Define $x : D \longrightarrow \mathbb{B}$ by $x^{-1}(1) = P$. Then $P \lhd D$ implies $\underline{x(c \wedge d) = x(c) \wedge x(d)}$. Also $x(0) = x(b) = 0$ and $x(1) = 1$. If $c \notin P$, then $I := \{ d \in D \mid \exists p \in P : d \geq c \wedge p \} \lhd D$, and $c = c \wedge 1 \in I \setminus P \xrightarrow{P \not\subseteq I} b \in I$. $\rightsquigarrow \exists p \in P : b \geq c \wedge p$. So for $d \notin P$ $\exists q \in P : b \geq d \wedge q$. $\rightsquigarrow$ $b \geq (c \wedge p \wedge q) \vee (d \wedge p \wedge q) = (c \vee d) \wedge p \wedge q$. $\rightsquigarrow c \vee d \notin P$. $\rightsquigarrow$ $\underline{x(c \vee d) = x(c) \vee x(d)}$, and $x(a) = 1$, $x(b) = 0$. □

<u>Cor.</u> Every distributive lattice is a sublattice of a Boolean algebra.

Now consider the embedding (as functions):

$$X(D) \hookrightarrow \mathbb{S}^D, \quad \mathbb{S} = \text{Sierpiński space} \quad \begin{matrix} 1 & \text{open} \\ | & \\ 0 & \end{matrix} \quad \tilde{\mathbb{S}} = \mathbb{B}$$

So $X(D)$ is a subspace of $\mathbb{S}^D$ with basic sets $\{ x \in X(D) \mid x(a_1) = \dots = x(a_n) = 1 \}$ for some $a_i \in D$. With $a := a_1 \wedge \dots \wedge a_n$, these sets are

$$\mathcal{U}(a) := \{ x \in X(D) \mid x(a) = 1 \}.$$

So $a \leq b \iff \hat{a} \leq \hat{b} \iff \forall x \in X(D) : x(a) \leq x(b) \iff \mathcal{U}(a) \subset \mathcal{U}(b)$. For $x, y \in X(D)$:

$$\overline{x} \subset \overline{y} \iff (\forall a \in D : x \in \mathcal{U}(a) \Rightarrow y \in \mathcal{U}(a)) \iff (\forall a \in D : x(a) = 1 \Rightarrow y(a) = 1) \iff x \leq y.$$

Note that the patch topology of the spectral space $\mathbb{S}$ is discrete. So $\mathbb{B}^D$ is compact.

<u>Prop. 27.</u> $X(D)$ is a closed subspace of $\mathbb{B}^D$.

<u>Pf.</u> Let $f \in \mathbb{B}^D$ be in the closure of $X(D)$. For $a, b \in D$, there exists $x \in X(D)$ with $x(a) = f(a)$, $x(b) = f(b)$, $x(a \vee b) = f(a \vee b)$, and $x(a \wedge b) = f(a \wedge b)$. Hence f is a lattice homomorphism. Also $f(1) = x(1) = 1$ for some $x \in X(D)$, and similarly, $f(0) = 0$. □

<u>Prop 28.</u> A compact space X is a Stone space iff $\mathcal{D}(X)$ is a basis.

<u>Pf.</u> $\mathcal{D}(X)$ consists of the closed open sets. So $\mathcal{D}(X)$ is a lattice. Since X is Hausdorff, the irreducible closed sets $\neq \emptyset$ are singletons. So X is sober. □

<u>Cor.</u> Up to isomorphism, $X \mapsto \mathcal{D}(X)$ gives a 1-1 correspondence between spectral spaces and bounded distributive lattices.

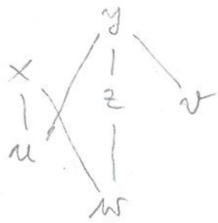
<u>Pf.</u> Let D be distributive. $\xrightarrow{\text{Prop. 27}}$ $X(D)$, as a subspace of $\mathbb{B}^D$, is compact, with basis $\{ x \in X(D) \mid x(a_1) = \dots = x(a_n) = 0,\ x(b_1) = \dots = x(b_n) = 1 \}$ for some $a_i, b_i \in D$. These sets are clopen, which makes $X(D)$ to a Stone space. For $x \not\leq y$ in $X(D)$, there exists $a \in D$ with $x(a) = 1$, $y(a) = 0$. $\rightsquigarrow x \in \mathcal{U}(a) \not\ni y$. So $X(D)$ is a Priestley space. □

$$X \longmapsto \mathcal{D}(X) \mapsto \mathcal{L}(X)$$

So we obtain equivalences:

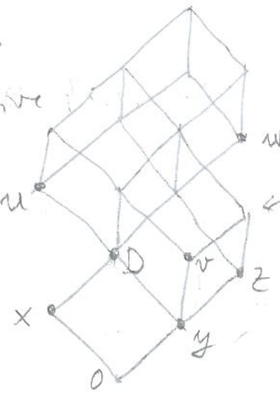
bounded distributive lattices $\leftrightarrow$ spectral spaces $\leftrightarrow$ Priestley spaces.

<u>Example.</u> For a finite spectral space X, the patch topology is discrete. So X is just a poset, e.g.



$X = \{x, y, z, u, v, w\}$.

$\mathcal{D}(X)$ is a distributive lattice (upper sets of X)



dense elements: $[D, 1]$

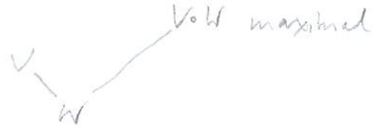
regular elements: $[0, D) \sqcup \{1\}$.

prime element in $\mathcal{D}(X)$ corresponding to x.

Partial order of X is opposite to the corresponding open sets in $\mathcal{D}(X)$.

For a finite spectral space X, $\mathcal{D}(X)$ is a Heyting algebra: $U \cap V \subset W \iff U \subset V \cdot W$. $\rightsquigarrow$ $V \cdot W = \bigcup \{U \mid U \cap V \subset W\}$.

Since $V \cdot W = V \cdot (V \cap W)$, we can assume that $W \subset V$. Then



In the example: $\uparrow v \cdot \uparrow y = u$.

By Prop. 22, the spectrum of a Heyting algebra A is a spectral space with

$$\mathcal{U}: A^{op} \xrightarrow{\sim} \mathcal{D}(\operatorname{Spec} A).$$

So $\mathcal{D}(\operatorname{Spec} A)$ is a dual Heyting algebra. For $a, b, c \in A$, $a \leq b \cdot c \iff a \wedge b \leq c$. $\iff a \wedge b \leq b \wedge c$. $\rightsquigarrow$ $b \cdot c = \bigvee \{a \in A \mid a \wedge b \leq b \wedge c\}$. So a distributive lattice D is a Heyting algebra iff $\forall\, b \geq c\ \exists \bigvee \{a \in A \mid a \wedge b \leq c\}$, i.e. iff

$$\bigcap \{\mathcal{U}(a) \mid \mathcal{U}(a) \cup \mathcal{U}(b) \supset \mathcal{U}(c)\} \in \mathcal{D}(\operatorname{Spec} A).$$

A spectral space X with this property

$$\forall\, U \subset V \text{ in } \mathcal{D}(X): \bigcap \{W \in \mathcal{D}(X) \mid W \cup U \supset V\} \in \mathcal{D}(X)$$

is called an <u>Esakia space</u>. The condition $W \cup U \supset V$ means that $V \setminus U \subset W$. So X is Esakia iff every $B \in \mathcal{L}(X)$ has a hull in $\mathcal{D}(X)$:

<u>Thm 6 (Esakia 1974).</u> A spectral space X is an Esakia space iff for all $B \in \mathcal{L}(X)$, $\uparrow B \in \mathcal{D}(X)$. Esakia spaces are equivalent to Heyting algebras via $\mathcal{U}: A^{op} \xrightarrow{\sim} \mathcal{D}(\operatorname{Spec} A)$.

<u>Pf.</u> For $B \in \mathcal{L}(X)$, the upper set $\uparrow B$ is closed in $\check{X}$: For $x \in B$ and $y \notin \uparrow B$. $\exists\, U \in \mathcal{D}(X): x \in U, y \notin U$. $\overset{\text{Bcp.}}{\Longrightarrow}$ $\exists\, U_1, \ldots, U_n \in \mathcal{D}(X): \uparrow B \subset U_1 \cup \ldots \cup U_n \not\ni y$. $\rightsquigarrow$ $\check{X} \setminus \uparrow B \in \mathcal{O}(\check{X})$.
$\rightsquigarrow$ B has a $\mathcal{D}(X)$-hull iff $\uparrow B \in \mathcal{D}(X)$. $\square$

<u>Example.</u> Let R be a commutative ring, and let $\mathcal{I}(R)$ be the poset of finitely generated ideals. For $I, J \in \mathcal{I}(R)$ we define the preorder

$$I \leq J :\iff \exists\, n \in \mathbb{N}: I^n \subset J.$$

$$I \equiv J :\iff I \leq J \leq I.$$

So the equivalence classes form a poset $\mathcal{D}(R)$. This is a distributive lattice, defining a spectral space $\operatorname{Spec} R$ with $\mathcal{D}(R)^{op} \cong \mathcal{D}(\operatorname{Spec} R)$, the <u>spectrum</u> of R. The lattice operations of $\mathcal{D}(R)$ are represented by $I \vee J = I + J$ and $I \wedge J = IJ$.

Spec R can be represented by the set of prime ideals:

$$I+J\subset P \Leftrightarrow I,J\subset P\ ;\quad IJ\subset P \Leftrightarrow I\subset P \text{ or } J\subset P.$$

Example. A partial order Ω with greatest element 1 is a Hilbert algebra with

$$x\cdot y = \begin{cases} 1 & \text{for } x\le y \\ y & \text{for } x\nleq y \end{cases}$$

Equivalently, every $x<1$ is prime. Furthermore $x\in\Omega$ is regular iff x is minimal. So the dense elements form an ideal $D(\Omega)$, and $\Omega/D(\Omega)$ is a discrete L-algebra. In particular, the prime elements of an L-algebra form an L-subalgebra (poset).

Let X be an L-algebra. If $x\in X$ is dense, then $x\cdot y=y$ for some $y\le x$. So $yx=(x\cdot y)x=x\wedge y=y$ in $S(X)$. So the map $q: X\to G(X)$ satisfies $q(x)=1$.

Prop. 25. Let X be a bounded Hilbert algebra. Then

$$q: X \twoheadrightarrow \overline{X} \hookrightarrow G(\overline{X})\cong G(X).$$

Pf. This follows by Glivenko's theorem. For the Boolean algebra $\overline{X}\cong X/D(X)$, the structure group consists of the integral steps functions. □

P: What about unbounded Hilbert algebras?

Now we consider the logical aspect of Hilbert algebras.

Let P be the set of terms in one operation $\cdot$ and variables $x,y,z,\ldots\in V$, e.g.

$$((x\cdot y)\cdot z)\cdot(y\cdot x)\in P.$$

For $A\subset P$ and $p\in P$ we write $A\vdash p$ if p can be derived from A and a fixed set of axioms $p_1,\ldots,p_n\in A$ by using the deduction rule

$$p,\ p\cdot q\vdash q. \quad (\text{MP}=\text{modus ponens})$$

Thm. 7 (Henkin 1950). The deduction theorem

$$(D)\quad A,p\vdash q \Longrightarrow A\vdash p\cdot q$$

holds iff the axioms imply

$$(K)\quad \vdash p\cdot(q\cdot p)$$

$$(S)\quad \vdash (p\cdot(q\cdot r))\cdot((p\cdot q)\cdot(p\cdot r)).$$

Pf. If (D) holds, then $p,q\vdash p \rightsquigarrow p\vdash q\cdot p \rightsquigarrow$ (K), and $p,\ p\cdot q,\ p\cdot(q\cdot r)\vdash q,\ q\cdot r\vdash r$. $\overset{(D)}{\rightsquigarrow} p\cdot(q\cdot r),\ p\cdot q\vdash p\cdot r \overset{(D)}{\rightsquigarrow} p\cdot(q\cdot r)\vdash (p\cdot q)\cdot(p\cdot r) \rightsquigarrow$ (S).

Conversely, assume (K) and (S), and that $\underline{A,p\vdash q}$ has been proved in n steps. We have (K) $\rightsquigarrow p\cdot((p\cdot p)\cdot p)\overset{(S)}{\rightsquigarrow}(p\cdot(p\cdot p))\cdot(p\cdot p)\overset{(K)}{\rightsquigarrow} p\cdot p$.

$n=1$: If $p=q$ (or $p\cdot q\in A$) then $\vdash p\cdot q$. Otherwise $q\in A$ or q is an axiom. $\overset{(K)}{\rightsquigarrow} A\vdash p\cdot q$.

$n>1$: Then q comes from $A,p\vdash r,\ r\cdot q\vdash q$. $\overset{\text{Induction}}{\rightsquigarrow} A\vdash p\cdot r,\ p\cdot(r\cdot q)\overset{(S)}{\vdash} p\cdot r, (p\cdot r)\cdot(p\cdot q)$ $\vdash p\cdot q$. □

Assume that axioms (K) and (S) hold. Then

$$p \equiv q \;:\Longleftrightarrow\; \vdash p\cdot q,\ q\cdot p \overset{(D)}{\Longleftrightarrow} p \vdash q \text{ and } q \vdash p.$$

is a <u>congruence relation</u>: $p \equiv q \Longrightarrow p\cdot r \equiv q\cdot r$ and $r\cdot p \equiv r\cdot q$. Indeed $p \equiv q$ gives $p\cdot r,\ q \vdash p\cdot r,\ p \vdash r \overset{(D)}{\rightsquigarrow} p\cdot r \vdash q\cdot r\ [\rightsquigarrow\ \vdash (p\cdot r)\cdot(q\cdot r)]$. By symmetry, $p\cdot r \equiv q\cdot r$. And $r\cdot p,\ r \vdash p \vdash q \overset{(D)}{\rightsquigarrow} r\cdot p \vdash r\cdot q.\ [\rightsquigarrow\ \vdash (r\cdot p)\cdot(r\cdot q)] \overset{\text{sym.}}{\rightsquigarrow} r\cdot p \equiv r\cdot q$.

$H(V) := P/\equiv$ is called the <u>Lindenbaum algebra</u>. Let $\mathbf{1} \in H(V)$ be the class of true propositions. Then $H(V)$ satisfies

$$(1)\quad x\cdot(y\cdot x) = 1$$

$$(2)\quad (x\cdot(y\cdot z))\cdot((x\cdot y)\cdot(x\cdot z)) = 1$$

$$(3)\quad x\cdot y = y\cdot x = 1 \Longrightarrow x = y.$$

<u>Thm. 8.</u> $H(V)$ is a Hilbert algebra.

<u>Pf.</u> $(1) \rightsquigarrow 1\cdot(y\cdot 1) = 1 \overset{(1)}{\rightsquigarrow} (y\cdot 1)\cdot 1 = (y\cdot 1)\cdot(1\cdot(y\cdot 1)) \overset{(1)}{=} 1. \overset{(3)}{\rightsquigarrow} \underline{y\cdot 1 = 1.} \rightsquigarrow$

$$1\cdot x = 1 \overset{(3)}{\Longrightarrow} x = 1.$$

$\underline{1}\cdot(\underline{1}\cdot(x\cdot x)) \overset{(1)}{=} (x\cdot(1\cdot x))\cdot((x\cdot 1)\cdot(x\cdot x)) \overset{(2)}{=} 1. \rightsquigarrow 1\cdot(x\cdot x) = 1 \rightsquigarrow \underline{x\cdot x = 1.} \rightsquigarrow$

$\underline{1}\cdot(\underline{1}\cdot((1\cdot x)\cdot x)) = ((1\cdot x)\cdot(1\cdot x))\cdot(((1\cdot x)\cdot 1)\cdot((1\cdot x)\cdot x)) \overset{(2)}{=} 1. \rightsquigarrow 1\cdot((1\cdot x)\cdot x) = 1 \rightsquigarrow$ $(1\cdot x)\cdot x = 1$. Since $x\cdot(1\cdot x) \overset{(1)}{=} 1, \overset{(3)}{\rightsquigarrow} \underline{1\cdot x = x.} \rightsquigarrow 1$ logical unit. Define

$$x \le y \;:\Longleftrightarrow\; x\cdot y = 1.$$

If $x \le y \le z$, then $x\cdot z = (x\cdot y)\cdot(x\cdot z) = (x\cdot(y\cdot z))\cdot((x\cdot y)\cdot(x\cdot z)) \overset{(2)}{=} 1. \rightsquigarrow x \le z$. Since $x \le x$ and $x \le y \le x \overset{(3)}{\Longrightarrow} x = y$, "$\le$" is a partial order. We prove

$$x \le y \Longrightarrow y\cdot z \le x\cdot z.$$

$x \le y \overset{(1)}{\rightsquigarrow} y\cdot z \le x\cdot(y\cdot z) \overset{(2)}{\le} (x\cdot y)\cdot(x\cdot z) = x\cdot z$. So $x \le y \Longrightarrow y\cdot z \le x\cdot z. \rightsquigarrow$

$\underline{x\cdot(y\cdot z)} \overset{(2)}{\le} \underline{(x\cdot y)\cdot(x\cdot z)} \overset{(1)}{\le} y\cdot(x\cdot z) \overset{(2)}{\le} \underline{(y\cdot x)\cdot(y\cdot z)} \overset{(1)}{\le} \underline{x\cdot(y\cdot z)} \rightsquigarrow H(V)$ Hilbert algebra. □

So the axioms (1)–(3) characterize Hilbert algebras which thus formalize the deduction theorem. $H(V)$ is the <u>free Hilbert algebra</u> with $|V|$ generators.

<u>Prop. 30.</u> If $X \in \mathbf{Top}$, then $\mathcal{O}(X)$ is commutative iff X is a <u>partition space</u>: $X = \bigsqcup_{i\in I} X_i$ with $\mathcal{O}(X) = \{\bigsqcup_{i\in J} X_i \mid J \subset I\} \cong \mathfrak{P}(I)$.

<u>Pf.</u> Let $\mathcal{O}(X)$ be commutative. $\overset{\text{Thm. 3}}{\rightsquigarrow} (U\cdot V)\cdot V = (V\cdot U)\cdot U$. For $V = \emptyset$, this gives $(U\cdot\emptyset)\cdot\emptyset = (X\setminus\overline{U})\cdot\emptyset = X\setminus\overline{X\setminus\overline{U}} = \overset{\circ}{\overline{U}} = (\emptyset\cdot U)\cdot U = X\cdot U = U$, i.e. all open sets are regular. For $U \in \mathcal{O}(X)$, the open set $U \vee (X\setminus\overline{U})$ is dense. Hence $U \vee (X\setminus\overline{U}) = X$, and thus $\overline{U} = U$. Consider the equivalence relation

$$x \sim y \;:\Longleftrightarrow\; \overline{x} = \overline{y}.$$

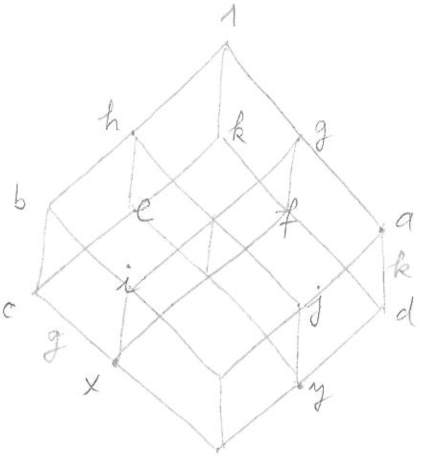
The equivalence classes $\overline{x}$ are clopen and form a basis for the topology. □

Since $H(V)$ is free, every valuation, i.e. map $v: V \to A$ into a Hilbert algebra A extends uniquely to an L-algebra morphism $v: H(V) \to A$. So the formulae p of the intuitionistic propositional calculus (IPC), i.e. the logical consequences of the axioms (K) + (S) hold in every Hilbert algebra A: If $p = p(x_1, \ldots, x_n)$, $x_i \in V$, is true, then $p(a_1, \ldots, a_n) = 1$ in A. We write $A \Vdash p$ if p holds in A; and simply $\Vdash p$ if p holds in each A. So we have

$$\vdash p \implies \Vdash p \quad \text{(soundness)}$$
$$\Vdash p \implies \vdash p \quad \text{(completeness)}.$$

Proofs in IPC are syntactical, while checking p in A belongs to semantics.

A similar completeness theorem holds for classical logic and Boolean algebras. Here we have $\Vdash p \iff \mathbb{B} \Vdash p$ which is quite convenient. In IPC, however, there is no finite Hilbert algebra with this property. Of course, $\Vdash p \iff H(\mathbb{N})$, but this algebra is very big. If $p = p(x_1, \ldots, x_n)$, we can choose $H_n := H(\{x_1, \ldots, x_n\})$. We have $H_1 = \mathbb{B}$, but $H_2 = H(\{x, y\})$ is already bigger (14 elements):



$a = x \cdot y$ $\qquad$ $b = y \cdot x$

$c = (x \cdot y) \cdot x$ $\qquad$ $d = (y \cdot x) \cdot y$

$e = (x \cdot y) \cdot y$ $\qquad$ $f = (y \cdot x) \cdot x$

$g = ((x \cdot y) \cdot x) \cdot x$ $\qquad$ $h = ((y \cdot x) \cdot y) \cdot y$

$i = ((x \cdot y) \cdot y) \cdot x$ $\qquad$ $j = ((y \cdot x) \cdot x) \cdot y$

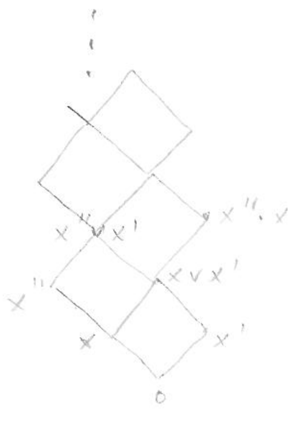
$k = (x \cdot y) \cdot ((y \cdot x) \cdot x) = (y \cdot x) \cdot ((x \cdot y) \cdot y)$

Der Verband ist $C(H_2)$.

(Since $(x \cdot y) \cdot x \le (x \cdot y) \cdot y$, we have $i \le g$.) We have $c \cdot x = g$, $a \cdot d = k$, etc.; so all edges are also to be coloured! We will see that H_n is finite. For the free Brouwerian semilattices $C(H_n)$, we have

$$|C(H_1)| = 2, \; |C(H_2)| = 18, \; |C(H_3)| = 623662965552330, \ldots$$

Though all $C(H_n)$ are even Heyting algebras, they are all non-free. The free Heyting algebra with 1 generator is already infinite.



Every poset Ω is a topological space with $\mathcal{O}(\Omega) := \text{upper sets}$. An L-algebra morphism
$v : H(V) \longrightarrow \mathcal{O}(\Omega)$ is called a <u>Kripke model</u>: To each formula p there is an upper set $v(p)$ of <u>states</u> x for which p holds:

$$x \Vdash p \;:\Longleftrightarrow\; x \in v(p).$$

So $x \Vdash p \cdot q \iff x \in v(p \cdot q) \iff \uparrow x \subset v(p) \cdot v(q) \iff \uparrow x \cap v(p) \subset v(q)$

$$\iff (\forall y \geqslant x : y \Vdash p \Rightarrow y \Vdash q).$$

For example, every $v : V \longrightarrow A$ into a finite Hilbert algebra A is a Kripke model: v extends to a morphism $v : H(V) \longrightarrow \mathcal{O}(\operatorname{Spec} A^{op}) \cong A$

<u>Thm. 9.</u> All H_n are finite. A proposition in IPC is valid iff it holds on $\mathcal{O}(I)$ with $I := [0,1]$.

<u>Pf.</u> For $a \in H_n$ there is a maximal ideal P with $a \notin P$ (Zorn). $\rightsquigarrow P \in \operatorname{Spec} H_n$. $\rightsquigarrow$ Every ideal $I \supsetneq P$ satisfies $a \in I$. So $A_a := H_n / P$ is <u>subdirectly irreducible</u>, i.e. A_a has a smallest $J \neq \{1\}$. Choose $c < 1$ in J. $\rightsquigarrow$ For all $b < 1$ in A_a, the ideal $\uparrow b$ contains J. $\rightsquigarrow c \geqslant b$. $\rightsquigarrow J = \{1, c\}$, and c is the greatest element of $A_a \setminus \{1\}$. Then $A_a \setminus \{c\}$ is an L-subalgebra of A_a. Indeed, assume that $b, d \neq c$ with $b \cdot d = c$. Then $1 = b \cdot c = b \cdot (b \cdot d) = (b \cdot b) \cdot (b \cdot d) = b \cdot d = c$. ↯.

Now A_a is n-generated. So c must be one of the generators. $\rightsquigarrow A_a \setminus \{1\}$ is $(n-1)$-generated, hence finite by induction. $\rightsquigarrow A_a$ is finite. Now

$$H_n \hookrightarrow \prod_{a \in H_n} A_a.$$

By induction, we can assume that H_i is finite for $i < n$. $\rightsquigarrow |A_a|$ bounded. $\rightsquigarrow$ For any $b \in H_n$, almost all components in $\prod A_a$ are equal. So there is a finite subset $S \subset H_n$ with $H_n \hookrightarrow \prod_{a \in S} A_a$. $\rightsquigarrow |H_n| < \infty$.

The second part was proved by Jaskowski 1936: It uses that I is a metric space with a countable basis, and follows the structure of H_n as an iterated subdirect product. This yields an L-algebra embedding

$$H_n \hookrightarrow \mathcal{O}(I).$$

for all n. □

For Boolean algebras, such an embedding is trivial, but for H_n one has to be careful. E.g.

$$H_1 = \mathcal{O}(\mathbb{S}) : \begin{array}{c} 1 \\ | \\ a \\ | \\ 0 \end{array} \qquad a \cdot 0 = 0.$$

So a has to be represented by a <u>dense</u> open set!

<u>Note:</u> The generators of H_n are exactly the minimal elements of H_n, since $a \cdot b \geqslant b$ etc.