot xy \in I$.
$\rightsquigarrow xy \in I$. (c) $\Rightarrow$ (d) is trivial.
(d) $\Rightarrow$ (b): If $x, x\cdot y \in I$, then $y \geqslant x \wedge y = (x\cdot y)x \in I$. $\square$

For a measure $\mu : X \to Y$, we call

$$\operatorname{Ker}\mu := \{x \in X \mid \mu(x) = 1\}$$

the <u>kernel</u> of μ.

<u>Prop. 36.</u> Let μ be a measure of MV-algebras. Then $\operatorname{Ker}\mu$ is an ideal, and μ factors through $X \twoheadrightarrow X/\operatorname{Ker}\mu$.

Pf. If $x, x\cdot y \in \mathrm{Ker}\,\mu$, then $\mu(x\wedge y) = \mu(x\cdot y)\mu(x) = 1 \rightsquigarrow \mu(y) \geq \mu(x\wedge y) = 1$.

If $x \equiv y \pmod{\mathrm{Ker}\,\mu}$, then $x\cdot y, y\cdot x \in \mathrm{Ker}\,\mu$. $\rightsquigarrow \mu(x) = \mu(x\cdot y)\mu(x) = \mu(y\cdot x)\mu(y) = \mu(y)$. So we have a factorization $\mu : X \twoheadrightarrow X/\mathrm{Ker}\,\mu \longrightarrow Y$. $\square$

For an MV-algebra X, let $G_0(X)$ be the group of invertible measures $\mu : X \to X$, a subgroup of $G(X)$.

<u>Def.</u> Let $\mu : X \longrightarrow Y$ be a measure of MV-algebras. We call μ <u>pure</u> if $\mathrm{Ker}\,\mu = \{1\}$.
We call $\mu : X \twoheadrightarrow Y$ a <u>weak covering</u> if the group

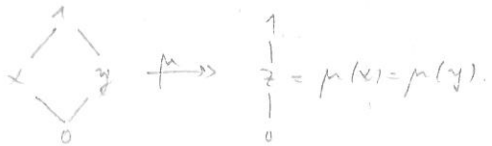
$$G(\mu) := \{\alpha \in G_0(X) \mid \mu\alpha = \mu\}$$

acts transitively on the fibres $\mu^{-1}(y)$, $y \in Y$. If $G := G(\mu)$ also satisfies

(G1) $\forall\, x \leq y$ in X, $\alpha, \beta \in G$, $\alpha x \leq \beta y\ \exists\, \gamma \in G : \gamma x = \alpha x,\ \gamma y = \beta y$.

we call μ a <u>covering.</u> If, in addition, there is a lattice embedding $i : Y \hookrightarrow X$ with $\mu i = 1_Y$, we call μ a <u>strong covering.</u>

Since $\alpha(1) = 1$ for all $\alpha \in G_0(X)$, every weak covering is pure.

<u>Example:</u> $X = \mathbb{B}^2$: 

is a strong covering. $G_0(X) = G(\mu) = \{1, \tau\}$ with $\tau : x \leftrightarrow y$.

<u>Problems.</u> • Are the implications strong cov. $\Rightarrow$ covering $\Rightarrow$ weak cov. strict?
• Let $\mu : X \twoheadrightarrow Y$ be a strong covering with $i : Y \hookrightarrow X$. Does Y generate X as an L-algebra with 0?

<u>Prop. 37.</u> Let $\mu : X \twoheadrightarrow Y$ be a strong covering. Then $G := G(\mu)$ satisfies

(G2) For $x, y \in X$ there exists a maximal $z := x \wedge \gamma(y) \geq x \wedge y$ with $\gamma \in G$.

<u>Pf.</u> Consider $t := \mu(x) \wedge \mu(y) \geq \mu(x\wedge y)$. $\rightsquigarrow i\mu(x) \wedge i\mu(y) = i(t) \geq i\mu(x\wedge y)$.
$\overset{(G1)}{\rightsquigarrow}\ \exists\, \alpha \in G : \alpha(i\mu(x)) = x,\ \alpha(i\mu(x\wedge y)) = x \wedge y$. $\rightsquigarrow z := \alpha i(t)$ satisfies $x \wedge y \leq z \leq x \wedge \alpha(i\mu(y))$. Choose $\beta \in G$ with $i\mu(y) = \beta(y)$. With $\gamma := \alpha\beta \rightsquigarrow \alpha(i\mu(y)) = \gamma(y)$. Suppose that $z \leq x \wedge \delta(y)$ for some $\delta \in G$. $\rightsquigarrow t = \mu(z) \leq \mu(x \wedge \delta(y)) \leq \mu(x) \wedge \mu(y) = t$. $\rightsquigarrow \mu(z) = \mu(x \wedge \delta(y)) \rightsquigarrow \exists\, \sigma \in G : \sigma(x \wedge \delta(y)) = z \rightsquigarrow \sigma((x\wedge\delta(y))\cdot z) = 1 \rightsquigarrow (x \wedge \delta(y))\cdot z = 1 \rightsquigarrow x \wedge \delta(y) = z$. $\rightsquigarrow z = x \wedge \gamma(y)$ maximal. $\square$

<u>Def.</u> Let X be an MV-algebra. We call a subgroup G of $G_0(X)$ <u>closed</u> if G satisfies (G1) and (G2).

Our next aim is to construct a universal covering of an MV-algebra and a corresponding „Galois theory".

Prop. 38. Let X be an MV-algebra.

(a) $y \mapsto x \cdot y$ is a lattice homomorphism $X \to X$.

(b) $x \mapsto x \cdot y$ is a lattice homomorphism $X^{op} \to X$.

(c) X is a distributive lattice.

Pf. We embed X into $G(X)$. Then $x \cdot y = y x^{-1} \wedge 1$.

a) $x \cdot (y \wedge z) = (y \wedge z) x^{-1} \wedge 1 = y x^{-1} \wedge z x^{-1} \wedge 1 = x \cdot y \wedge x \cdot z$. $\rightsquigarrow$

$$(x \vee y) \wedge (x \vee z) = (x \cdot y) \cdot y \wedge (x \cdot z) \cdot z \overset{(K)}{\leq} \Big[\big((x \cdot y) \cdot (x \cdot z)\big) \cdot \big((x \cdot y) \cdot y\big)\Big] \wedge \Big[\underbrace{\big((x \cdot z) \cdot (x \cdot y)\big) \cdot \big((x \cdot z) \cdot z\big)}_{\big((x \cdot y) \cdot (x \cdot z)\big) \cdot \big((x \cdot y) \cdot z\big)}\Big]$$

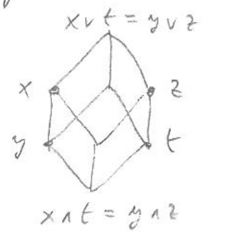
$$= (x \cdot y \wedge x \cdot z) \cdot y \wedge (x \cdot y \wedge x \cdot z) \cdot z = \big(x \cdot (y \wedge z)\big) \cdot (y \wedge z) = x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z). \rightsquigarrow (c).$$

$$\rightsquigarrow \big(x \cdot (y \vee z)\big) x = x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) = (x \cdot y) x \vee (x \cdot z) x = (x \cdot y \vee x \cdot z) x \rightsquigarrow (a).$$

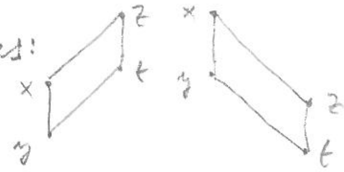
b) $(x \vee y) \cdot z = z' \cdot (x \vee y)' = z' \cdot (x' \wedge y') = z' \cdot x' \wedge z' \cdot y' = x \cdot z \wedge y \cdot z$

$(x \wedge y) \cdot z = z' \cdot (x \wedge y)' = z' \cdot (x' \vee y') = z' \cdot x' \vee z' \cdot y' = x \cdot z \vee y \cdot z$ $\square$

Def. Let D be a distributive lattice. We call two intervals $[y, x]$ and $[t, z]$ *equivalent* if $x \vee t = y \vee z$ and $x \wedge t = y \wedge z$:



special cases:

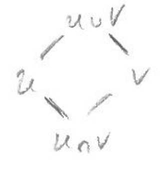


The equivalence class of $[y, x]$ will be denoted by x/y.

Note: If X is an MV-algebra, $\boxed{x/y = z/t \Longrightarrow x \cdot y = z \cdot t}$.

Indeed, $(x \vee y) \cdot y = \big((x \cdot y) \cdot y\big) \cdot y = x \cdot y = x \cdot (x \wedge y)$, where $\big((x \cdot y) \cdot y\big) \cdot y = x \cdot y$ follows from $x \leq (x \cdot y) \cdot y \rightsquigarrow \big((x \cdot y) \cdot y\big) \cdot y \leq x \cdot y \leq \big((x \cdot y) \cdot y\big) \cdot y$.

Let $B(D)$ be the Boolean algebra generated by D. It can be realised as $\mathfrak{B}(X)$ if $X = \operatorname{Spec} D$ and thus $D = \mathfrak{D}(X)$. For $U, V \in \mathfrak{D}(X)$ we have



$(U \cup V) \setminus V = U \setminus V = U \setminus (U \cap V)$. Conversely, assume that $U_1 \supseteq U_2$ and $V_1 \supseteq V_2$. Then $U_1 \setminus U_2 = V_1 \setminus V_2$ implies $U_1 = U_2 \cup (U_1 \setminus U_2) = U_2 \cup (V_1 \setminus V_2)$. $\rightsquigarrow U_1 \cup V_2 = U_2 \cup V_1$ and $U_1 \cap V_2 = U_2 \cap V_2$. Thus, by symmetry, $U_1 \cap V_2 = U_2 \cap V_1$. Hence

$$U_1 / U_2 = V_1 / V_2 \iff U_1 \setminus U_2 = V_1 \setminus V_2.$$

Denote the complement of $a \in B(D)$ by a^c. So we obtain for $x_1 \geq x_2$, $y_1 \geq y_2$ in D:

$$x_1 / x_2 = y_1 / y_2 \iff x_1 \wedge x_2^c = y_1 \wedge y_2^c \text{ in } B(D).$$

Prop. 39. Let D be a distributive lattice. Every $a \in B(D)$ is a disjoint union

$$a = (x_1 \wedge y_1^c) \vee \dots \vee (x_n \wedge y_n^c), \quad x_i, y_i \in D. \qquad (1)$$

Pf. $(x \wedge y^c) \wedge (z \wedge t^c)^c = x \wedge y^c \wedge (z^c \vee t) = (x \wedge y^c \wedge z^c) \vee (x \wedge y^c \wedge t) = (x \wedge (y \vee z)^c) \vee (x \wedge t \wedge y^c)$,

disjoint since $z^c \wedge t \leq z^c \wedge z = 0$. Furthermore,

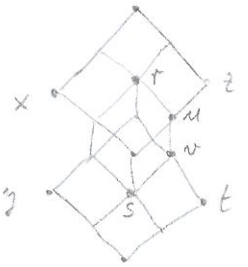
$$(x \wedge y^c) \wedge (z \wedge t^c) = (x \wedge z) \wedge (y \vee t)^c. \qquad (2)$$

Thus, if $a = a_1 \vee \dots \vee a_n$ is disjoint with $a_i = x_i \wedge y_i^c$ and $b = x \wedge y^c$, then $a \wedge b^c = (a_1 \wedge b^c) \vee \dots \vee (a_n \wedge b^c)$ is in normal form, and also $a \vee b = (a \wedge b^c) \vee b$. □

Replacing y_i by $x_i \wedge y_i$, we can assume that $y_i < x_i$ (removing zero terms). So the $x_i \setminus y_i := x_i \wedge y_i^c$ correspond to quotients x_i / y_i. Eq. (2) gives

$$(x \setminus y) \wedge (z \setminus t) = 0 \iff x \wedge z \leq y \vee t.$$

The sublattice generated by x, y, z, t is



So u/v corresponds to $(x \setminus y) \wedge (z \setminus t)$. We call x/y and z/t *disjoint* if $x \wedge z \leq y \vee t$. (i.e. u/v trivial). $u = (x \wedge z) \vee t$; $v = (y \wedge z) \vee t$.

If $x \setminus y$ and $z \setminus t$ are disjoint, we write $(x \setminus y) + (z \setminus t) := (x \setminus y) \vee (z \setminus t)$. Accordingly, we have a *sum* $x/y + z/t$ of disjoint quotients. So (1) can be written as

$$a = (x_1 \setminus y_1) + \dots + (x_n \setminus y_n). \qquad (3)$$

If x/y and z/t are disjoint, we have $x/y + z/t = s/(y \wedge t) + (x \vee z)/r$, where

$$y \wedge t \leq s \leq r \leq x \vee z.$$

By induction, every sum (3) can be put into a form where $x_1 > y_1 \geq x_2 > y_2 \geq \dots$. So

Cor. Let D be a distributive lattice. Every $a \in B(D)$ is a disjoint union

$$\boxed{a = (x_1 \setminus x_2) + \dots + (x_{2n-1} \setminus x_{2n})}$$

with $x_1 > x_2 \geq \dots \geq x_{2n}$ in D.

Def. We call an embedding $i : D \hookrightarrow B$ of distributive lattices *tight* if every a/b in B splits into $a/b = x_1/y_1 + \dots + x_n/y_n$ with $x_i, y_i \in D$.

Thus $i : D \hookrightarrow B(D)$ is tight.

Problem. Let $X \hookrightarrow Y$ be a tight lattice embedding of MV algebras, and $x > y$, $z > t$ in X with $x \cdot y = z \cdot t$ in Y. Does this imply that $x \cdot y = z \cdot t$ holds in X?

Prop. 40. Let $i : D \hookrightarrow B$ be a tight embedding of distributive lattices. If $x \geq y \geq z$ with $x, z \in D$, there exist $u, v \in D$ with $u \geq v \geq z$ and $y \vee u = x$, $y \wedge u = v$.