Quasi-model-categories

University of Stuttgart
Germany

Master's Thesis

Mathias Ritter
March 2018

Contents

Chapter 0

Introduction

0.1 Quillen model categories

In his seminal monograph “Homotopical Algebra” [**12**] Daniel Quillen introduced *model categories* as an axiomatic framework for homotopy theory of topological spaces, simplicial sets and chain complexes [**12,** Ex. on pp. I.2f]. With a slight variation, Bousfield and Friedlander applied the concept of model categories to spectra [**1,** §2].

Roughly speaking, a model category in the sense of Quillen is a category $\mathcal{M}$ that has finite limits and colimits and that is equipped with subsets $\operatorname{Cof}\mathcal{M} \subseteq \operatorname{Mor}\mathcal{M}$ of *cofibrations*, $\operatorname{Fib}\mathcal{M} \subseteq \operatorname{Mor}\mathcal{M}$ of *fibrations* and $\operatorname{Qis}\mathcal{M} \subseteq \operatorname{Mor}\mathcal{M}$ of *quasi-isomorphisms*, also known as *weak equivalences*, fulfilling certain axioms [**12,** Def. I.1.1].

In particular, all pushouts and pullbacks exist. Furthermore, pushouts of cofibrations along arbitrary morphisms are required to be cofibrations. Dually, pullbacks of fibrations along arbitrary morphisms are required to be fibrations.

The notion of a model category is self-dual, which is not a priori evident when considering the examples above.

Quillen introduces the homotopy category $\mathcal{M}[\mathrm{Qis}^{-1}]$ of $\mathcal{M}$ as localisation of $\mathcal{M}$ with respect to $\operatorname{Qis}\mathcal{M}$, in which the quasi-isomorphisms become formally inverted in the sense of Gabriel and Zisman [**6**].

Under further assumptions on $\mathcal{M}$, Dwyer, Hirschhorn, Kan and Smith showed that the morphisms of $\mathcal{M}[\mathrm{Qis}^{-1}]$ can be represented by 3-arrows, allowing for a calculus of fractions on $\mathcal{M}[\mathrm{Qis}^{-1}]$; cf. [**5,** Cor. 10.9]. For a more general approach to this 3-arrow calculus, we refer to [**16**] by Sebastian Thomas.

To avoid a calculus of fractions, Quillen uses bifibrant objects as follows.

To indicate that $i \in \operatorname{Mor}\mathcal{M}$ is a cofibration, we often write $X \overset{i}{\rightarrowtail} Y$.

To indicate that $p \in \operatorname{Mor}\mathcal{M}$ is a fibration, we often write $X \overset{p}{\twoheadrightarrow} Y$.

To indicate that $w \in \operatorname{Mor}\mathcal{M}$ is a quasi-isomorphism, we often write $X \xrightarrow[\approx]{w} Y$.

A model category $\mathcal{M}$ has a terminal object ! and an initial object ¡. Thus, one can define full subcategories $\mathcal{M}_{\text{cof}}$ of *cofibrant* objects, $\mathcal{M}_{\text{fib}}$ of *fibrant* objects and $\mathcal{M}_{\text{bif}}$ of *bifibrant* objects by letting

$$\begin{aligned}
\operatorname{Ob}\mathcal{M}_{\text{cof}} &:= \{X \in \operatorname{Ob}\mathcal{M} : \text{¡} \rightarrowtail X\} \\
\operatorname{Ob}\mathcal{M}_{\text{fib}} &:= \{X \in \operatorname{Ob}\mathcal{M} : X \twoheadrightarrow \,!\} \\
\operatorname{Ob}\mathcal{M}_{\text{bif}} &:= \operatorname{Ob}\mathcal{M}_{\text{cof}} \cap \operatorname{Ob}\mathcal{M}_{\text{fib}}\,.
\end{aligned}$$

In [**12,** Def. I.1.3], Quillen introduced *left-homotopy* and, dually, *right-homotopy* of morphisms. He showed that right-homotopy yields a congruence on $\mathcal{M}_{\text{cof}}$, that left-homotopy yields a congruence on $\mathcal{M}_{\text{fib}}$ and that left- and right-homotopy coincide on $\mathcal{M}_{\text{bif}}$; cf. [**12,** pp. I.8-11]. In consequence, we obtain factor categories $\overline{\mathcal{M}_{\text{cof}}}$, $\overline{\mathcal{M}_{\text{fib}}}$ and $\overline{\mathcal{M}_{\text{bif}}}$.

The homotopy and factor categories introduced by Quillen are related by the commutative diagram of functors

$$\begin{tikzcd}
\overline{\mathcal{M}_{\text{cof}}} \arrow[r, "\gamma_{\text{cof}}"] & \mathcal{M}_{\text{cof}}[\text{Qis}^{-1}] \arrow[d, "\wr"] \\
\overline{\mathcal{M}_{\text{bif}}} \arrow[u] \arrow[d] \arrow[r, "\gamma", "\sim"'] & \mathcal{M}[\text{Qis}^{-1}] \\
\overline{\mathcal{M}_{\text{fib}}} \arrow[r, "\gamma_{\text{fib}}"'] & \mathcal{M}_{\text{fib}}[\text{Qis}^{-1}], \arrow[u, "\wr"']
\end{tikzcd}$$

where $\sim$ denotes an equivalence of categories [**12,** Th. I.1]. Thus, already $\mathcal{M}_{\text{cof}}$, $\mathcal{M}_{\text{fib}}$ and $\mathcal{M}_{\text{bif}}$ contain the essential homotopy-theoretic information of $\mathcal{M}$.

Therefore, one can choose to work in $\mathcal{M}_{\text{cof}}$, $\mathcal{M}_{\text{fib}}$ or $\mathcal{M}_{\text{bif}}$ only.

However, e.g. in $\mathcal{M}_{\text{bif}}$, we have neither pushouts along cofibrations nor pullbacks along fibrations at our disposal.

0.2 Working with $\mathcal{M}_{\text{fib}}$ or $\mathcal{M}_{\text{cof}}$

In [**2**], Kenneth Brown introduced *categories of fibrant objects* as an axiomatization of the full subcategory of fibrant objects in a model category.

Roughly speaking, a category of fibrant objects is a category $\mathcal{B}$ with finite products and terminal object ! equipped with subsets $\operatorname{Fib}\mathcal{B} \subseteq \operatorname{Mor}\mathcal{B}$ of *fibrations* and $\operatorname{Qis}\mathcal{B} \subseteq \operatorname{Mor}\mathcal{B}$ of *quasi-isomorphisms* fulfilling certain axioms [**2,** pp. 420f]. Pullbacks of fibrations along arbitrary morphisms are required to exist and to yield a fibration. Furthermore, every object is fibrant.

The notion of a category with fibrant objects is not self-dual.

Brown showed that the morphisms of $\mathcal{B}[\text{Qis}^{-1}]$ for a category of fibrant objects $\mathcal{B}$ can be represented by 2-arrows, allowing for a calculus of fractions on $\mathcal{B}[\text{Qis}^{-1}]$; cf. [**2,** Th. 1].

This calculus of fractions was simplified by Sebastian Thomas by considering only certain well-behaved 2-arrows [**17,** Ch. II]. In particular, he made equality of morphisms tractable.

For an overview of different approaches to fibration categories, we refer to [**13**] by Andrei Radulescu-Banu.

0.3 Working with $\mathcal{M}_{\mathrm{bif}}$

0.3.1 Quasi-model-categories

In this work, we introduce *quasi-model-categories* as an axiomatization of $\mathcal{M}_{\mathrm{bif}}$ for a model category $\mathcal{M}$.

Roughly speaking, a quasi-model-category is a category $\mathcal{C}$ with terminal object $!$ and initial object $¡$ equipped with subsets $\operatorname{Cof}\mathcal{C} \subseteq \operatorname{Mor}\mathcal{C}$ of *cofibrations*, $\operatorname{Fib}\mathcal{C} \subseteq \operatorname{Mor}\mathcal{C}$ of *fibrations* and $\operatorname{Qis}\mathcal{C} \subseteq \operatorname{Mor}\mathcal{C}$ of *quasi-isomorphisms* fulfilling certain axioms; cf. Definition 100. Furthermore, all objects are bifibrant.

We add *quasi-pushouts* $\operatorname{QPO}\mathcal{C}$ and *quasi-pullbacks* $\operatorname{QPB}\mathcal{C}$ as data for a quasi-model-category $\mathcal{C}$. They are weak pushouts resp. weak pullbacks satisfying further properties; cf. Definitions 96 and 97, and Lemma 104. They serve as a practical replacement for the missing pushouts and pullbacks.

The notion of a quasi-model-category is self-dual; cf. Remark 102.

0.3.2 The basic example

Let $\mathcal{M}$ be a model category in the sense of Definition 172, which is a slight variation of Quillen's original definition. We additionally require that in $\mathcal{M}$, pushouts of quasi-isomorphisms along cofibrations yield quasi-isomorphisms and, dually, pullbacks of quasi-isomorphisms along fibrations yield quasi-isomorphisms. I.e. $\mathcal{M}$ is *proper* in the sense of Bousfield and Friedlander [**1,** Def. 1.2]. On the other hand, we only require pushouts of arbitrary morphisms along cofibrations and pullbacks of arbitrary morphisms along fibrations to exist.

Theorem 193. Suppose that $\mathcal{M}$ is weakly pointed, i.e. that $¡ \xrightarrow{\;+\;} !$ and $¡ \xrightarrow{\;\bullet\;} !$.

Then $\mathcal{M}_{\mathrm{bif}}$ carries the structure of a quasi-model-category.

0.3.3 Results

For this §0.3.3, let $\mathcal{C}$ be a quasi-model-category.

0.3.3.1 The homotopy category

Let $X, Y \in \operatorname{Ob}\mathcal{C}$. We call $f_0,\ f_1 \in {}_{\mathcal{C}}(X, Y)$ *homotopic*, written $f_0 \sim f_1$, if there exists a commutative diagram as follows.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, "i_0 \wr"'] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X & \hat{X} \arrow[l, "t"', "\approx"] \arrow[r, "\hat{f}"] & \hat{Y} \arrow[u, "\wr p_0"'] \arrow[d, "\wr p_1"] & Y \arrow[l, "s"', "\approx"] \\
X \arrow[r, equal] \arrow[u, equal] & X \arrow[r, "f_1"'] \arrow[u, "i_1 \wr"] & Y \arrow[r, equal] & Y \arrow[u, equal]
\end{tikzcd}$$

Proposition 52. The homotopy relation $(\sim)$ is a congruence on $\mathcal{C}$; cf. Definition 108.

Let $\operatorname{Ho}\mathcal{C} := \mathcal{C}/(\sim)$ be the factor category. Let $\mathrm{L}_{\mathcal{C}} : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ be the residue class functor.

Let $\mathcal{D}$ be a category. Write $[\mathcal{C}, \mathcal{D}]$ for the category of functors from $\mathcal{C}$ to $\mathcal{D}$. We consider the full subcategories ${}_{(\sim)}[\mathcal{C}, \mathcal{D}]$ and ${}_{\mathrm{Loc}}[\mathcal{C}, \mathcal{D}]$ of $[\mathcal{C}, \mathcal{D}]$ defined by

$$\begin{aligned} \operatorname{Ob}\, {}_{(\sim)}[\mathcal{C}, \mathcal{D}] &:= \{F \in \operatorname{Ob}[\mathcal{C}, \mathcal{D}] : \text{for } f, g \in \operatorname{Mor}\mathcal{C} \text{ with } f \sim g, \text{ we have } Ff = Fg\} \\ \operatorname{Ob}\, {}_{\mathrm{Loc}}[\mathcal{C}, \mathcal{D}] &:= \{F \in \operatorname{Ob}[\mathcal{C}, \mathcal{D}] : F(\operatorname{Qis}\mathcal{C}) \subseteq \operatorname{Iso}\mathcal{D}\}. \end{aligned}$$

Theorem 62. We have the isomorphism of categories

$$\begin{array}{ccc} {}_{(\sim)}[\mathcal{C}, \mathcal{D}] & \longleftarrow & [\operatorname{Ho}\mathcal{C}, \mathcal{D}] \\ (U \circ \mathrm{L}_{\mathcal{C}} \xrightarrow{\beta * \mathrm{L}_{\mathcal{C}}} V \circ \mathrm{L}_{\mathcal{C}}) & \hookleftarrow & (U \xrightarrow{\beta} V). \end{array}$$

In particular, we have the universal property of the factor category; i.e. every functor $F \in \operatorname{Ob}\, {}_{(\sim)}[\mathcal{C}, \mathcal{D}]$ factors uniquely over $\mathrm{L}_{\mathcal{C}}$ as $F = \overline{F} \circ \mathrm{L}_{\mathcal{C}}$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\mathrm{L}_{\mathcal{C}}"'] & \mathcal{D} \\ \operatorname{Ho}\mathcal{C} \arrow[ur, dotted, "\exists!\, \overline{F}"'] & \end{tikzcd}$$

Proposition 59. We have ${}_{(\sim)}[\mathcal{C}, \mathcal{D}] = {}_{\mathrm{Loc}}[\mathcal{C}, \mathcal{D}]$.

The proof of the inclusion $\operatorname{Ob}\, {}_{(\sim)}[\mathcal{C}, \mathcal{D}] \subseteq \operatorname{Ob}\, {}_{\mathrm{Loc}}[\mathcal{C}, \mathcal{D}]$ uses weak pushouts and weak pullbacks; cf. Lemma 53.

As a consequence of this lemma, the residue class functor $\mathrm{L}_{\mathcal{C}} : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ also has the universal property of the localisation of $\mathcal{C}$ with respect to the quasi-isomorphisms.

Theorem 63. We have the isomorphism of categories

$$\begin{array}{ccc} {}_{\mathrm{Loc}}[\mathcal{C}, \mathcal{D}] & \longleftarrow & [\operatorname{Ho}\mathcal{C}, \mathcal{D}] \\ (U \circ \mathrm{L}_{\mathcal{C}} \xrightarrow{\beta * \mathrm{L}_{\mathcal{C}}} V \circ \mathrm{L}_{\mathcal{C}}) & \hookleftarrow & (U \xrightarrow{\beta} V). \end{array}$$

In particular, every functor $F \in \operatorname{Ob}\, {}_{\mathrm{Loc}}[\mathcal{C}, \mathcal{D}]$ factors uniquely over $\mathrm{L}_{\mathcal{C}}$ as $F = \overline{F} \circ \mathrm{L}_{\mathcal{C}}$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\mathrm{L}_{\mathcal{C}}"'] & \mathcal{D} \\ \operatorname{Ho}\mathcal{C} \arrow[ur, dotted, "\exists!\, \overline{F}"'] & \end{tikzcd}$$

Technically speaking, we establish the homotopy category and its universal property more generally for *categories with split denominators*; cf. Definition 39. The notion of a category with split denominators is a precursor to the notion of a quasi-model-category; cf. Definition 108. A category with split denominators is in particular a *uni-fractionable category* in the sense of Sebastian Thomas [**16,** Def. 3.1]. We only need weak pushouts and weak pullbacks for the construction of the homotopy category, so that no further properties of quasi-pushouts and quasi-pullbacks were needed.

0.3.3.2 Hirschhorn replacement

We have the following *Hirschhorn replacement lemma.*

Proposition 112 (Cf. [**9,** Cor. 7.3.12]). The following assertions (1, 2) hold in $\mathcal{C}$.

(1) Suppose given

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \\ B \arrow[ur, "g"'] & \end{tikzcd}$$

such that $ig \sim f$. There exists $B \xrightarrow{g'} X$ with $g \sim g'$ and $ig' = f$.

(2) Suppose given

$$\begin{tikzcd} & X \arrow[d, "p"] \\ A \arrow[ur, "f"] \arrow[r, "g"'] & Y \end{tikzcd}$$

such that $fp \sim g$. There exists $A \xrightarrow{f'} X$ with $f \sim f'$ and $f'p = g$.

Using this replacement lemma, we obtain the following corollaries.

Corollary 113. A quasi-pushout yields a weak pushout in $\operatorname{Ho}\mathcal{C}$.

Corollary 114. A quasi-pullback yields a weak pullback in $\operatorname{Ho}\mathcal{C}$.

0.3.3.3 Loop and suspension functor

We construct the loop and suspension functors

$$\begin{tikzcd} \operatorname{Ho}\mathcal{C} \arrow[r, shift left, "\Sigma_{\mathcal{C}}"] & \operatorname{Ho}\mathcal{C}. \arrow[l, shift left, "\Omega_{\mathcal{C}}"] \end{tikzcd}$$

Cf. Definitions 141 and 153.

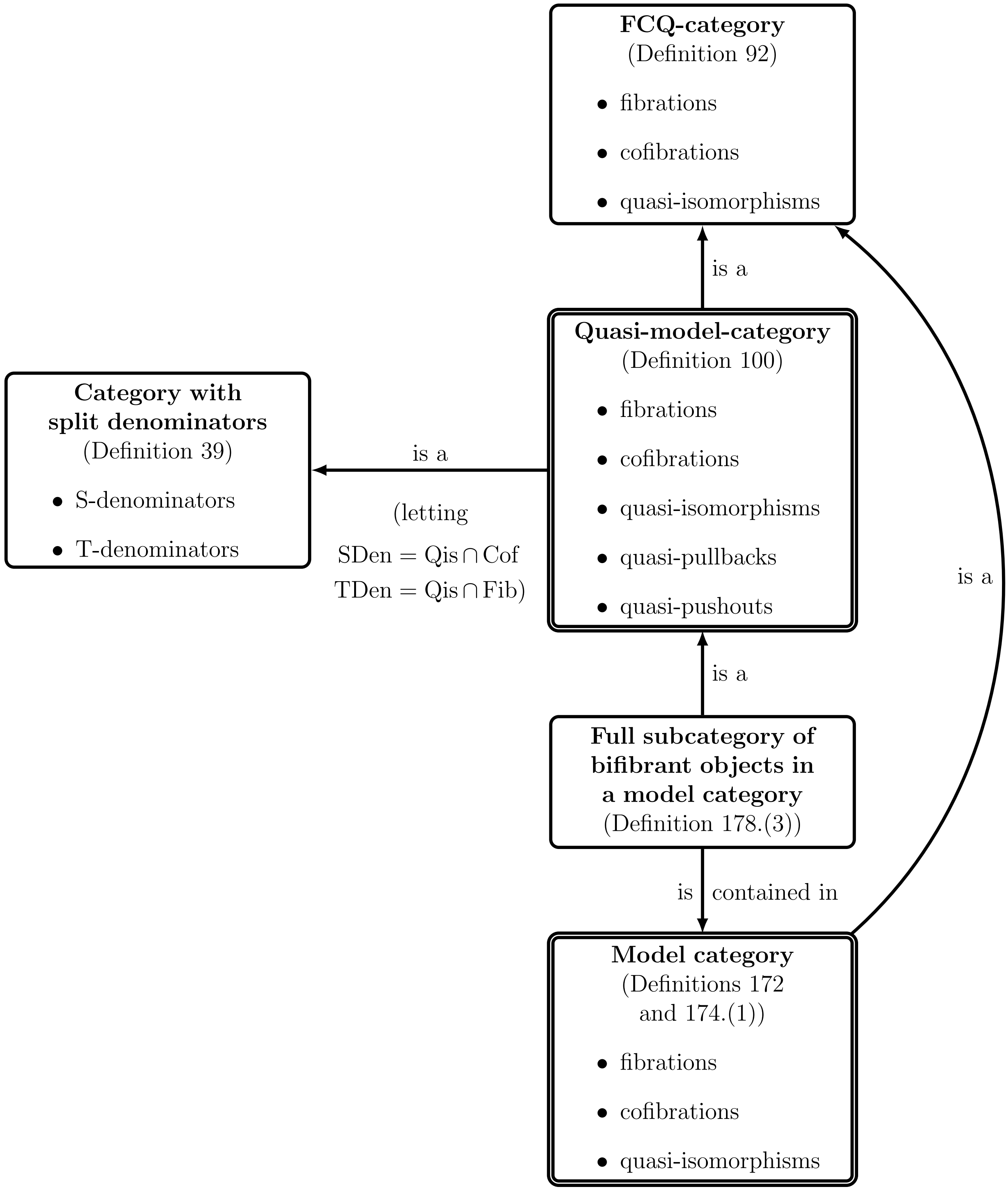
Theorem 160. Suppose that $\mathcal{C}$ is pointed, i.e. that $! \cong \text{¡}$.

Then the suspension functor $\Sigma_{\mathcal{C}}$ is left adjoint to the loop functor $\Omega_{\mathcal{C}}$.

In the context of model categories, this has been shown by Quillen [**12,** pp. I.2.9ff]. Having only a quasi-model-category at our disposal forced us to follow an entirely different path compared to Quillen's original line of arguments. We hope to have achieved an accessible proof.

0.4 An overview diagram

We provide a diagram relating the main notions considered in this work.



0.5 Acknowledgments

I thank Matthias Künzer for introducing me to homotopical algebra and for countless discussions on the subject of this thesis. Furthermore, I thank Sebastian Thomas for lessons in abstract homotopy theory, in particular, for removing an obstacle in Chapter 2.

0.6 Conventions

We assume the reader to be familiar with elementary category theory. An introduction to this subject can be found in [**11**] or [**15**]. Some basic definitions and notations are given below.

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be categories.

1. All categories are supposed to be small (with respect to a sufficiently big universe); cf. [**15,** §3.2 and §3.3].

2. We write $\operatorname{Ob}\mathcal{A}$ for the set of objects and $\operatorname{Mor}\mathcal{A}$ for the set of morphisms of $\mathcal{A}$.

   Given $A, B \in \operatorname{Ob}\mathcal{A}$, we denote the set of morphisms from A to B by ${}_{\mathcal{A}}(A, B)$.

   The identity morphism of $A \in \operatorname{Ob}\mathcal{A}$ is written as 1_A or $1_A^{\mathcal{A}}$. If unambiguous, we often write $1 := 1_A$.

   Let $f \in \operatorname{Mor}\mathcal{A}$. We write $\operatorname{Source} f \in \operatorname{Ob}\mathcal{A}$ for its source and $\operatorname{Target} f \in \operatorname{Ob}\mathcal{A}$ for its target.

3. The composition of morphisms in $\mathcal{A}$ is written naturally:

$$\left(A \xrightarrow{f} B \xrightarrow{g} C\right) = \left(A \xrightarrow{fg} C\right) = \left(A \xrightarrow{f \cdot g} C\right).$$

4. The composition of functors is written traditionally:

$$\left(\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}\right) = \left(\mathcal{A} \xrightarrow{G \circ F} \mathcal{C}\right) = \left(\mathcal{A} \xrightarrow{GF} \mathcal{C}\right).$$

5. The opposite category of $\mathcal{A}$ is denoted by $\mathcal{A}^\circ$. To a morphism $X \xrightarrow{f} Y$ in $\mathcal{A}$ corresponds the morphism $Y \xrightarrow{f^\circ} X$ in $\mathcal{A}^\circ$.

6. We write $\operatorname{Coret}\mathcal{A}$ for the set of coretractions and $\operatorname{Ret}\mathcal{A}$ for the set of retractions of $\mathcal{A}$. We write $\operatorname{Iso}\mathcal{A} = (\operatorname{Coret}\mathcal{A}) \cap (\operatorname{Ret}\mathcal{A})$ for the set of isomorphisms in $\mathcal{A}$.

7. Suppose given $A, B \in \operatorname{Ob}\mathcal{A}$. If A and B are isomorphic in $\mathcal{A}$, we write $A \cong B$.

   To indicate that $\varphi \in {}_{\mathcal{A}}(A, B)$ is an isomorphism, we often write $A \xrightarrow[\sim]{\varphi} B$. Given an isomorphism $f \in {}_{\mathcal{A}}(A, B)$, we often write $f^- \in {}_{\mathcal{A}}(B, A)$ for its inverse.

8. A commutative diagram in $\mathcal{A}$ is a functor from a category associated to a poset to the category $\mathcal{A}$.

9. If we work in a category $\mathcal{A}$, diagrams are meant to be diagrams in $\mathcal{A}$, unless specified otherwise.

10. A commutative quadrangle

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A' \arrow[r, "g"'] & B' \end{tikzcd}$$

is sometimes denoted by (A, B, A', B'). Note that (A, B, A', B') is not the same diagram as (A, A', B, B'), the latter being a mirror image of the former.

11. To indicate that a commutative quadrangle is a pushout, we often write

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] \arrow[ur, phantom, very near end, "\lrcorner"] & B'. \end{tikzcd}$$

To indicate that a commutative quadrangle is a pullback, we often write

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, very near start, "\ulcorner"] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B'. \end{tikzcd}$$

12. Let $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} X$ in $\mathcal{A}$. We say that the quadrangle

$$\begin{tikzcd} X \arrow[r, equal] & X \\ Y \arrow[u, "g"] & X \arrow[l, "f"] \arrow[u, equal] \end{tikzcd}$$

is commutative if the quadrangle

$$\begin{tikzcd} X & X \arrow[l, "1"'] \\ Y \arrow[u, "g"] & X \arrow[l, "f"] \arrow[u, "1"'] \end{tikzcd}$$

is commutative, i.e. if $fg = 1_X$.

13. A functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is called an equivalence if there exists a functor $\mathcal{B} \xrightarrow{G} \mathcal{A}$ such that $(F \circ G) \cong 1_{\mathcal{A}}$ and $(G \circ F) \cong 1_{\mathcal{B}}$. Recall that a functor is an equivalence if and only if it is full, faithful and dense.

14. A functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is called an isomorphism of categories, if F is an isomorphism in the (1-)category of categories. Recall that a functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is an isomorphsim of categories if and only if it is full, faithful and bijective on objects.

15. By $[\mathcal{A},\mathcal{B}]$ we denote the *functor category* whose objects are the functors from $\mathcal{A}$ to $\mathcal{B}$ and whose morphisms are the transformations between such functors.

16. Suppose given

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "J"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{C}. \end{tikzcd}$$

We have the transformation $(H \circ F) \overset{\beta*\alpha}{\Longrightarrow} (J \circ G)$ with

$$(\beta * \alpha)_X := H\alpha_X \cdot \beta_{GX} = \beta_{FX} \cdot J\alpha_X$$

for $X \in \operatorname{Ob}\mathcal{A}$.

Furthermore, we write $\beta * F := \beta * 1_F$. Thus, we have $(\beta * F)_X = \beta_{FX}$ for $X \in \operatorname{Ob}\mathcal{A}$.

Moreover, we write $H * \alpha := 1_H * \alpha$. Thus, we have $(H * \alpha)_X = H\alpha_X$ for $X \in \operatorname{Ob}\mathcal{A}$.

If unabiguous, we write $\beta F := \beta * F$ and $H\alpha := H * \alpha$.

17. Let $\mathcal{A} \xrightarrow{F} \mathcal{B}$ and $\mathcal{B} \xrightarrow{G} \mathcal{A}$ be functors. Let $1_\mathcal{A} \xrightarrow{\eta} (G \circ F)$ and $(F \circ G) \xrightarrow{\varepsilon} 1_\mathcal{B}$ be transformations. We call $(F, G, \eta, \varepsilon)$ an adjunction, if the following diagrams commute.

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, equal] & F \circ G \circ F \arrow[d, "\varepsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, equal] & G \circ F \circ G \arrow[d, "G\varepsilon"] \\ & G \end{tikzcd}$$

In this case, F is called left adjoint to G and G is called right adjoint to F. We also write $F \dashv G$. Furthermore, we call η the unit and ε the counit of the adjunction.

Chapter 1

Preliminaries

1.1 The factor category

In this §1.1 we review the factor category $\mathcal{C}/(\sim)$ of a category $\mathcal{C}$ with respect to a congruence $(\sim)$ on $\mathcal{C}$, the class functor $R_{(\sim)} : \mathcal{C} \to \mathcal{C}/(\sim)$ and its (2-)universal property; cf. Definitions 3 and 4 and Proposition 12 below. This wellknown construction can for example be found in [**11,** III.8] or (without the universal property) in [**15,** 6.4].

1.1.1 Congruences

For this §1.1.1 let $\mathcal{C}$ be a category.

Definition 1.

(1) Let $(\sim)$ be an equivalence relation on $\operatorname{Mor}\mathcal{C}$. We call $(\sim)$ *categorical* if the following assertion holds.

Suppose given $f, g \in \operatorname{Mor}\mathcal{C}$ with $f \sim g$.
Then we have $\operatorname{Source} f = \operatorname{Source} g$ and $\operatorname{Target} f = \operatorname{Target} g$, i.e.

$$\operatorname{Source} f \underset{g}{\overset{f}{\rightrightarrows}} \operatorname{Target} f.$$

For $X, Y \in \operatorname{Ob}\mathcal{C}$ we write $(\sim)_{X,Y} := (\sim) \cap {}_{\mathcal{C}}(X,Y)^{\times 2}$.

(2) Let $(\sim)$ be a categorical equivalence relation on $\operatorname{Mor}\mathcal{C}$. We call $(\sim)$ a *congruence* on $\mathcal{C}$ if the following assertion holds.

Suppose given $X \underset{f_1}{\overset{f_0}{\rightrightarrows}} Y \underset{g_1}{\overset{g_0}{\rightrightarrows}} Z$ in $\mathcal{C}$ with $f_0 \sim f_1$ and $g_0 \sim g_1$.
Then we have $f_0 g_0 \sim f_1 g_1$.

Let $f \in \operatorname{Mor}\mathcal{C}$. We write $[f]$ for its equivalence class.

(3) Let $(\sim)$ be a congruence on $\mathcal{C}$. We call $(\mathcal{C}, (\sim))$ a *category with congruence*.

Remark 2. Let $(\sim)$ be a categorical equivalence relation on $\operatorname{Mor}\mathcal{C}$. The following assertions (1, 2) are equivalent.

(1) The relation $(\sim)$ is a congruence on $\mathcal{C}$.

(2) The following assertions (a, b) hold.

(a) Suppose given $X \underset{f_1}{\overset{f_0}{\rightrightarrows}} Y \xrightarrow{g} Z$ in $\mathcal{C}$ with $f_0 \sim f_1$. Then we have $f_0 g \sim f_1 g$.

(b) Suppose given $X \xrightarrow{f} Y \underset{g_1}{\overset{g_0}{\rightrightarrows}} Z$ in $\mathcal{C}$ with $g_0 \sim g_1$. Then we have $f g_0 \sim f g_1$.

1.1.2 Factor category and class functor

For this §1.1.2 let $\mathcal{C}$ be a category and let $(\sim)$ be a congruence on $\mathcal{C}$.

Definition 3 (and Lemma)**.** We shall define a category $\mathcal{C}/(\sim)$ as follows.

Let $\operatorname{Ob}(\mathcal{C}/(\sim)) := \operatorname{Ob}\mathcal{C}$.

For $X, Y \in \operatorname{Ob}(\mathcal{C}/(\sim))$ we define ${}_{(\mathcal{C}/(\sim))}(X,Y) := {}_{\mathcal{C}}(X,Y)/(\sim)_{X,Y}$.

For X, Y and Z in $\operatorname{Ob}(\mathcal{C}/(\sim))$, $[f] \in {}_{(\mathcal{C}/(\sim))}(X,Y)$ and $[g] \in {}_{(\mathcal{C}/(\sim))}(Y,Z)$ we define

$$[f] \cdot [g] := [fg].$$

For $X \in \operatorname{Ob}(\mathcal{C}/(\sim))$ we define $1_X^{\mathcal{C}/(\sim)} := \left[1_X^{\mathcal{C}}\right]$.

This defines a category $\mathcal{C}/(\sim)$*.*

We call $\mathcal{C}/(\sim)$ the *factor category* of $\mathcal{C}$ *modulo* $(\sim)$.

If unambiguous, we often write $\overline{\mathcal{C}} := \mathcal{C}/(\sim)$.

Proof. Suppose given $X \underset{f_1}{\overset{f_0}{\rightrightarrows}} Y \underset{g_1}{\overset{g_0}{\rightrightarrows}} Z$ in $\mathcal{C}$ with $[f_0] = [f_1]$ and $[g_0] = [g_1]$, i.e. $f_0 \sim f_1$ and $g_0 \sim g_1$.

Since $(\sim)$ is a congruence, we have $f_0 g_0 \sim f_1 g_1$, i.e. $[f_0 g_0] = [f_1 g_1]$. Thus, composition is welldefined.

Suppose given $X \xrightarrow{[f]} Y \xrightarrow{[g]} Z \xrightarrow{[h]} W$ in $\mathcal{C}/(\sim)$. We have

$$[f] \cdot ([g] \cdot [h]) = [f] \cdot [gh] = [f(gh)] = [(fg)h] = [fg] \cdot [h] = ([f] \cdot [g]) \cdot [h].$$

Furthermore, we have $[f] \cdot [1_Y] = [f 1_Y] = [f]$ and $[1_Y] \cdot [g] = [1_Y g] = [g]$.

Thus, we have a category indeed. $\square$

Definition 4 (and Lemma)**.** *We have the functor*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{R_{(\sim)}} & \mathcal{C}/(\sim) = \overline{\mathcal{C}} \\ (X \xrightarrow{f} Y) & \mapsto & (X \xrightarrow{[f]} Y). \end{array}$$

We call $R_{(\sim)}$ the *class functor* of $(\sim)$.

If unambiguous, we often write $R := R_{\mathcal{C}} := R_{(\sim)}$.

Proof. Suppose given $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$.

We have $R(fg) = [fg] = [f] \cdot [g] = Rf \cdot Rg$ and $R\big(1_X^{\mathcal{C}}\big) = \big[1_X^{\mathcal{C}}\big] = 1_X^{\overline{\mathcal{C}}} = 1_{RX}^{\overline{\mathcal{C}}}$.

Thus, we have a functor indeed. □

Lemma 5. *The following assertions* (1, 2) *hold.*

(1) *The class functor $R_{(\sim)}$ is full.*

(2) *The class functor $R_{(\sim)}$ is bijective on objects. In particular, $R_{(\sim)}$ is dense.*

Proof. Ad (1). Suppose given $X, Y \in \operatorname{Ob} \mathcal{C}$. Suppose given

$$\varphi \in {}_{\overline{\mathcal{C}}}(RX, RY) \overset{\text{D4}}{=} {}_{\overline{\mathcal{C}}}(X, Y).$$

By Definition 3, there exists $f \in {}_{\mathcal{C}}(X, Y)$ with $\varphi = [f] \overset{\text{D4}}{=} Rf$. Thus, R is full.

Ad (2). We have $\operatorname{Ob} \overline{\mathcal{C}} \overset{\text{D3}}{=} \operatorname{Ob} \mathcal{C}$ and $RX \overset{\text{D4}}{=} X$ for $X \in \operatorname{Ob} \mathcal{C}$. □

1.1.3 Universal property

For this §1.1.3 let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories and let $(\sim)$ be a congruence on $\mathcal{C}$.

Definition 6. Let ${}_{(\sim)}[\mathcal{C}, \mathcal{D}]$ be the full subcategory of $[\mathcal{C}, \mathcal{D}]$ with

$$\operatorname{Ob} {}_{(\sim)}[\mathcal{C}, \mathcal{D}] := \{F \in \operatorname{Ob}[\mathcal{C}, \mathcal{D}] : \text{for } f, g \in \operatorname{Mor} \mathcal{C} \text{ with } f \sim g, \text{ we have } Ff = Fg\}.$$

Remark 7. The following assertions (1, 2) hold.

(1) We have $R_{(\sim)} \in \operatorname{Ob} {}_{(\sim)}[\mathcal{C}, \overline{\mathcal{C}}]$; cf. Definition 4.

(2) Suppose given $F \in \operatorname{Ob} {}_{(\sim)}[\mathcal{C}, \mathcal{D}]$ and $G \in \operatorname{Ob}[\mathcal{D}, \mathcal{E}]$.
Then we have $(G \circ F) \in \operatorname{Ob} {}_{(\sim)}[\mathcal{C}, \mathcal{E}]$.

Definition 8 (and Lemma)**.** Let $F \in \operatorname{Ob}_{(\sim)}[\mathcal{C},\mathcal{D}]$.

The following assertions (1, 2, 3) *hold.*

(1) *We have the functor*

$$\begin{array}{rcl} \overline{\mathcal{C}} = \mathcal{C}/(\sim) & \xrightarrow{\overline{F}} & \mathcal{D} \\ (X \xrightarrow{[f]} Y) & \mapsto & (FX \xrightarrow{Ff} FY). \end{array}$$

(2) *We have* $\overline{F} \circ R_{(\sim)} = F$.

(3) *Suppose given* $\tilde{F} \in [\overline{\mathcal{C}},\mathcal{D}]$ *with* $\tilde{F} \circ R_{(\sim)} = F$. *Then we have* $\tilde{F} = \overline{F}$.

Proof. Ad (1). Suppose given $X \overset{f}{\underset{\tilde{f}}{\rightrightarrows}} Y \xrightarrow{g} Z$ in $\mathcal{C}$ with $f \sim \tilde{f}$.

Since $F \in \operatorname{Ob}_{(\sim)}[\mathcal{C},\mathcal{D}]$, we have $Ff = F\tilde{f}$. Thus, $\overline{F}$ is welldefined on morphisms. Furthermore, we have

$$\overline{F}1_X^{\overline{\mathcal{C}}} \overset{\text{D3}}{=} \overline{F}[1_X^{\mathcal{C}}] = F1_X^{\mathcal{C}} = 1_{FX}^{\mathcal{D}} = 1_{\overline{F}X}^{\mathcal{D}}$$

and

$$\overline{F}\left([f]\cdot[g]\right) = \overline{F}[fg] = F(fg) = Ff \cdot Fg = \overline{F}[f]\cdot\overline{F}[g].$$

Thus, $\overline{F}$ is a functor indeed.

Ad (2). Suppose given $X \xrightarrow{f} Y$ in $\mathcal{C}$. We have

$$(\overline{F}\circ R)(X \xrightarrow{f} Y) \overset{\text{D4}}{=} \overline{F}(X \xrightarrow{[f]} Y) = (FX \xrightarrow{Ff} FY) = F(X \xrightarrow{f} Y).$$

Ad (3). Suppose given $X \xrightarrow{[f]} Y$ in $\overline{\mathcal{C}}$. We have

$$\overline{F}(X \xrightarrow{[f]} Y) \overset{\text{D4}}{=} (\overline{F}\circ R)(X \xrightarrow{f} Y) = (\tilde{F}\circ R)(X \xrightarrow{f} Y) \overset{\text{D4}}{=} \tilde{F}(X \xrightarrow{[f]} Y).$$

□

Lemma 9. *Suppose given* $F \in \operatorname{Ob}_{(\sim)}[\mathcal{C},\mathcal{D}]$. *The following assertions* (1, 2) *hold.*

(1) *Suppose that* F *is full. Then* $\overline{F}$ *is full.*

(2) *Suppose that* F *is dense. Then* $\overline{F}$ *is dense.*

Proof. Ad (1). Suppose given $X, Y \in \operatorname{Ob}\overline{\mathcal{C}}$.

Suppose given $\varphi \in {}_{\mathcal{D}}(\overline{F}X, \overline{F}Y) = {}_{\mathcal{D}}(FX, FY)$. Since F is full, there exists $f \in {}_{\mathcal{C}}(X,Y)$ with $Ff = \varphi$. We have $\overline{F}[f] = Ff = \varphi$. Thus, $\overline{F}$ is full.

Ad (2). Suppose given $Z \in \operatorname{Ob}\mathcal{D}$. Since F is dense, there exists $X \in \operatorname{Ob}\mathcal{C}$ with $FX \cong Z$. We have $\overline{F}X = FX \cong Z$. Thus, $\overline{F}$ is dense. □

Definition 10 (and Lemma)**.** Let $F \xrightarrow{\alpha} G$ in ${}_{(\sim)}[\mathcal{C},\mathcal{D}]$. Define $\overline{\alpha} := (\alpha_X)_{X\in\mathrm{Ob}\,\overline{\mathcal{C}}}$.

The following assertions (1, 2, 3) *hold.*

(1) *We have* $\overline{F} \xrightarrow{\overline{\alpha}} \overline{G}$ *in* $[\overline{\mathcal{C}},\mathcal{D}]$.

(2) *We have* $\overline{\alpha} * R_{(\sim)} = \alpha$.

(3) *Suppose given* $\overline{F} \xrightarrow{\tilde{\alpha}} \overline{G}$ *with* $\tilde{\alpha} * R_{(\sim)} = \alpha$. *Then we have* $\tilde{\alpha} = \overline{\alpha}$.

Proof. Ad (1). For $X \in \mathrm{Ob}\,\overline{\mathcal{C}} = \mathrm{Ob}\,\mathcal{C}$, we have $(FX \xrightarrow{\alpha_X} GX) \overset{\mathrm{D8}}{=} (\overline{F}X \xrightarrow{\alpha_X} \overline{G}X)$.

We show that $\overline{\alpha}$ is natural. Suppose given $X \xrightarrow{[f]} Y$ in $\overline{\mathcal{C}}$. We have

$$\alpha_X \cdot \overline{G}[f] \overset{\mathrm{D8}}{=} \alpha_X \cdot Gf = Ff \cdot \alpha_Y \overset{\mathrm{D8}}{=} \overline{F}[f] \cdot \alpha_Y .$$

$$\begin{tikzcd}
\overline{F}X = FX \arrow[r, "Ff=\overline{F}[f]"] \arrow[d, "\alpha_X"'] & FY = \overline{F}Y \arrow[d, "\alpha_Y"] \\
\overline{G}X = GX \arrow[r, "Gf=\overline{G}[f]"'] & GY = \overline{G}Y
\end{tikzcd}$$

Ad (2). Suppose given $X \in \mathrm{Ob}\,\mathcal{C}$. We have

$$(\overline{\alpha} * R)_X = \overline{\alpha}_{RX} \overset{\mathrm{D4}}{=} \overline{\alpha}_X = \alpha_X .$$

Ad (3). Suppose given $X \in \mathrm{Ob}\,\overline{\mathcal{C}}$. We have

$$\tilde{\alpha}_X \overset{\mathrm{D4}}{=} \tilde{\alpha}_{RX} = (\tilde{\alpha} * R)_X = \alpha_X = \overline{\alpha}_X .$$

□

Remark 11. *Suppose given an isotransformation* $F \xrightarrow[\sim]{\alpha} G$ *in* ${}_{(\sim)}[\mathcal{C},\mathcal{D}]$. *Then we have an isotransformation* $\overline{F} \xrightarrow[\sim]{\overline{\alpha}} \overline{G}$.

Proof. This follows since $\overline{\alpha}_X = \alpha_X$ for $X \in \mathrm{Ob}\,\mathcal{C} = \mathrm{Ob}\,\overline{\mathcal{C}}$; cf. Definition 10. □

Proposition 12. Recall that $\mathcal{C}$ and $\mathcal{D}$ are categories.

Recall that $(\sim)$ is a congruence on $\mathcal{C}$ and that $\overline{\mathcal{C}} = \mathcal{C}/(\sim)$. Cf. Definitions 1 and 3.

The following assertions (1, 2) *hold.*

(1) *We have* $R_{(\sim)} \in \mathrm{Ob}\ {}_{(\sim)}[\mathcal{C},\overline{\mathcal{C}}]$*;* cf. Definitions 4 and 6.
Suppose given $F \xrightarrow{\alpha} G$ *in* ${}_{(\sim)}[\mathcal{C},\mathcal{D}]$.
We have unique $\overline{F},\overline{G} \in [\overline{\mathcal{C}},\mathcal{D}]$ *with* $\overline{F}\circ R_{(\sim)} = F$ *and* $\overline{G}\circ R_{(\sim)} = G$*;* cf. Definition 8.
We have a unique transformation $\overline{F} \overset{\overline{\alpha}}{\Longrightarrow} \overline{G}$ *with* $\overline{\alpha} * R_{(\sim)} = \alpha$*;* cf. Definition 10.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "R_{(\sim)}"] \arrow[dr, bend left, "G"] \arrow[dr, bend right, "F"'] \arrow[dr, phantom, "\Uparrow\alpha"] & \overline{\mathcal{C}} \arrow[d, bend right, "\overline{F}"'] \arrow[d, bend left, "\overline{G}"] \arrow[d, phantom, "\overset{\overline{\alpha}}{\Rightarrow}"] \\
& \mathcal{D}
\end{tikzcd}$$

(2) *We have the isomorphism of categories*

$$\begin{array}{rcl} {}_{(\sim)}[\mathcal{C},\mathcal{D}] & \longleftarrow & [\overline{\mathcal{C}},\mathcal{D}] \\ (U \circ R_{(\sim)} \xrightarrow{\beta * R_{(\sim)}} V \circ R_{(\sim)}) & \leftarrow\!\shortmid & (U \xrightarrow{\beta} V) \end{array}$$

with inverse

$$\begin{array}{rcl} {}_{(\sim)}[\mathcal{C},\mathcal{D}] & \longrightarrow & [\overline{\mathcal{C}},\mathcal{D}] \\ (F \xrightarrow{\alpha} G) & \mapsto & (\overline{F} \xrightarrow{\overline{\alpha}} \overline{G})\,. \end{array}$$

Proof. *Ad* (1). This follows from Definitions 8 and 10.

Ad (2). This follows from (1) and Remark 7. □

1.1.4 Functoriality

For this §1.1.4, let $\Big(\mathcal{C}, (\overset{\mathcal{C}}{\sim})\Big)$, $\Big(\mathcal{D}, (\overset{\mathcal{D}}{\sim})\Big)$ and $\Big(\mathcal{E}, (\overset{\mathcal{E}}{\sim})\Big)$ be categories with congruences.

By abuse of notation, we write $(\sim) = (\overset{\mathcal{C}}{\sim})$ and $(\sim) = (\overset{\mathcal{D}}{\sim})$, etc.

Definition 13. Let ${}_{\mathrm{Ho}}[\mathcal{C},\mathcal{D}]$ be the full subcategory of $[\mathcal{C},\mathcal{D}]$ with

$$\begin{aligned} \mathrm{Ob}\ {}_{\mathrm{Ho}}[\mathcal{C},\mathcal{D}] &:= \{F \in \mathrm{Ob}[\mathcal{C},\mathcal{D}] : R_{\mathcal{D}} \circ F \in \mathrm{Ob}\ {}_{(\sim)}[\mathcal{C},\overline{\mathcal{D}}]\} \\ &= \{F \in \mathrm{Ob}[\mathcal{C},\mathcal{D}] : \text{for } f,g \in \mathrm{Mor}\,\mathcal{C} \text{ with } f \sim g, \text{ we have } Ff \sim Fg\}. \end{aligned}$$

Definition 14 (and Lemma)**.** Let $F \in \mathrm{Ob}\ {}_{\mathrm{Ho}}[\mathcal{C},\mathcal{D}]$.

Let $\mathrm{Ho}\,F := \overline{(R_{\mathcal{D}} \circ F)}$; cf. Definition 8.(1).

The following assertions (1, 2, 3) *hold.*

(1) *We have*

$$\begin{array}{rcl} \overline{\mathcal{C}} = \mathcal{C}/(\sim) & \xrightarrow{\mathrm{Ho}\,F} & \mathcal{D}/(\sim) = \overline{\mathcal{D}} \\ (X \xrightarrow{[f]} Y) & \mapsto & (FX \xrightarrow{[Ff]} FY). \end{array}$$

(2) *We have* $R_{\mathcal{D}} \circ F = (\mathrm{Ho}\,F) \circ R_{\mathcal{C}}$ *.*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "R_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, "R_{\mathcal{D}}"] \\ \overline{\mathcal{C}} \arrow[r, "\mathrm{Ho}\,F"'] & \overline{\mathcal{D}} \end{tikzcd}$$

(3) *Suppose given* $\tilde{F} \in \mathrm{Ob}[\overline{\mathcal{C}},\overline{\mathcal{D}}]$ *with* $R_{\mathcal{D}} \circ F = \tilde{F} \circ R_{\mathcal{C}}$*. Then we have* $\tilde{F} = \mathrm{Ho}\,F$*.*

Proof. By Definition 8.(2), there exists a functor

$$\Big(\overline{\mathcal{C}} \xrightarrow{\mathrm{Ho}\,F} \overline{\mathcal{D}}\Big) = \Big(\overline{\mathcal{C}} \xrightarrow{\overline{(R_{\mathcal{D}} \circ F)}} \overline{\mathcal{D}}\Big)$$

with $R_{\mathcal{D}} \circ F = (\operatorname{Ho} F) \circ R_{\mathcal{C}}$.

By Definition 8.(3), this equality determines $\operatorname{Ho} F$ uniquely.

Furthermore, we have

$$(\operatorname{Ho} F)\Big(X \xrightarrow{[f]} Y\Big) \overset{\text{D8.(1)}}{=} \Big((R_{\mathcal{D}} \circ F)X \xrightarrow{(R_{\mathcal{D}} \circ F)f} (R_{\mathcal{D}} \circ F)Y\Big) \overset{\text{D4}}{=} \Big(FX \xrightarrow{[Ff]} FY\Big)$$

for $X \xrightarrow{[f]} Y$ in $\overline{\mathcal{C}}$. □

Lemma 15. *Suppose given* $F \in \operatorname{Ob} {}_{\operatorname{Ho}}[\mathcal{C}, \mathcal{D}]$ *and* $G \in \operatorname{Ob} {}_{\operatorname{Ho}}[\mathcal{D}, \mathcal{E}]$.

The following assertions (1, 2) *hold.*

(1) *We have* $\operatorname{Ho} 1_{\mathcal{C}} = 1_{\overline{\mathcal{C}}}$.

(2) *We have* $G \circ F \in \operatorname{Ob} {}_{\operatorname{Ho}}[\mathcal{C}, \mathcal{E}]$ *and* $\operatorname{Ho}(G \circ F) = \operatorname{Ho} G \circ \operatorname{Ho} F$.

Proof. Ad (1). Suppose given $X \xrightarrow{[f]} Y$ in $\overline{\mathcal{C}}$. We have

$$(\operatorname{Ho} 1_{\mathcal{C}})\Big(X \xrightarrow{[f]} Y\Big) = \Big(1_{\mathcal{C}}X \xrightarrow{[1_{\mathcal{C}} f]} 1_{\mathcal{C}}Y\Big) = \Big(X \xrightarrow{[f]} Y\Big) = 1_{\overline{\mathcal{C}}}\Big(X \xrightarrow{[f]} Y\Big).$$

Ad (2). Suppose given $X \xrightarrow{[f]} Y$ in $\overline{\mathcal{C}}$. We have

$$\begin{aligned}(\operatorname{Ho}(G \circ F))\Big(X \xrightarrow{[f]} Y\Big) &= \Big((G \circ F)X \xrightarrow{[(G \circ F)f]} (G \circ F)Y\Big) \\ &= (\operatorname{Ho} G)\Big(FX \xrightarrow{[Ff]} FY\Big) \\ &= ((\operatorname{Ho} G) \circ (\operatorname{Ho} F))\Big(X \xrightarrow{[f]} Y\Big).\end{aligned}$$

□

Definition 16 (and Lemma)**.** Suppose given $F \xrightarrow{\alpha} G$ in ${}_{\operatorname{Ho}}[\mathcal{C}, \mathcal{D}]$. Define

$$\operatorname{Ho} \alpha := \overline{(R_{\mathcal{D}} * \alpha)} : \operatorname{Ho} F \longrightarrow \operatorname{Ho} G;$$

cf. Definition 10. *The following assertions* (1, 2, 3) *hold.*

(1) *We have* $(\operatorname{Ho} \alpha)_X = [\alpha_X]$ *for* $X \in \operatorname{Ob} \overline{\mathcal{C}}$.

(2) *We have* $(\operatorname{Ho} \alpha) * R_{\mathcal{C}} = R_{\mathcal{D}} * \alpha$.

(3) *Suppose given* $\operatorname{Ho} F \xrightarrow{\tilde{\alpha}} \operatorname{Ho} G$ *with* $\tilde{\alpha} * R_{\mathcal{C}} = R_{\mathcal{D}} * \alpha$. *Then we have* $\tilde{\alpha} = \operatorname{Ho} \alpha$.

Proof. We have the transformation $(R_{\mathcal{D}} \circ F) \xrightarrow{R_{\mathcal{D}} * \alpha} (R_{\mathcal{D}} \circ G)$.

By Definition 10.(2), there exists a unique transformation

$$\Big(\operatorname{Ho} F \xrightarrow{\operatorname{Ho} \alpha} \operatorname{Ho} G\Big) \overset{\text{D14}}{=} \Big(\overline{(R_{\mathcal{D}} \circ F)} \xrightarrow{\overline{(R_{\mathcal{D}} * \alpha)}} \overline{(R_{\mathcal{D}} \circ G)}\Big)$$

with $(\operatorname{Ho} \alpha) * R_{\mathcal{C}} = R_{\mathcal{D}} * \alpha$.

By Definition 10.(3), this equality defines $\operatorname{Ho} \alpha$ uniquely.

Furthermore, we have $(\operatorname{Ho} \alpha)_X \overset{\text{D10}}{=} (R_{\mathcal{D}} * \alpha)_X = R_{\mathcal{D}} \alpha_X \overset{\text{D4}}{=} [\alpha_X]$ for $X \in \operatorname{Ob} \overline{\mathcal{C}}$. □

Lemma 17. *Suppose given* $F \xrightarrow{\alpha} F'$ *and* $F' \xrightarrow{\alpha'} F''$ *in* ${}_{\mathrm{Ho}}[\mathcal{C}, \mathcal{D}]$.

The following assertions (1, 2) *hold.*

(1) *We have* $\mathrm{Ho}\, 1_F = 1_{\mathrm{Ho}\, F}$.

(2) *We have* $\mathrm{Ho}(\alpha\alpha') = \mathrm{Ho}\,\alpha \cdot \mathrm{Ho}\,\alpha'$.

Proof. *Ad* (1). Suppose given $X \in \mathrm{Ob}\,\overline{\mathcal{C}}$. We have

$$(\mathrm{Ho}\, 1_F)_X = [(1_F)_X] = [1^{\mathcal{D}}_{FX}] = 1^{\overline{\mathcal{D}}}_{FX} = 1^{\overline{\mathcal{D}}}_{(\mathrm{Ho}\, F)X} = (1_{\mathrm{Ho}\, F})_X\ .$$

Ad (2). Suppose given $X \in \mathrm{Ob}\,\overline{\mathcal{C}}$. We have

$$(\mathrm{Ho}(\alpha\alpha'))_X = [(\alpha\alpha')_X] = [\alpha_X \cdot \alpha'_X] = [\alpha_X] \cdot [\alpha'_X] = (\mathrm{Ho}\,\alpha)_X \cdot (\mathrm{Ho}\,\alpha')_X = ((\mathrm{Ho}\,\alpha) \cdot (\mathrm{Ho}\,\alpha'))_X\ .$$

□

Lemma 18. *Suppose given* $F \xrightarrow{\alpha} F'$ *in* ${}_{\mathrm{Ho}}[\mathcal{C}, \mathcal{D}]$ *and* $G \xrightarrow{\beta} G'$ *in* ${}_{\mathrm{Ho}}[\mathcal{D}, \mathcal{E}]$.

We have $\mathrm{Ho}(\beta * \alpha) = (\mathrm{Ho}\,\beta) * (\mathrm{Ho}\,\alpha)$.

Proof. Suppose given $X \in \mathrm{Ob}\,\overline{\mathcal{C}}$. We have

$$\begin{aligned}
(\mathrm{Ho}(\beta * \alpha))_X &\overset{\mathrm{D16.(1)}}{=} [(\beta * \alpha)_X] \\
&= [G\alpha_X \cdot \beta_{F'X}] \\
&= [G\alpha_X] \cdot [\beta_{F'X}] \\
&\overset{\mathrm{D14.(1),D16.(1)}}{=} (\mathrm{Ho}\, G)[\alpha_X] \cdot (\mathrm{Ho}\,\beta)_{F'X} \\
&\overset{\mathrm{D14.(1),D16.(1)}}{=} (\mathrm{Ho}\, G)(\mathrm{Ho}\,\alpha)_X \cdot (\mathrm{Ho}\,\beta)_{(\mathrm{Ho}\, F')X} \\
&= (\mathrm{Ho}\,\beta * \mathrm{Ho}\,\alpha)_X\ .
\end{aligned}$$

□

1.2 A remark on isomorphisms

For this §1.2, let $\mathcal{C}$ be a category.

Remark 19. *Suppose given* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ *in* $\mathcal{C}$.

Suppose that $fg, gh \in \mathrm{Iso}\,\mathcal{C}$. *Then we have* $f, g, h, fgh \in \mathrm{Iso}\,\mathcal{C}$.

$$\begin{tikzcd}
& B \arrow[rr, "gh", "\sim"'] \arrow[dr, "g"] & & D \\
A \arrow[ur, "f"] \arrow[rr, "\sim", "fg"'] \arrow[urrr, "fgh"] & & C \arrow[ur, "h"'] &
\end{tikzcd}$$

Proof. It suffices to show that $f, g, h \in \operatorname{Iso}\mathcal{C}$.

We have
$$g \cdot h(gh)^- = 1_B \qquad \text{and} \qquad (fg)^- f \cdot g = 1_C\,.$$
Thus, we have $g \in (\operatorname{Coret}\mathcal{C}) \cap (\operatorname{Ret}\mathcal{C}) = \operatorname{Iso}\mathcal{C}$. Furthermore, we have

$$f \cdot g(fg)^- = 1_A \qquad \text{and} \qquad g(fg)^- \cdot f = g(fg)^- \cdot f \cdot gg^- = gg^- = 1_B\,.$$

Similarily, we have

$$(gh)^- g \cdot h = 1_D \qquad \text{and} \qquad h \cdot (gh)^- g = g^- g \cdot h \cdot (gh)^- g = g^- g = 1_C\,.$$

□

1.3 Retracts and lifting properties

In this §1.3 we introduce rectracts and the extension properties. Furthermore, we establish the well known *rectract argument*; cf. for example [**9,** Prop. 7.2.2].

For this §1.3 let $\mathcal{C}$ be a category.

Definition 20. Let $A \xrightarrow{a} A'$ and $B \xrightarrow{b} B'$ in $\mathcal{C}$.

We call a a *retract* of b if there exists a commutative diagram as follows.

$$\begin{tikzcd}
A \arrow[r, "f"'] \arrow[rr, bend left, "1"] \arrow[d, "a"'] & B \arrow[r, "g"'] \arrow[d, "b"] & A \arrow[d, "a"] \\
A' \arrow[r, "f'"] \arrow[rr, bend right, "1"'] & B' \arrow[r, "g'"] & A'
\end{tikzcd}$$

Definition 21. Let $A \xrightarrow{a} A'$ and $B \xrightarrow{b} B'$ in $\mathcal{C}$.

We call (a, b) an *extension pair* if the following assertion $(*)$ holds.

$(*)$ Suppose given $A \xrightarrow{f} B$ and $A' \xrightarrow{f'} B'$ with $af' = fb$.

Then there exists $A' \xrightarrow{h} B$ with $ah = f$ and $hb = f'$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] \arrow[ur, dotted, "h"] & B'
\end{tikzcd}$$

Lemma 22. *Suppose given* $A \xrightarrow{i} B \xrightarrow{j} C$ *and* $X \xrightarrow{p} Y \xrightarrow{q} Z$*.*

The follwing assertions (1, 2) *hold.*

(1) *Suppose that* (i, p) *and* (j, p) *are extension pairs. Then* (ij, p) *is an extension pair.*

(2) *Suppose that* (i, p) *and* (i, q) *are extension pairs. Then* (i, pq) *is an extension pair.*

Proof. Ad (1). Suppose given $A \xrightarrow{f} X$ and $C \xrightarrow{g} Z$ with $ijg = fp$.

Since (i, p) is an extension pair, there exists $B \xrightarrow{k} X$ with $ik = f$ and $jg = kp$.

Since (j, p) is an extension pair, there exists $C \xrightarrow{h} X$ with $jh = k$ and $hp = g$.

Thus, we have $hp = g$ and $(ij)h = ik = f$. Therefore, (ij, p) is an extension pair.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[dd, "p"] \\
B \arrow[ur, dotted, "k"] \arrow[d, "j"'] & \\
C \arrow[uur, dotted, "h"] \arrow[r, "g"'] & Z
\end{tikzcd}$$

Ad (2). This is dual to (1). $\square$

Lemma 23 (The retract argument, [**9**, Prop. 7.2.2])**.**

Suppose given $X \xrightarrow{i} Y \xrightarrow{p} Z$ *in* $\mathcal{C}$*. The following assertions* (1, 2) *hold.*

(1) *Suppose that* (ip, p) *is an extension pair. Then* ip *is a retract of* i*.*

(2) *Suppose that* (i, ip) *is an extension pair. Then* ip *is a retract of* p*.*

Proof. Ad (1). Since (ip, p) is an extension pair, there exists a commutative diagram as follows.

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "ip"'] & Y \arrow[d, "p"] \\
Z \arrow[ur, dotted, "h"] \arrow[r, equal] & Z
\end{tikzcd}$$

Thus, we have the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "ip"'] & X \arrow[r, equal] \arrow[d, "i"'] & X \arrow[d, "ip"] \\
Z \arrow[r, "h"] \arrow[rr, bend right, "1"'] & Y \arrow[r, "p"] & Z
\end{tikzcd}$$

Ad (2). This is dual to (1). $\square$

1.4 (Weak) pushouts and (weak) pullbacks

For this §1.4, let $\mathcal{C}$ be a category.

Definition 24. A commutative quadrangle

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

is called a *weak pushout*, if the following assertion $(*)$ holds.

$(*)$ Suppose given $A' \xrightarrow{x} T \xleftarrow{y} B$ with $ax = fy$.

Then there exists $B' \xrightarrow{u} T$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \arrow[ddr, bend left, "y"] & \\ A' \arrow[r, "f'"'] \arrow[drr, bend right, "x"'] & B' \arrow[dr, dotted, "u"] & \\ & & T \end{tikzcd}$$

To indicate that a commutative quadrangle (A, B, A', B') is a weak pushout, we often write

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{w}"] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B'. \end{tikzcd}$$

Definition 25. A commutative quadrangle

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

is called a *weak pullback*, if the following assertion $(*)$ holds.

$(*)$ Suppose given $A' \xleftarrow{x} T \xrightarrow{y} B$ with $xf' = yb$.

Then there exists $T \xrightarrow{u} A$ such that the following diagram commutes.

$$\begin{tikzcd} T \arrow[dr, dotted, "u"] \arrow[drr, bend left, "y"] \arrow[ddr, bend right, "x"'] & & \\ & A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ & A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

To indicate that a commutative quadrangle (A, B, A', B') is a weak pullback, we often write

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{W}", near start] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B'. \end{tikzcd}$$

Remark 26. Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

be a commutative diagram in $\mathcal{C}$.

The following assertions (1, 2) are equivalent.

(1) The quadrangle (A, B, A', B') is a weak pushout.

(2) The quadrangle (A, A', B, B') is a weak pushout.

The following assertions (3, 4) are equivalent.

(3) The quadrangle (A, B, A', B') is a weak pullback.

(4) The quadrangle (A, A', B, B') is a weak pullback.

Lemma 27. *Suppose given*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{W}", near end] & B \arrow[r, "g"] \arrow[d, "b"] \arrow[dr, phantom, "\mathrm{W}", near end] & C \arrow[d, "c"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C'. \end{tikzcd}$$

Then (A, C, A', C') is a weak pushout.

Proof. Suppose given $A' \xrightarrow{x} T \xleftarrow{y} C$ with $ax = fgy$.

Since (A, B, A', B') is a weak pushout, there exists a commutative diagram as follows.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \arrow[ddr, bend left, "gy"] & \\ A' \arrow[r, "f'"'] \arrow[drr, bend right, "x"'] & B' \arrow[dr, dotted, "k"] & \\ & & T \end{tikzcd}$$

Since (B, C, B', C') is a weak pushout, there exists a commutative diagram as follows.

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[d, "c"] \arrow[ddr, bend left, "y"] & \\
B' \arrow[r, "g'"'] \arrow[drr, bend right, "k"'] & C' \arrow[dr, dotted, "h"] & \\
& & T
\end{tikzcd}$$

Thus, the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "fg"] \arrow[d, "a"'] & C \arrow[d, "c"] \arrow[ddr, bend left, "y"] & \\
A' \arrow[r, "f'g'"'] \arrow[drr, bend right, "x"'] & C' \arrow[dr, "h"] & \\
& & T
\end{tikzcd}$$

$\square$

Lemma 28. *Suppose given*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{W}"] & B \arrow[r, "g"] \arrow[d, "b"'] \arrow[dr, phantom, "\mathrm{W}"] & C \arrow[d, "c"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C'.
\end{tikzcd}$$

Then (A, C, A', C') is a weak pullback.

Proof. This is dual to Lemma 27. $\square$

Definition 29. A weak pushout

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{W}"] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

is called a *pushout*, if the following assertion $(*)$ holds.

$(*)$ Suppose given $B' \overset{u}{\underset{v}{\rightrightarrows}} T$ with $f'u = f'v$ and $bu = bv$.

Then we have $u = v$.

To indicate that a commutative quadrangle (A, B, A', B') is a pushout, we often write

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] & B'.
\end{tikzcd}$$

Definition 30. A weak pullback

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{W}", near start] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

is called a *pullback*, if the following assertion (∗) holds.

(∗) Suppose given $T \overset{u}{\underset{v}{\rightrightarrows}} A$ with $uf = vf$ and $ua = va$.

Then we have $u = v$.

To indicate that a commutative quadrangle (A, B, A', B') is a pullback, we often write

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\ulcorner", near start] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B'. \end{tikzcd}$$

Remark 31. Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

be a commutative diagram in $\mathcal{C}$. The following assertions (1, 2) are equivalent.

(1) The quadrangle (A, B, A', B') is a pushout.

(2) The quadrangle (A, A', B, B') is a pushout.

The following assertions (3, 4) are equivalent.

(3) The quadrangle (A, B, A', B') is a pullback.

(4) The quadrangle (A, A', B, B') is a pullback.

Lemma 32. *Suppose given*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\lrcorner", near end] & B \arrow[r, "g"] \arrow[d, "b"'] \arrow[dr, phantom, "\lrcorner", near end] & C \arrow[d, "c"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C'. \end{tikzcd}$$

Then (A, C, A', C') *is a pushout.*

Proof. By Lemma 27, the quadrangle (A, C, A', C') is a weak pushout.

Suppose given $C' \overset{u}{\underset{v}{\rightrightarrows}} T$ with $f'g'u = f'g'v$ and $cu = cv$.

Since $f' \cdot g'u = f' \cdot g'v$ and $b \cdot g'u = b \cdot g'v$, we obtain $g'u = g'v$; cf. Definition 29.

Since $g'u = g'v$ and $cu = cv$, we obtain $u = v$; cf. Definition 29. □

Lemma 33. *Suppose given*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\ulcorner" very near start] & B \arrow[r, "g"] \arrow[d, "b"'] \arrow[dr, phantom, "\ulcorner" very near start] & C \arrow[d, "c"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C'. \end{tikzcd}$$

Then (A, C, A', C') is a pullback.

Proof. This is dual to Lemma 32. □

Lemma 34. *Suppose given the following commutative diagram.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[rr, bend left, "h"] \arrow[dr, phantom, "\lrcorner" very near end] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[d, "c"] \\ A' \arrow[r, "f'"'] \arrow[rr, bend right, "h'"'] & B' \arrow[r, "g'"] \arrow[r, phantom, "\lrcorner" near end, shift right=2ex] & C' \end{tikzcd}$$

Then (B, C, B', C') is a pushout.

Proof. Suppose given $B' \xrightarrow{x} T \xleftarrow{y} C$ with $bx = gy$.

Since (A, C, A', C') is a weak pushout, we have a commutative diagram as follows.

$$\begin{tikzcd} A \arrow[r, "fg"] \arrow[d, "a"'] & C \arrow[d, "c"] \arrow[ddr, bend left, "y"] & \\ A' \arrow[r, "f'g'"'] \arrow[drr, bend right, "f'x"'] & C' \arrow[dr, dotted, "u"] & \\ & & T \end{tikzcd}$$

Cf. Definitions 24 and 29.

Since we have $f' \cdot g'u = f' \cdot x$ and $b \cdot g'u = gcu = gy = b \cdot x$, and since (A, B, A', B') is a pushout, we obtain $g'u = x$.

Since the diagram

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[d, "c"] \arrow[ddr, bend left, "y"] & \\ B' \arrow[r, "g'"'] \arrow[drr, bend right, "x"'] & C' \arrow[dr, "u"] & \\ & & T \end{tikzcd}$$

commutes, (B, C, B', C') is a weak pushout.

It remains to show that (B, C, B', C') fullfills $(*)$ from Definition 29.

Suppose given $C' \underset{v}{\overset{u}{\rightrightarrows}} T$ with $cu = cv$ and $g'u = g'v$.

Since $f'g' \cdot u = f'g' \cdot v$ and $cu = cv$, we obtain $u = v$; cf. Definition 29. $\square$

Lemma 35. *Suppose given*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "\hat{b}"] \\ A' \arrow[r, "\hat{f}"'] & \hat{B}. \end{tikzcd}$$

The following assertions (1, 2, 3) *hold.*

(1) *There exists a commutative diagram as follows.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "b"] \arrow[ddr, bend left, "\hat{b}"] & \\ A' \arrow[r, "f'"'] \arrow[drr, bend right, "\hat{f}"'] & B' \arrow[dr, dotted, "u"] & \\ & & \hat{B} \end{tikzcd}$$

(2) *Suppose given commutative diagrams*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "b"] \arrow[ddr, bend left, "\hat{b}"] & \\ A' \arrow[r, "f'"'] \arrow[drr, bend right, "\hat{f}"'] & B' \arrow[dr, "u"] & \\ & & \hat{B} \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "\hat{b}"] \arrow[ddr, bend left, "b"] & \\ A' \arrow[r, "\hat{f}"'] \arrow[drr, bend right, "f'"'] & \hat{B} \arrow[dr, "v"] & \\ & & B'. \end{tikzcd}$$

Then we have $uv = 1_{B'}$ *and* $vu = 1_{\hat{B}}$.

(3) *Suppose given a commutative diagram as follows.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "b"] \arrow[ddr, bend left, "\hat{b}"] & \\ A' \arrow[r, "f'"'] \arrow[drr, bend right, "\hat{f}"'] & B' \arrow[dr, "u"] & \\ & & \hat{B} \end{tikzcd}$$

Then we have $u \in \operatorname{Iso} \mathcal{C}$.

Proof. *Ad* (1). This holds since (A, B, A', B') is a weak pushout; cf. Definitions 24 and 29.

Ad (2). By symmetry, it suffices to show that $uv = 1_{B'}$.

Since $f' \cdot uv = \hat{f}v = f'$ and $b \cdot uv = \hat{b}v = b$, we obtain $uv = 1_{B'}$; cf. Definition 29.

Ad (3). This follows from (1) and (2). $\square$

1.5 Equivalences and adjoint functors

In this §1.5, we recapitulate elementary facts about equivalences of categories and adjoint functors. In particular, we recall that mutually inverse equivalences are also adjoint functors.

For this §1.5, let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories.

Lemma 36. *Suppose given mutually inverse equivalences $F \in [\mathcal{C}, \mathcal{D}]$ and $G \in [\mathcal{D}, \mathcal{C}]$.*

Suppose given $F \circ G \xrightarrow[\sim]{\beta} 1_{\mathcal{D}}$. Then there exists $1_{\mathcal{C}} \xrightarrow[\sim]{\alpha} G \circ F$ such that (F, G, α, β) is an adjunction.

Proof. Suppose given $X \in \operatorname{Ob} \mathcal{C}$.

Since F is full and faithful, there uniquely exists $\alpha_X \in {}_{\mathcal{C}}(X, GFX)$ with

$$F\alpha_X = \beta^{-}_{FX} .$$

We show that $\alpha := (\alpha_Y)_{Y \in \operatorname{Ob} \mathcal{C}}$ is natural, i.e. that $\alpha \in {}_{[\mathcal{C},\mathcal{C}]}(1_{\mathcal{C}}, G \circ F)$.

Suppose given $Y \xrightarrow{f} Z$ in $\mathcal{C}$. We show that the following diagram is commutative.

$$\begin{tikzcd} Y \arrow[r, "\alpha_Y"] \arrow[d, "f"'] & GFY \arrow[d, "GFf"] \\ Z \arrow[r, "\alpha_Z"] & GFZ \end{tikzcd}$$

Applying F, we obtain the following diagram.

$$\begin{tikzcd} FY \arrow[r, "F\alpha_Y = \beta^{-}_{FY}"] \arrow[d, "Ff"'] & FGFY \arrow[d, "FGFf"] \\ FZ \arrow[r, "F\alpha_Z = \beta^{-}_{FZ}"] & FGFZ \end{tikzcd}$$

Since β is natural, the latter diagram commutes. Since F is full and faithful, the former diagram commutes as well. Therefore, α is natural.

By construction, we have $F\alpha \cdot \beta F = 1_F$.

We show that $\alpha G \cdot G\beta = 1_G$.

Suppose given $Y \in \operatorname{Ob} \mathcal{D}$. We have to show that $\alpha_{GY} \cdot G\beta_Y = 1_{GY}$.

Since F is faithful, it suffices to show that $F\alpha_{GY} \cdot FG\beta_Y \overset{!}{=} F1_{GY} = 1_{FGY}$.

Thus, it suffices to show that $\beta^{-}_{FGY} \cdot FG\beta_Y = 1_{FGY}$.

Since β is natural, the following diagram commutes.

$$\begin{tikzcd} FGFGY \arrow[r, "\beta_{FGY}"] \arrow[d, "FG\beta_Y"'] & FGY \arrow[d, "\beta_Y"] \\ FGY \arrow[r, "\beta_Y"'] & Y \end{tikzcd}$$

Since β is an isotransformation, we obtain $\beta_{FGY} = FG\beta_Y$, as needed. □

Lemma 37. *Let $(F, G, \eta, \varepsilon)$ and $(F, \tilde{G}, \tilde{\eta}, \tilde{\varepsilon})$ be adjunctions with $F \in \mathrm{Ob}[\mathcal{C}, \mathcal{D}]$.*
Let $\gamma := (\tilde{\eta}G) \cdot (\tilde{G}\varepsilon) \in {}_{[\mathcal{D},\mathcal{C}]}(G, \tilde{G})$ and $\tilde{\gamma} := (\eta\tilde{G}) \cdot (G\tilde{\varepsilon}) \in {}_{[\mathcal{D},\mathcal{C}]}(\tilde{G}, G)$.
Then the following assertions (1, 2) hold.

(1) *We have $\gamma \in \mathrm{Iso}[\mathcal{D}, \mathcal{C}]$ with $\gamma^- = \tilde{\gamma}$. In particular, we have $G \xrightarrow[\sim]{\gamma} \tilde{G}$.*

(2) *The following diagrams commute.*

$$\begin{tikzcd} 1_{\mathcal{C}} \arrow[r, "\eta"] \arrow[dr, "\tilde{\eta}"'] & G \circ F \arrow[d, "\gamma F"] \\ & \tilde{G} \circ F \end{tikzcd} \qquad \begin{tikzcd} F \circ G \arrow[r, "\varepsilon"] \arrow[d, "F\gamma"'] & 1_{\mathcal{D}} \\ F \circ \tilde{G} \arrow[ur, "\tilde{\varepsilon}"'] & \end{tikzcd}$$

Proof. *Ad* (1). We show have to show that $\gamma \cdot \tilde{\gamma} = 1_G$ and $\tilde{\gamma} \cdot \gamma = 1_{\tilde{G}}$. By symmetry, it suffices to show that $\gamma \cdot \tilde{\gamma} = 1_G$.

Suppose given $Y \in \mathrm{Ob}\,\mathcal{D}$. We have to show that

$$1_{GY} \stackrel{!}{=} \gamma_Y \cdot \tilde{\gamma}_Y = \tilde{\eta}_{GY} \cdot \tilde{G}\varepsilon_Y \cdot \eta_{\tilde{G}Y} \cdot G\tilde{\varepsilon}_Y\,.$$

Since η and $\tilde{\varepsilon}$ are natural, we have the following commutative diagram.

$$\begin{tikzcd} GY \arrow[r, "\eta_{GY}"] \arrow[d, "\tilde{\eta}_{GY}"'] & GFGY \arrow[d, "GF\tilde{\eta}_{GY}"] & \\ \tilde{G}FGY \arrow[r, "\eta_{\tilde{G}FGY}"'] \arrow[d, "\tilde{G}\varepsilon_Y"'] & GF\tilde{G}FGY \arrow[r, "G\tilde{\varepsilon}_{FGY}"] \arrow[d, "GF\tilde{G}\varepsilon_Y"] & GFGY \arrow[d, "G\varepsilon_Y"] \\ \tilde{G}Y \arrow[r, "\eta_{\tilde{G}Y}"'] & GF\tilde{G}Y \arrow[r, "G\tilde{\varepsilon}_Y"'] & GY \end{tikzcd}$$

Thus, it suffices to show that

$$\eta_{GY} \cdot GF\tilde{\eta}_{GY} \cdot G\tilde{\varepsilon}_{FGY} \cdot G\varepsilon_Y \stackrel{!}{=} 1_{GY}\,.$$

Since $\eta G \cdot G\varepsilon = 1_G$, it suffices to show that

$$GF\tilde{\eta}_{GY} \cdot G\tilde{\varepsilon}_{FGY} \stackrel{!}{=} 1_{GFGY}\,.$$

This holds, since we have $F\tilde{\eta} \cdot \tilde{\varepsilon}F = 1_F$.

Ad (2). Suppose given $X \in \mathrm{Ob}\,\mathcal{C}$. We have to show that

$$\tilde{\eta}_X \stackrel{!}{=} \eta_X \cdot \gamma_{FX} = \eta_X \cdot \tilde{\eta}_{GFX} \cdot \tilde{G}\varepsilon_{FX}\,.$$

Since $\tilde{\eta}$ is natural, the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "\tilde{\eta}_X"'] & GFX \arrow[d, "\tilde{\eta}_{GFX}"] \\ \tilde{G}FX \arrow[r, "\tilde{G}F\eta_X"'] & \tilde{G}FGFX \end{tikzcd}$$

Thus, it suffices to show that

$$\tilde{\eta}_X \cdot \tilde{G}F\eta_X \cdot \tilde{G}\varepsilon_{FX} \overset{!}{=} \tilde{\eta}_X\,.$$

This holds, since $F\eta \cdot \varepsilon F = 1_F$.

Suppose given $Y \in \operatorname{Ob}\mathcal{D}$. We have to show that

$$\varepsilon_Y \overset{!}{=} F\gamma_Y \cdot \tilde{\varepsilon}_Y = F\tilde{\eta}_{GY} \cdot F\tilde{G}\varepsilon_Y \cdot \tilde{\varepsilon}_Y\,.$$

Since $\tilde{\varepsilon}$ is natural, the following diagram commutes.

$$\begin{tikzcd} F\tilde{G}FGY \arrow[r, "F\tilde{G}\varepsilon_Y"] \arrow[d, "\tilde{\varepsilon}_{FGY}"'] & F\tilde{G}Y \arrow[d, "\tilde{\varepsilon}_Y"] \\ FGY \arrow[r, "\varepsilon_Y"'] & Y \end{tikzcd}$$

Thus, it suffices to show that

$$F\tilde{\eta}_{GY} \cdot \tilde{\varepsilon}_{FGY} \cdot \varepsilon_Y \overset{!}{=} \varepsilon_Y\,.$$

This holds, since $F\tilde{\eta} \cdot \tilde{\varepsilon}F = 1_F$. $\square$

Lemma 38. *Suppose given adjunctions $(F, G, \eta, \varepsilon)$ and $(F', G', \eta', \varepsilon')$ with $F \in \operatorname{Ob}[\mathcal{C}, \mathcal{D}]$ and $F' \in \operatorname{Ob}[\mathcal{D}, \mathcal{E}]$.*

We have the adjunction $(F' \circ F, G \circ G', \eta \cdot G\eta'F, F'\varepsilon G' \cdot \varepsilon')$.

Proof. We have the following transformations.

$$1_{\mathcal{C}} \xrightarrow{\eta} GF \xrightarrow{G\eta'F} GG'F'F$$

$$F'FGG' \xrightarrow[F'\varepsilon G']{} F'G' \xrightarrow[\varepsilon']{} 1_{\mathcal{E}}$$

Thus, we need to show that the following diagrams commute.

$$\begin{tikzcd} F'F \arrow[r, "F'F(\eta\cdot G\eta'F)"] \arrow[dr, equal] & F'FGG'F'F \arrow[d, "(F'\varepsilon G'\cdot\varepsilon')F'F"] \\ & F'F \end{tikzcd} \qquad \begin{tikzcd} GG' \arrow[r, "(\eta\cdot G\eta'F)GG'"] \arrow[dr, equal] & GG'F'FGG' \arrow[d, "GG'(F'\varepsilon G'\cdot\varepsilon')"] \\ & GG' \end{tikzcd}$$

Since ε' is natural and $(F, G, \eta, \varepsilon)$ and $(F', G', \eta', \varepsilon')$ are adjunctions, the diagram

$$\begin{tikzcd} F'F \arrow[r, "F'F\eta"] \arrow[dr, equal] & F'FGF \arrow[r, "F'FG\eta'F"] \arrow[d, "F'\varepsilon F"] & F'FGG'F'F \arrow[d, "F'\varepsilon G'F'F"] \\ & F'F \arrow[r, "F'\eta'F"'] \arrow[dr, equal] & F'G'F'F \arrow[d, "\varepsilon'F'F"] \\ & & F'F \end{tikzcd}$$

commutes.

Since η' is natural and $(F, G, \eta, \varepsilon)$ and $(F', G', \eta', \varepsilon')$ are adjunctions, the diagram

$$\begin{tikzcd}
GG' \arrow[r, "\eta GG'"] \arrow[dr, equal] & GFGG' \arrow[r, "G\eta' FGG'"] \arrow[d, "G\varepsilon G'"'] & GG'F'FGG' \arrow[d, "GG'F'\varepsilon G'"] \\
 & GG' \arrow[r, "G\eta' G'"'] \arrow[dr, equal] & GG'F'G' \arrow[d, "GG'\varepsilon'"] \\
 & & GG'
\end{tikzcd}$$

commutes. $\square$

Chapter 2

Categories with split denominators

In this chapter 2, we introduce categories with split denominators; cf. Definition 39 below. For these categories we establish a notion of homotopy, the homotopy category and its 2-universal property; cf. Definitions 50 and 55 and Theorem 62 below.

The notion of a category with split denominators is a precursor to the notion of a quasi-model-category; cf. Definitions 100 and 108 below.

2.1 Axioms for categories with split denominators

Definition 39. Let $\mathcal{C}$ be a category.

Let $\operatorname{SDen}\mathcal{C}$ and $\operatorname{TDen}\mathcal{C}$ be subsets of $\operatorname{Mor}\mathcal{C}$.

The elements of $\operatorname{SDen}\mathcal{C}$ are called *S-denominators*. To indicate that $s \in \operatorname{Mor}\mathcal{C}$ is an S-denominator, we often write

$$X \xrightarrow[\circ]{s} Y.$$

The elements of $\operatorname{TDen}\mathcal{C}$ are called *T-denominators*. To indicate that $t \in \operatorname{Mor}\mathcal{C}$ is a T-denominator, we often write

$$X \xrightarrow[\square]{t} Y.$$

Let

$$\operatorname{Den}\mathcal{C} := \{f \in \operatorname{Mor}\mathcal{C} : \exists\,(s,t) \in \operatorname{SDen}\mathcal{C} \times \operatorname{TDen}\mathcal{C} \text{ with } f = st\}.$$

The elements of $\operatorname{Den}\mathcal{C}$ are called *denominators*. To indicate that $d \in \operatorname{Mor}\mathcal{C}$ is a denominator, we often write

$$X \xrightarrow[\approx]{d} Y.$$

We call $(\mathcal{C}, \operatorname{SDen}\mathcal{C}, \operatorname{TDen}\mathcal{C})$ a *category with split denominators* if the following axioms $(\mathbf{S}_{\text{SDen}}, \mathbf{S}_{\text{TDen}}, \mathbf{S}_{\text{Den}})$ hold. We often refer to just $\mathcal{C}$ as a category with split denominators.

$\mathbf{S}_{\mathrm{SDen}}$ The following assertions (1, 2, 3, 4) hold.

(1) Suppose given $X \xrightarrow{s'} Y \xrightarrow{s''} Z$ in $\mathcal{C}$. Then we have $s's'' \in \operatorname{SDen}\mathcal{C}$.

(2) Suppose given $X \in \operatorname{Ob}\mathcal{C}$. We have $1_X \in \operatorname{SDen}\mathcal{C}$.

(3) We have $\operatorname{SDen}\mathcal{C} \subseteq \operatorname{Coret}\mathcal{C}$.

(4) Suppose given $X' \xleftarrow{s} X \xrightarrow{f} Y$ in $\mathcal{C}$. There exists a weak pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] \arrow[dr, phantom, "\mathrm{W}"] & Y \arrow[d, "s'"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

in $\mathcal{C}$.

$\mathbf{S}_{\mathrm{TDen}}$ The following assertions (1, 2, 3, 4) hold.

(1) Suppose given $X \xrightarrow{t'} Y \xrightarrow{t''} Z$ in $\mathcal{C}$. Then we have $t't'' \in \operatorname{TDen}\mathcal{C}$.

(2) Suppose given $X \in \operatorname{Ob}\mathcal{C}$. We have $1_X \in \operatorname{TDen}\mathcal{C}$.

(3) We have $\operatorname{TDen}\mathcal{C} \subseteq \operatorname{Ret}\mathcal{C}$.

(4) Suppose given $X' \xrightarrow{f'} Y' \xleftarrow{t'} Y$ in $\mathcal{C}$. There exists a weak pullback

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\mathrm{W}"] & Y \arrow[d, "t'"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

in $\mathcal{C}$.

$\mathbf{S}_{\mathrm{Den}}$ Suppose given $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$. The following assertions (1, 2, 3) hold.

(1) Suppose that $f, g \in \operatorname{Den}\mathcal{C}$. Then we have $fg \in \operatorname{Den}\mathcal{C}$.

(2) Suppose that $f, fg \in \operatorname{Den}\mathcal{C}$. Then we have $g \in \operatorname{Den}\mathcal{C}$.

(3) Suppose that $g, fg \in \operatorname{Den}\mathcal{C}$. Then we have $f \in \operatorname{Den}\mathcal{C}$.

Remark 40. A category with split denominators is a uni-fractionable category as defined by Thomas [**16,** Def. 3.1] such that all S-denominators are coretractions and all T-denominators are retractions. Whereas in our context, it is no longer necessary to work with fractions, we have kept the notions of denominators, S-denominators and T-denominators.

Remark 41. The full subcategory of bifibrant objects $\mathcal{M}_{\mathrm{bif}}$ in a model category

$$(\mathcal{M}, \operatorname{Qis}\mathcal{M}, \operatorname{Cof}\mathcal{M}, \operatorname{Fib}\mathcal{M})$$

is a category with split denominators for

$$\operatorname{SDen}(\mathcal{M}_{\mathrm{bif}}) = \operatorname{Qis}\mathcal{M} \cap \operatorname{Cof}\mathcal{M} \cap \operatorname{Mor}(\mathcal{M}_{\mathrm{bif}})$$

and

$$\operatorname{TDen}(\mathcal{M}_{\mathrm{bif}}) = \operatorname{Qis}\mathcal{M} \cap \operatorname{Fib}\mathcal{M} \cap \operatorname{Mor}(\mathcal{M}_{\mathrm{bif}}).$$

Furthermore, we have

$$\operatorname{Den}(\mathcal{M}_{\mathrm{bif}}) = \operatorname{Qis}\mathcal{M} \cap \operatorname{Mor}(\mathcal{M}_{\mathrm{bif}}).$$

Cf. Definition 108 and Theorem 193 below.

Remark 42. Let $\mathcal{C}$ be a category. Let

$$\begin{aligned} \operatorname{SDen}\mathcal{C} &:= \{1_X \ : X \in \operatorname{Ob}\mathcal{C}\} \\ \operatorname{TDen}\mathcal{C} &:= \{1_X \ : X \in \operatorname{Ob}\mathcal{C}\}. \end{aligned}$$

Then $\mathcal{C}$ is a category with split denominators. So, in general $\operatorname{Iso}\mathcal{C} \not\subseteq \operatorname{Den}\mathcal{C}$.

2.2 Elementary properties

In the sequel, the results of this §2.2 are often used tacitly.

For this §2.2 let $\mathcal{C}$ be a category with split denominators.

Remark 43. Suppose given $X \xrightarrow[\approx]{q} Y$ in $\mathcal{C}$.

Then there exists a commutative diagram as follows.

$$\begin{tikzcd} X \arrow[rr, "q"', "\approx"] \arrow[dr, "i"', circ] & & Y \\ & Z \arrow[ur, "p"'] & \end{tikzcd}$$

Cf. Definition 39.

Remark 44. *The following assertions* (1, 2, 3) *hold.*

(1) *We have* $\operatorname{SDen}\mathcal{C} \subseteq \operatorname{Den}\mathcal{C}$.

(2) *We have* $\operatorname{TDen}\mathcal{C} \subseteq \operatorname{Den}\mathcal{C}$.

(3) *Suppose given* $X \in \operatorname{Ob}\mathcal{C}$. *We have* $1_X \in \operatorname{Den}\mathcal{C}$.

Proof. *Ad* (1). This follows from $\mathbf{S}_{\mathrm{TDen}}$.(2).

Ad (2). This follows from $\mathbf{S}_{\mathrm{SDen}}$.(2).

Ad (3). This follows from $\mathbf{S}_{\mathrm{SDen}}$.(2) and $\mathbf{S}_{\mathrm{TDen}}$.(2). □

Remark 45. *The following assertions* (1, 2) *hold.*

(1) *Suppose given* $X \xrightarrow{s} Y$ *in* $\mathcal{C}$*. Then there exists a commutative diagram as follows.*

$$\begin{tikzcd} X \arrow[rr, "1"] \arrow[dr, "s"'] & & X \\ & Y \arrow[ur, "s'"', "\approx"] & \end{tikzcd}$$

(2) *Suppose given* $X \xrightarrow{t} Y$ *in* $\mathcal{C}$*. Then there exists a commutative diagram as follows.*

$$\begin{tikzcd} Y \arrow[rr, "1"] \arrow[dr, "t'"', "\approx"] & & Y \\ & X \arrow[ur, "t"'] & \end{tikzcd}$$

Proof. Ad (1). This follows from Remark 44.(1,3) and $\mathbf{S}_{\mathrm{Den}}$.(2).

Ad (2). This follows from Remark 44.(2,3) and $\mathbf{S}_{\mathrm{Den}}$.(3). □

Remark 46. *Suppose given the following commutative diagram.*

$$\begin{tikzcd} X \arrow[r, "d", "\approx"'] \arrow[d, "x"', "\approx"] & Y \\ X' \arrow[r, "d'"'] & Y' \arrow[u, "y'", "\approx"'] \end{tikzcd}$$

Then we have $d' \in \mathrm{Den}\,\mathcal{C}$*.*

Proof. Since we have $xd' \cdot y' = d$, we obtain $xd' \in \mathrm{Den}\,\mathcal{C}$; cf. $\mathbf{S}_{\mathrm{Den}}$.(3).

Thus, we obtain $d' \in \mathrm{Den}\,\mathcal{C}$; cf. $\mathbf{S}_{\mathrm{Den}}$.(2). □

2.3 A lemma on factorisations

In this §2.3 we establish a lemma in categories with split denominators which could be expressed by saying that they admit a weaker variant of functorial factorisations of denominators.

For this §2.3 let $\mathcal{C}$ be a category with split denominators.

Lemma 47 (Cf. also [**16,** Lem. 5.1])**.** *The following assertions* (1, 2) *hold.*

(1) *Suppose given the following commutative diagram.*

$$\begin{tikzcd} & X \arrow[dr, "p"] & \\ A \arrow[ur, "i"] \arrow[rr, "u", "\approx"'] \arrow[d, "f"'] & & C \arrow[d, "g"] \\ B \arrow[rr, "\approx", "v"'] & & D \end{tikzcd}$$

Then there exists a commutative diagram as follows.

$$\begin{tikzcd}
& X \arrow[dr, "p"] \arrow[dd, "h"] & \\
A \arrow[ur, "i"] \arrow[rr, "\approx", "u"'] \arrow[dd, "f"'] & & C \arrow[dd, "g"] \\
& Y \arrow[dr, "q"] & \\
B \arrow[ur, "j"] \arrow[rr, "\approx", "v"'] & & D
\end{tikzcd}$$

(2) *Suppose given the following commutative diagram.*

$$\begin{tikzcd}
A \arrow[rr, "u", "\approx"'] \arrow[dd, "f"'] & & C \arrow[dd, "g"] \\
& Y \arrow[dr, "q"] & \\
B \arrow[ur, "j"] \arrow[rr, "\approx", "v"'] & & D
\end{tikzcd}$$

Then there exists a commutative diagram as follows.

$$\begin{tikzcd}
& X \arrow[dr, "p"] \arrow[dd, "h"] & \\
A \arrow[ur, "i"] \arrow[rr, "\approx", "u"'] \arrow[dd, "f"'] & & C \arrow[dd, "g"] \\
& Y \arrow[dr, "q"] & \\
B \arrow[ur, "j"] \arrow[rr, "\approx", "v"'] & & D
\end{tikzcd}$$

Proof. *Ad* (1). We have a commutative diagram as follows.

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f"'] \arrow[dr, phantom, "\mathrm{W}"] & X \arrow[d, "\tilde{f}"] \arrow[ddr, bend left, "pg"] & \\
B \arrow[r, "m"'] \arrow[drr, bend right, "\approx", "v"'] & \tilde{X} \arrow[dr, dotted, "k", "\approx"'] & \\
& & D
\end{tikzcd}$$

Cf. $\mathbf{S}_{\mathrm{SDen}}$.(4) and $\mathbf{S}_{\mathrm{Den}}$.(2). There exists a commutative diagram as follows.

$$\begin{tikzcd}
\tilde{X} \arrow[rr, "k", "\approx"'] \arrow[dr, "l"'] & & D \\
& Y \arrow[ur, "q"'] &
\end{tikzcd}$$

Finally, we have the following commutative diagram.

$$
\begin{tikzcd}
 & X \arrow[dr, "p"] \arrow[dd, "\tilde{f}l"] & \\
A \arrow[ur, "i"] \arrow[rr, "\approx"', "u"'] \arrow[dd, "f"'] & & C \arrow[dd, "g"] \\
 & Y \arrow[dr, "q"] & \\
B \arrow[ur, "ml"] \arrow[rr, "\approx"', "v"'] & & D
\end{tikzcd}
$$

Cf. $\mathbf{S}_{\mathrm{SDen}}.(1)$.

Ad (2). This is dual to (1). $\square$

Corollary 48. *Let $\mathcal{C}$ be a category with split denominators.*

Suppose given a commutative diagram as follows.

$$
\begin{tikzcd}
A \arrow[r, "u", "\approx"'] \arrow[d, "f"'] & C \arrow[d, "g"] \\
B \arrow[r, "\approx", "v"'] & D
\end{tikzcd}
$$

Then there exists a commutative diagram as follows.

$$
\begin{tikzcd}
 & X \arrow[dr, "p"] \arrow[dd, "h"] & \\
A \arrow[ur, "i"] \arrow[rr, "\approx"', "u"'] \arrow[dd, "f"'] & & C \arrow[dd, "g"] \\
 & Y \arrow[dr, "q"] & \\
B \arrow[ur, "j"] \arrow[rr, "\approx"', "v"'] & & D
\end{tikzcd}
$$

Proof. There exists a commutative diagram as follows.

$$
\begin{tikzcd}
A \arrow[rr, "u", "\approx"'] \arrow[dr, "i"'] & & C \\
 & X \arrow[ur, "p"'] &
\end{tikzcd}
$$

Thus, the assertion follows from Lemma 47.(1). $\square$

2.4 Homotopy

In this §2.4, we establish the homotopy relation for a category with split denominators and show that it is a congruence; cf. Definition 50 and Proposition 52 below.

For this §2.4 let $\mathcal{C}$ be a category with split denominators.

Lemma 49. *Let $X, Y \in \operatorname{Ob} \mathcal{C}$. Suppose given $X \underset{f_1}{\overset{f_0}{\rightrightarrows}} Y$ in $\mathcal{C}$. The following assertions* (1, 2, 3′, 3″) *on f_0 and f_1 are equivalent.*

(1) *There exists a commutative diagram as follows.*

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, "i_0\,\wr"'] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[d, equal] & \hat{X} \arrow[l, "t"', "\approx"] \arrow[r, "\hat{f}"] & \hat{Y} \arrow[u, "\wr\, p_0"'] \arrow[d, "\wr\, p_1"] & Y \arrow[l, "s"', "\approx"] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f_1"'] \arrow[u, "i_1\,\wr"] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

(2) *There exists a commutative diagram as follows.*

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, equal] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "\hat{f}_0"] \arrow[d, "j_0\,\wr"'] & Y' \arrow[u, "\wr\, q_0"'] \arrow[d, "\wr\, q_1"] & Y \arrow[l, "y"', "\approx"] \arrow[d, equal] \\
X \arrow[d, equal] & X' \arrow[l, "x"', "\approx"] \arrow[r, "\hat{f}_1"] & Y \arrow[r, equal] \arrow[d, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f_1"'] \arrow[u, "j_1\,\wr"] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

(3′) *There exists a commutative diagram as follows.*

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, "k_0\,\wr"'] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[d, equal] & \hat{X} \arrow[l, "\tau"'] \arrow[r, "\hat{f}"] & \hat{Y} \arrow[u, "r_0"'] \arrow[d, "\wr\, r_1"] & Y \arrow[l, "\sigma"'] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f_1"'] \arrow[u, "k_1"] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

(3″) *There exists a commutative diagram as follows.*

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, "k_0"'] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[d, equal] & \hat{X} \arrow[l, "\tau"'] \arrow[r, "\hat{f}"] & \hat{Y} \arrow[u, "\wr\, r_0"'] \arrow[d, "r_1"] & Y \arrow[l, "\sigma"'] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f_1"'] \arrow[u, "k_1\,\wr"] & Y \arrow[r, equal] & Y
\end{tikzcd}
$$

Proof. Ad (1) ⇒ (2). By assumption, we have a commutative diagram as follows.

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, "i_0\,\wr"'] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[d, equal] & \hat{X} \arrow[l, "t"', "\approx"] \arrow[r, "\hat{f}"] & \hat{Y} \arrow[u, "\wr\, p_0"'] \arrow[d, "\wr\, p_1"] & Y \arrow[l, "s"', "\approx"] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f_1"'] \arrow[u, "i_1\,\wr"] & Y \arrow[r, equal] & Y
\end{tikzcd}
$$

Thus, we have the following commutative diagram.

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, equal] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "i_0 \hat{f}"] \arrow[d, "i_0\,\wr"'] & \hat{Y} \arrow[u, "\wr\, p_0"'] \arrow[d, "\wr\, p_1"] & Y \arrow[l, "s"', "\approx"] \arrow[d, equal] \\
X \arrow[d, equal] & \hat{X} \arrow[l, "t"', "\approx"] \arrow[r, "\hat{f} p_1"] & Y \arrow[r, equal] \arrow[d, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f_1"'] \arrow[u, "i_1\,\wr"] & Y \arrow[r, equal] & Y
\end{tikzcd}
$$

Ad (2) ⇒ (3″). By assumption, we have a commutative diagram as follows.

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, equal] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "\hat{f}_0"] \arrow[d, "j_0\,\wr"'] & Y' \arrow[u, "\wr\, q_0"'] \arrow[d, "\wr\, q_1"] & Y \arrow[l, "y"', "\approx"] \arrow[d, equal] \\
X \arrow[d, equal] & X' \arrow[l, "x"', "\approx"] \arrow[r, "\hat{f}_1"] & Y \arrow[r, equal] \arrow[d, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f_1"'] \arrow[u, "j_1\,\wr"] & Y \arrow[r, equal] & Y
\end{tikzcd}
$$

Using Corollary 48, we obtain a commutative diagram as follows.

$$\begin{tikzcd}
 & \tilde{X} \arrow[dr, "b"] \arrow[dd, "m"] & \\
X \arrow[ur, "a"] \arrow[rr, "\approx", "j_0"'] \arrow[dd, "\hat{f}_0"'] & & X' \arrow[dd, "\hat{f}_1"] \\
 & \tilde{Y} \arrow[dr, "d"] & \\
Y' \arrow[ur, "c"] \arrow[rr, "\approx", "q_1"'] & & Y
\end{tikzcd}$$

By Remark 45, there exist commutative diagrams as follows.

$$\begin{tikzcd}
X' \arrow[rr, "1"] \arrow[dr, "\approx", "b'"'] & & X' \\
 & \tilde{X} \arrow[ur, "b"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
Y' \arrow[rr, "1"] \arrow[dr, "c"'] & & Y' \\
 & \tilde{Y} \arrow[ur, "\approx", "c'"'] &
\end{tikzcd}$$

Thus, we have the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, "a"'] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[d, equal] & \tilde{X} \arrow[l, "\approx", "bx"'] \arrow[r, "m"] & \tilde{Y} \arrow[u, "\approx", "c'q_0"'] \arrow[d, "d"] & Y \arrow[l, "\approx", "yc"'] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[u, "\approx", "j_1b'"'] \arrow[r, "f_1"'] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

Cf. $\mathbf{S}_{\mathrm{Den}}.(1)$. There exist commutative diagrams as follows.

$$\begin{tikzcd}
\tilde{X} \arrow[rr, "\approx", "bx"'] \arrow[dr, "s"'] & & X \\
 & \hat{X} \arrow[ur, "\tau"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[rr, "\approx", "yc"'] \arrow[dr, "\sigma"'] & & \tilde{Y} \\
 & \hat{Y} \arrow[ur, "t"'] &
\end{tikzcd}$$

By Remark 45, there exist commutative diagrams as follows.

$$\begin{tikzcd}
\tilde{X} \arrow[rr, "1"] \arrow[dr, "s"'] & & \tilde{X} \\
 & \hat{X} \arrow[ur, "\approx", "s'"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{Y} \arrow[rr, "1"] \arrow[dr, "\approx", "t'"'] & & \tilde{Y} \\
 & \hat{Y} \arrow[ur, "t"'] &
\end{tikzcd}$$

Finally, we have the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, "as"'] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[d, equal] & \hat{X} \arrow[l, "\tau"'] \arrow[r, "s'mt'"] & \hat{Y} \arrow[u, "\approx", "tc'q_0"'] \arrow[d, "td"] & Y \arrow[l, "\sigma"'] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[u, "\approx", "j_1b's"'] \arrow[r, "f_1"'] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

Cf. $\mathbf{S}_{\mathrm{SDen}}.(1)$, $\mathbf{S}_{\mathrm{TDen}}.(1)$ and $\mathbf{S}_{\mathrm{Den}}.(1)$.

Ad $(3'') \Rightarrow (1)$. This follows from Remark 44.(1, 2).

Ad $(1) \Leftrightarrow (3')$. This follows by symmetry from $(1) \Leftrightarrow (3'')$. $\square$

Definition 50 (and Lemma)**.** Let $X, Y \in \operatorname{Ob}\mathcal{C}$.

We call $f_0,\, f_1 \in {}_{\mathcal{C}}(X,Y)$ *homotopic*, written $f_0{\sim}f_1$, if there exists a commutative diagram as follows.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, "i_0\,\wr"'] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[d, equal] & \hat{X} \arrow[l, "t"', "\approx"] \arrow[r, "\hat{f}"] & \hat{Y} \arrow[u, "\wr\, p_0"'] \arrow[d, "\wr\, p_1"] & Y \arrow[l, "s"', "\approx"] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f_1"'] \arrow[u, "i_1\,\wr"] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

Cf. Lemma 49.(1).

The relation $(\sim)$ *on* ${}_{\mathcal{C}}(X,Y)$ *is an equivalence relation.*

The equivalence class of $f \in {}_{\mathcal{C}}(X,Y)$ is denoted by $[f]$.

Proof. Suppose given $X \xrightarrow{f} Y$. We have the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, "1\,\wr"'] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[d, equal] & X \arrow[l, "1"', "\approx"] \arrow[r, "f"] & Y \arrow[u, "\wr\, 1"'] \arrow[d, "\wr\, 1"] & Y \arrow[l, "1"', "\approx"] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f"'] \arrow[u, "1\,\wr"] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

Cf. Remark 44.(3). Thus, $(\sim)$ is reflexive.

Symmetry of $(\sim)$ follows from the symmetry of the defining diagram.

We show that $(\sim)$ is transitive.

Suppose given $f, g, h \in {}_{\mathcal{C}}(X,Y)$ with $f \sim g$ and $g \sim h$.

We have a commutative diagram as follows.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "k'"] \arrow[d, "j_0", "\wr"'] & Y' \arrow[u, "\wr", "q_0"'] \arrow[d, "\wr", "q_1"'] & Y \arrow[l, "y'"', "\approx"] \arrow[d, equal] \\
X \arrow[d, equal] & X' \arrow[l, "x'"', "\approx"] \arrow[r, "l'"] & Y \arrow[r, equal] \arrow[d, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "g"'] \arrow[u, "j_1", "\wr"'] \arrow[d, equal] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "k''"] \arrow[d, "j_2", "\wr"'] & Y'' \arrow[u, "\wr", "q_2"'] \arrow[d, "\wr", "q_3"'] & Y \arrow[l, "y''"', "\approx"] \arrow[d, equal] \\
X \arrow[d, equal] & X' \arrow[l, "x''"', "\approx"] \arrow[r, "l''"] & Y \arrow[r, equal] \arrow[d, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "h"'] \arrow[u, "j_3", "\wr"'] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

Cf. Lemma 49.(2). Using Corollary 48, we obtain a commutative diagram as follows.

$$\begin{tikzcd}
& \hat{X} \arrow[dr, "b"] \arrow[dd, "m"] & \\
X \arrow[ur, "a"] \arrow[rr, "\approx", "j_1"'] \arrow[dd, "k''"'] & & X' \arrow[dd, "l'"] \\
& \hat{Y} \arrow[dr, "d"] & \\
Y'' \arrow[ur, "c"] \arrow[rr, "\approx", "q_2"'] & & Y
\end{tikzcd}$$

By Remark 45, there exist commutative diagrams as follows.

$$\begin{tikzcd}
X' \arrow[rr, "1"] \arrow[dr, "\wr", "b'"'] & & X' \\
& \hat{X} \arrow[ur, "b"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
Y'' \arrow[rr, "1"] \arrow[dr, "c"'] & & Y'' \\
& \hat{Y} \arrow[ur, "\wr", "c'"'] &
\end{tikzcd}$$

There exist commutative diagrams as follows.

$$\begin{tikzcd}
X \arrow[dr, dotted, "k"] \arrow[drr, bend left, "j_0 b' m"] \arrow[ddr, bend right, "k'"'] & & \\
& \tilde{Y} \arrow[r, "\tilde{q}_1", "\approx"'] \arrow[d, "\tilde{d}"'] \arrow[dr, phantom, "\mathrm{W}"] & \hat{Y} \arrow[d, "d"] \\
& Y' \arrow[r, "\approx", "q_1"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
Y' \arrow[dr, dotted, "\tilde{d}'", "\wr"'] \arrow[drr, bend left, "q_1 y'' c", "\approx"'] \arrow[ddr, bend right, "\wr", "1"'] & & \\
& \tilde{Y} \arrow[r, "\tilde{q}_1", "\approx"'] \arrow[d, "\tilde{d}"'] \arrow[dr, phantom, "\mathrm{W}"] & \hat{Y}d \arrow[d] \\
& Y' \arrow[r, "\approx", "q_1"'] & Y
\end{tikzcd}$$

Cf. $\mathbf{S}_{\mathrm{TDen}}$.(4) and $\mathbf{S}_{\mathrm{Den}}$. Furthermore, there exist commutative diagrams as follows.

$$
\begin{tikzcd}
X \arrow[r, "a", "\circ" marking] \arrow[d, "j_2", "\wr"'] & \hat{X} \arrow[d, "\wr", "\tilde{j}_2"'] \arrow[ddr, bend left, "mc'q_3"] & \\
X' \arrow[r, "\tilde{a}"', "\circ" marking] \arrow[rr, bend right, "l''"'] & \tilde{X} \arrow[dr, dotted, "l"] \arrow[l, phantom, "\mathrm{W}", near start] & \\
& & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "a", "\circ" marking] \arrow[d, "j_2", "\wr"'] & \hat{X} \arrow[d, "\wr", "\tilde{j}_2"'] \arrow[ddr, bend left, "bx'j_2", "\wr"'] & \\
X' \arrow[r, "\tilde{a}"', "\circ" marking] \arrow[rr, bend right, "1"', "\approx"] & \tilde{X} \arrow[dr, dotted, "\tilde{a}'", "\wr"'] \arrow[l, phantom, "\mathrm{W}", near start] & \\
& & X'
\end{tikzcd}
$$

Cf. $\mathbf{S}_{\mathrm{SDen}}$.(4) and $\mathbf{S}_{\mathrm{Den}}$. Finally, we have the following commutative diagram.

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "k"] \arrow[d, "j_0 b' \tilde{j}_2"', "\wr"] & \tilde{Y} \arrow[u, "\wr", "\tilde{d} q_0"'] \arrow[d, "\wr", "\tilde{q}_1 c' q_3"'] & Y \arrow[l, "y'\tilde{d}'"', "\approx"] \arrow[d, equal] \\
X \arrow[d, equal] & \tilde{X} \arrow[l, "\tilde{a}' x''"', "\approx"] \arrow[r, "l"] & Y \arrow[r, equal] \arrow[d, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "h"'] \arrow[u, "j_3 \tilde{a}"', "\wr"] & Y \arrow[r, equal] & Y
\end{tikzcd}
$$

Cf. $\mathbf{S}_{\mathrm{Den}}$.(1). By Lemma 49, we have $f \sim h$. $\square$

Lemma 51. *Suppose given X, Y and Z in $\operatorname{Ob} \mathcal{C}$. The following assertions (1, 2) hold.*

(1) *Suppose given* $X \underset{f_1}{\overset{f_0}{\rightrightarrows}} Y \xrightarrow{g} Z$ *with $f_0 \sim f_1$. Then we have $f_0 g \sim f_1 g$.*

(2) *Suppose given* $X \xrightarrow{f} Y \underset{g_1}{\overset{g_0}{\rightrightarrows}} Z$ *with $g_0 \sim g_1$. Then we have $f g_0 \sim f g_1$.*

Ad (1). We have a commutative diagram as follows.

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, "k_0"', "\wr"] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[d, equal] & \hat{X} \arrow[l, "\tau"', "\shortmid" marking] \arrow[r, "\hat{f}"] & \hat{Y} \arrow[u, "r_0"', "\shortmid" marking] \arrow[d, "\wr", "r_1"'] & Y \arrow[l, "\sigma"', "\circ" marking] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f_1"'] \arrow[u, "k_1"', "\circ" marking] & Y \arrow[r, equal] & Y
\end{tikzcd}
$$

Cf. Lemma 49.(3′). Furthermore, we have commutative diagrams as follows.

$$
\begin{tikzcd}
Y \arrow[r, "\sigma"] \arrow[d, "g"'] & \hat{Y} \arrow[d, "\hat{g}"] \arrow[ddr, bend left, "r_0 g"] & \\
Z \arrow[r, "\tilde{\sigma}"'] \arrow[drr, bend right, "1"', "\wr"] & \hat{Z} \arrow[dr, dotted, "\rho_0", "\wr"'] \arrow[ul, phantom, "\mathrm{W}"] & \\
 & & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "\sigma"] \arrow[d, "g"'] & \hat{Y} \arrow[d, "\hat{g}"] \arrow[ddr, bend left, "r_1 g"] & \\
Z \arrow[r, "\tilde{\sigma}"'] \arrow[drr, bend right, "1"', "\wr"] & \hat{Z} \arrow[dr, dotted, "\rho_1", "\wr"'] \arrow[ul, phantom, "\mathrm{W}"] & \\
 & & Z
\end{tikzcd}
$$

Cf. $\mathbf{S}_{\mathrm{SDen}}.(4)$ and $\mathbf{S}_{\mathrm{Den}}.(2)$. Thus, we have the following commutative diagram.

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0 g"] \arrow[d, "k_0", "\wr"'] & Z \arrow[r, equal] & Z \arrow[d, equal] \\
X \arrow[d, equal] & \hat{X} \arrow[l, "\tau"'] \arrow[r, "\hat{f}\hat{g}"] & \hat{Z} \arrow[u, "\rho_0"', "\wr"] \arrow[d, "\rho_1", "\wr"'] & Z \arrow[l, "\tilde{\sigma}"'] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f_1 g"'] \arrow[u, "k_1"] & Z \arrow[r, equal] & Z
\end{tikzcd}
$$

Therefore, we have $f_0 g \sim f_1 g$.

Ad (2). This is dual to (1).

Proposition 52. Recall that $\mathcal{C}$ is a category with split denominators; cf. Definition 39.

Homotopy is a congruence on $\mathcal{C}$; cf. Definition 50*.*

Proof. Homotopy is an equivalence relation; cf. Definition 50. Moreover, it is a congruence by Lemma 51 and Remark 2. □

Lemma 53. *The following assertions* (1, 2) *hold.*

(1) *Suppose given a commutative diagram in $\mathcal{C}$ as follows.*

$$
\begin{tikzcd}
X \arrow[rr, "1"] \arrow[dr, "s"'] & & X \\
 & Y \arrow[ur, "s'"', "\wr"] &
\end{tikzcd}
$$

Then we have $s's \sim 1_Y$.

(2) *Suppose given a commutative diagram in $\mathcal{C}$ as follows.*

$$
\begin{tikzcd}
Y \arrow[rr, "1"] \arrow[dr, "t'"', "\wr"] & & Y \\
 & X \arrow[ur, "t"'] &
\end{tikzcd}
$$

Then we have $tt' \sim 1_X$.

Proof. *Ad* (1). There exist commutative diagrams as follows

$$
\begin{tikzcd}
X \arrow[r, "s"] \arrow[d, "s"'] & Y \arrow[d, "s_0"] \arrow[ddr, bend left, "1", "\approx"'] \arrow[dl, phantom, "\mathrm{W}"] & \\
Y \arrow[r, "\approx", "s_1"'] \arrow[drr, bend right, "\approx", "s's"'] & \hat{Y} \arrow[dr, dotted, "u", "\approx"'] & \\
& & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "s"] \arrow[d, "s"'] & Y \arrow[d, "s_0"] \arrow[ddr, bend left, "1", "\approx"'] \arrow[dl, phantom, "\mathrm{W}"] & \\
Y \arrow[r, "\approx", "s_1"'] \arrow[drr, bend right, "\approx", "1"'] & \hat{Y} \arrow[dr, dotted, "v", "\approx"'] & \\
& & Y
\end{tikzcd}
$$

Cf. $\mathbf{S}_{\mathrm{SDen}}$.(4) and $\mathbf{S}_{\mathrm{Den}}$. Thus, we have the following commutative diagram.

$$
\begin{tikzcd}
Y \arrow[r, equal] \arrow[d, equal] & Y \arrow[r, "s's"] \arrow[d, "s_1"', "\approx"] & Y \arrow[r, equal] \arrow[d, equal] & Y \arrow[d, equal] \\
Y \arrow[d, equal] & \hat{Y} \arrow[l, "v"', "\approx"] \arrow[r, "u"] & Y \arrow[r, equal] \arrow[d, equal] & Y \arrow[d, equal] \\
Y \arrow[r, equal] & Y \arrow[r, "1"'] \arrow[u, "s_0"] & Y \arrow[r, equal] & Y
\end{tikzcd}
$$

Therefore, we have $s's \sim 1_Y$.

Ad (2). This is dual to (1). $\square$

Lemma 54. *Suppose given $X \xrightarrow{f} Y$ in $\mathcal{C}$ with $f \sim f' \in \operatorname{Den}\mathcal{C}$. Then we have $f \in \operatorname{Den}\mathcal{C}$.*

Proof. By assumption, there exists a commutative diagram as follows.

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, "i_0"', "\approx"] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[d, equal] & \hat{X} \arrow[l, "t"', "\approx"] \arrow[r, "\hat{f}"] & \hat{Y} \arrow[u, "\approx", "p_0"'] \arrow[d, "\approx"', "p_1"] & Y \arrow[l, "s"', "\approx"] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "\approx", "f'"'] \arrow[u, "i_1"', "\approx"] & Y \arrow[r, equal] & Y
\end{tikzcd}
$$

Since we have $i_1 \cdot \hat{f} \cdot p_1 = f'$, we obtain $\hat{f} \in \operatorname{Den}\mathcal{C}$; cf. Remark 46.

Thus, we have $f = i_0 \hat{f} p_0 \in \operatorname{Den}\mathcal{C}$; cf. $\mathbf{S}_{\mathrm{Den}}$.(1). $\square$

2.5 The homotopy category

2.5.1 Homotopy category and localisation functor

In this §2.5.1, we establish the homotopy category $\operatorname{Ho}\mathcal{C}$ of a category with split denominators $\mathcal{C}$, the localisation functor $\mathrm{L}_{\mathcal{C}} : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ and their elementary properties; cf. Definitions 55 and 56 below.

For this §2.5.1, let $\mathcal{C}$ be a category with split denominators; cf. Definition 39.

Definition 55. Recall that the homotopy relation $(\sim)$ is a congruence on $\mathcal{C}$; cf. Proposition 52.

We define $\operatorname{Ho}\mathcal{C} := \mathcal{C}/(\sim)$; cf. Definition 3.

We have

$$\operatorname{Ob}\operatorname{Ho}\mathcal{C} = \operatorname{Ob}\mathcal{C}.$$

For X and Y in $\operatorname{Ob}\operatorname{Ho}\mathcal{C}$, we have

$$_{\operatorname{Ho}\mathcal{C}}(X,Y) = {}_{\mathcal{C}}(X,Y)/(\sim)_{X,Y}.$$

Recall that for $f \in {}_{\mathcal{C}}(X,Y)$ we write $[f] \in {}_{\operatorname{Ho}\mathcal{C}}(X,Y)$; cf. Definition 50.

For X, Y and Z in $\operatorname{Ob}\operatorname{Ho}\mathcal{C}$, $[f] \in {}_{\operatorname{Ho}\mathcal{C}}(X,Y)$ and $[g] \in {}_{\operatorname{Ho}\mathcal{C}}(Y,Z)$, we have

$$[f]\cdot[g] = [fg].$$

For $X \in \operatorname{Ob}\operatorname{Ho}\mathcal{C}$, we have

$$1_X^{\operatorname{Ho}\mathcal{C}} = [1_X^{\mathcal{C}}].$$

We call $\operatorname{Ho}\mathcal{C}$ the *homotopy category* of $\mathcal{C}$.

This defines a category $\operatorname{Ho}\mathcal{C}$; Definition 3.

Definition 56. We define $\mathrm{L}_{\mathcal{C}} := R_{(\sim)}$; cf. Definition 4.

We have

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathrm{L}_{\mathcal{C}}} & \operatorname{Ho}\mathcal{C} \\ (X \xrightarrow{f} Y) & \mapsto & (X \xrightarrow{[f]} Y). \end{array}$$

We call $\mathrm{L}_{\mathcal{C}}$ the *localisation functor* of $\mathcal{C}$.

If unambiguous, we often write $\mathrm{L} := \mathrm{L}_{\mathcal{C}}$.

This defines a functor $\mathrm{L}_{\mathcal{C}} : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$; cf. Definition 4.

Lemma 57. *The following assertions* (1, 2) *hold.*

(1) *The localisation functor* $\mathrm{L}_{\mathcal{C}}$ *is full.*

(2) *The localisation functor* $\mathrm{L}_{\mathcal{C}}$ *is bijective on objects. In particular,* $\mathrm{L}_{\mathcal{C}}$ *is dense.*

Proof. This follows from Lemma 5. □

2.5.2 Universal property

In this §2.5.2 we establish the (2-)universal property of the homotopy category $\operatorname{Ho}\mathcal{C}$ of a category with split denominators $\mathcal{C}$; cf. Theorems 62 and 63 below.

For this §2.5.2 let $\mathcal{C}$ be a category with split denominators; cf. Definition 39. Let $\mathcal{D}$ be a category.

Definition 58. Let ${}_{\mathrm{Loc}}[\mathcal{C},\mathcal{D}]$ be the full subcategory of $[\mathcal{C},\mathcal{D}]$ with

$$\mathrm{Ob}\,{}_{\mathrm{Loc}}[\mathcal{C},\mathcal{D}] := \{F \in \mathrm{Ob}[\mathcal{C},\mathcal{D}] : F(\mathrm{Den}\,\mathcal{C}) \subseteq \mathrm{Iso}\,\mathcal{D}\}.$$

The elements of $\mathrm{Ob}\,{}_{\mathrm{Loc}}[\mathcal{C},\mathcal{D}]$ are called *localising* functors.

Proposition 59. *We have* $\mathrm{Ob}\,{}_{(\sim)}[\mathcal{C},\mathcal{D}] = \mathrm{Ob}\,{}_{\mathrm{Loc}}[\mathcal{C},\mathcal{D}]$; cf. Definition 6.

Proof. *Ad* $\subseteq$. Let $F \in \mathrm{Ob}\,{}_{(\sim)}[\mathcal{C},\mathcal{D}]$. We show that F is localising.

It suffices to show that $F(\mathrm{SDen}\,\mathcal{C}) \subseteq \mathrm{Iso}\,\mathcal{D}$ and $F(\mathrm{TDen}\,\mathcal{C}) \subseteq \mathrm{Iso}\,\mathcal{D}$.

By duality, it suffices to show that $F(\mathrm{SDen}\,\mathcal{C}) \subseteq \mathrm{Iso}\,\mathcal{D}$.

Suppose given $X \xrightarrow{s} Y$ in $\mathcal{C}$. There exists a commutative diagram as follows.

$$\begin{tikzcd}
X \arrow[rr, "1"] \arrow[dr, "s"'] & & X \\
& Y \arrow[ur, "s'"'] &
\end{tikzcd}$$

By Lemma 53.(1), we have $s's \sim 1_Y$. Since $F \in \mathrm{Ob}\,{}_{(\sim)}[\mathcal{C},\mathcal{D}]$, we have $Fs' \cdot Fs = 1_{FY}$.

Furthermore, we have $Fs \cdot Fs' = 1_X$. Thus, we have $Fs \in \mathrm{Iso}\,\mathcal{D}$.

Ad $\supseteq$. Let $F \in \mathrm{Ob}\,{}_{\mathrm{Loc}}[\mathcal{C},\mathcal{D}]$.

Suppose given $X \overset{f_0}{\underset{f_1}{\rightrightarrows}} Y$ with $f_0 \sim f_1$ in $\mathcal{C}$. We show that $Ff_0 = Ff_1$.

There exists a commutative diagram in $\mathcal{C}$ as follows.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, "i_0 \approx"'] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X & \hat{X} \arrow[l, "t"', "\approx"] \arrow[r, "\hat{f}"] & \hat{Y} \arrow[u, "\approx p_0"'] \arrow[d, "\approx p_1"'] & Y \arrow[l, "s"', "\approx"] \\
X \arrow[u, equal] \arrow[r, equal] & X \arrow[u, "i_1 \approx"'] \arrow[r, "f_1"'] & Y \arrow[r, equal] & Y \arrow[u, equal]
\end{tikzcd}$$

Cf. Definition 50.

Since $F \in \mathrm{Ob}\,{}_{\mathrm{Loc}}[\mathcal{C},\mathcal{D}]$, we have the following commutative diagram in $\mathcal{D}$.

$$\begin{tikzcd}
FX \arrow[r, equal] \arrow[d, equal] & FX \arrow[r, "Ff_0"] \arrow[d, "Fi_0"', "\wr"] & FY \arrow[r, equal] & FY \arrow[d, equal] \\
FX & F\hat{X} \arrow[l, "Ft"', "\sim"] \arrow[r, "F\hat{f}"] & F\hat{Y} \arrow[u, "\wr", "Fp_0"'] \arrow[d, "\wr", "Fp_1"'] & FY \arrow[l, "Fs"', "\sim"] \\
FX \arrow[u, equal] \arrow[r, equal] & FX \arrow[u, "Fi_1"', "\wr"] \arrow[r, "Ff_1"'] & FY \arrow[r, equal] & FY \arrow[u, equal]
\end{tikzcd}$$

We have

$$Fi_0 = Fi_0 \cdot Ft \cdot (Ft)^- = (Ft)^- = Fi_1 \cdot Ft \cdot (Ft)^- = Fi_1\,.$$

Similarly, we have $Fp_0 = Fp_1$. Thus, we have

$$Ff_0 = Fi_0 \cdot F\hat{f} \cdot Fp_0 = Fi_1 \cdot F\hat{f} \cdot Fp_1 = Ff_1\,.$$

□

Corollary 60. *We have* $\mathrm{L}_\mathcal{C} \in \mathrm{Ob}\,{}_{\mathrm{Loc}}[\mathcal{C}, \mathrm{Ho}\,\mathcal{C}]$.

Proof. This follows from Proposition 59; cf. Definition 56 and Remark 7.(1). □

Lemma 61. *Suppose given* $F \in \mathrm{Ob}\,{}_{(\sim)}[\mathcal{C}, \mathcal{D}]$. *The following assertions* (1, 2) *hold.*

(1) *Suppose that F is full. Then $\overline{F}$ is full.*

(2) *Suppose that F is dense. Then $\overline{F}$ is dense.*

Proof. This follows from Lemma 9. □

We have the following (2-)universal property of the factor category with respect to homotopy.

Theorem 62. Recall that $\mathcal{C}$ is a category with split denominators.

Recall that the homotopy relation $(\sim)$ is a congruence on $\mathcal{C}$; cf. Proposition 52.

Recall that $\mathcal{D}$ is a category.

The following assertions (1, 2) *hold.*

(1) *We have* $\mathrm{L}_\mathcal{C} \in \mathrm{Ob}\,{}_{(\sim)}[\mathcal{C}, \mathrm{Ho}\,\mathcal{C}]$*;* cf. Definition 56.

Suppose given $F \xrightarrow{\alpha} G$ *in* ${}_{(\sim)}[\mathcal{C}, \mathcal{D}]$*;* cf. Definition 6.

We have unique functors $\overline{F}, \overline{G} : \mathrm{Ho}\,\mathcal{C} \to \mathcal{D}$ *with* $\overline{F} \circ \mathrm{L}_\mathcal{C} = F$ *and* $\overline{G} \circ \mathrm{L}_\mathcal{C} = G$*;* cf. Definition 8.

We have a unique transformation $\overline{F} \overset{\overline{\alpha}}{\Rightarrow} \overline{G}$ *with* $\overline{\alpha} * \mathrm{L}_\mathcal{C} = \alpha$*;* cf. Definition 10.

Specifically, we have

$$\overline{F}[f] = [Ff]$$

for $X \xrightarrow{f} Y$ *in* $\mathcal{C}$ *and*

$$\overline{\alpha} = (\alpha_X)_{X \in \mathrm{Ob}\,\mathrm{Ho}\,\mathcal{C}}\,.$$

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "\mathrm{L}_\mathcal{C}"] \arrow[drr, bend right, "F"'] \arrow[drr, bend left=10, "G"] \arrow[drr, phantom, "\Uparrow\alpha"] & & \mathrm{Ho}\,\mathcal{C} \arrow[d, bend right, "\overline{F}"'] \arrow[d, bend left, "\overline{G}"] \arrow[d, phantom, "\overset{\overline{\alpha}}{\Rightarrow}"] \\
& & \mathcal{D}
\end{tikzcd}$$

(2) *We have the isomorphism of categories*

$$
\begin{array}{rcl}
{}_{(\sim)}[\mathcal{C},\mathcal{D}] & \longleftarrow & [\mathrm{Ho}\,\mathcal{C},\mathcal{D}] \\
(U\circ \mathrm{L}_\mathcal{C} \xrightarrow{\beta * \mathrm{L}_\mathcal{C}} V\circ \mathrm{L}_\mathcal{C}) & \leftarrow\!\shortmid & (U \xrightarrow{\beta} V)
\end{array}
$$

with inverse

$$
\begin{array}{rcl}
{}_{(\sim)}[\mathcal{C},\mathcal{D}] & \longrightarrow & [\mathrm{Ho}\,\mathcal{C},\mathcal{D}] \\
(F \xrightarrow{\alpha} G) & \mapsto & (\overline{F} \xrightarrow{\overline{\alpha}} \overline{G})\,.
\end{array}
$$

Proof. This follows from Definitions 8 and 10, and Proposition 12. □

For sake of easier use, we use Proposition 59 to reformulate Theorem 62 as the (2-)universal property of the localisation with respect to denominators.

Theorem 63. Recall that $\mathcal{C}$ is a category with split denominators.

Recall that $\mathcal{D}$ is a category.

The following assertions (1, 2) *hold.*

(1) *We have* $\mathrm{L}_\mathcal{C} \in \mathrm{Ob}\ {}_{\mathrm{Loc}}[\mathcal{C},\mathrm{Ho}\,\mathcal{C}]$*;* cf. Corollary 60.

Suppose given $F \xrightarrow{\alpha} G$ *in* ${}_{\mathrm{Loc}}[\mathcal{C},\mathcal{D}]$*;* cf. Definition 58.

We have unique functors $\overline{F},\overline{G} : \mathrm{Ho}\,\mathcal{C} \to \mathcal{D}$ *with* $\overline{F}\circ \mathrm{L}_\mathcal{C} = F$ *and* $\overline{G}\circ \mathrm{L}_\mathcal{C} = G$.

We have a unique transformation $\overline{F} \overset{\overline{\alpha}}{\Rightarrow} \overline{G}$ *with* $\overline{\alpha} * \mathrm{L}_\mathcal{C} = \alpha$.

Specifically, we have

$$\overline{F}[f] = [Ff]$$

for $X \xrightarrow{f} Y$ *in* $\mathcal{C}$ *and*

$$\overline{\alpha} = (\alpha_X)_{X\in \mathrm{Ob}\,\mathrm{Ho}\,\mathcal{C}}\,.$$

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "\mathrm{L}_\mathcal{C}"] \arrow[dr, bend left=20, "G"] \arrow[dr, bend right=20, "F"'] \arrow[dr, phantom, "\overset{\alpha}{\Uparrow}"] & \mathrm{Ho}\,\mathcal{C} \arrow[d, bend right, "\overline{F}"'] \arrow[d, bend left, "\overline{G}"] \arrow[d, phantom, "\overset{\overline{\alpha}}{\Rightarrow}"] \\
& \mathcal{D}
\end{tikzcd}
$$

(2) *We have the isomorphism of categories*

$$
\begin{array}{rcl}
{}_{\mathrm{Loc}}[\mathcal{C},\mathcal{D}] & \longleftarrow & [\mathrm{Ho}\,\mathcal{C},\mathcal{D}] \\
(U\circ \mathrm{L}_\mathcal{C} \xrightarrow{\beta * \mathrm{L}_\mathcal{C}} V\circ \mathrm{L}_\mathcal{C}) & \leftarrow\!\shortmid & (U \xrightarrow{\beta} V)
\end{array}
$$

with inverse

$$
\begin{array}{rcl}
{}_{\mathrm{Loc}}[\mathcal{C},\mathcal{D}] & \longrightarrow & [\mathrm{Ho}\,\mathcal{C},\mathcal{D}] \\
(F \xrightarrow{\alpha} G) & \mapsto & (\overline{F} \xrightarrow{\overline{\alpha}} \overline{G})\,.
\end{array}
$$

Proof. This follows from Proposition 59 and Theorem 62. □

2.5.3 Saturatedness

For this §2.5.3, let $\mathcal{C}$ be a category with split denominators.

Remark 64. By Corollary 60, we have $\mathrm{L}_{\mathcal{C}}\, f \in \operatorname{Iso} \operatorname{Ho} \mathcal{C}$ for $f \in \operatorname{Den} \mathcal{C}$.

Definition 65. We call $\mathcal{C}$ *saturated*, if the following assertion $(\star)$ holds.

$(\star)$ Suppose given $f \in \operatorname{Mor} \mathcal{C}$. Then $\mathrm{L}_{\mathcal{C}}\, f \in \operatorname{Iso} \operatorname{Ho} \mathcal{C}$ if and only if $f \in \operatorname{Den} \mathcal{C}$.

Remark 66. *Let $\mathcal{C}$ be saturated. Then we have* $\operatorname{Iso} \mathcal{C} \subseteq \operatorname{Den} \mathcal{C}$. Cf. also Remark 42.

Proof. Suppose given $f \in \operatorname{Iso} \mathcal{C}$. Since L is a functor, we have $\mathrm{L}\, f \in \operatorname{Iso} \operatorname{Ho} \mathcal{C}$.
Thus, we have $f \in \operatorname{Den} \mathcal{C}$; cf. Definition 65. □

Remark 67. *The following assertions* (1, 2) *are equivalent.*

(1) *The category with split denominators $\mathcal{C}$ is saturated.*

(2) *Suppose given $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ in $\mathcal{C}$.*

Suppose that $fg, gh \in \operatorname{Den} \mathcal{C}$. Then we have $f, g, h, fgh \in \operatorname{Den} \mathcal{C}$.

$$\begin{tikzcd}
& B \arrow[rr, "gh", "\approx"' description] \arrow[dr, "g"] & & D \\
A \arrow[ur, "f"] \arrow[urrr, "fgh"] \arrow[rr, "fg"', "\approx" description] & & C \arrow[ur, "h"'] &
\end{tikzcd}$$

Proof. Ad (1) $\Rightarrow$ (2). After applying L, this follows from Remark 19.

Ad (2) $\Rightarrow$ (1). Suppose given $X \xrightarrow{f} Y$ in $\mathcal{C}$ with $\mathrm{L}\, f \in \operatorname{Iso} \operatorname{Ho} \mathcal{C}$.

Thus, there exists $Y \xrightarrow{g} X$ with $fg \sim 1_X$ and $gf \sim 1_Y$; cf. Definition 55.

By Lemma 54, we obtain $fg \in \operatorname{Den} \mathcal{C}$ and $gf \in \operatorname{Den} \mathcal{C}$.

$$\begin{tikzcd}
& Y \arrow[rr, "gf", "\approx"' description] \arrow[dr, "g"] & & Y \\
X \arrow[ur, "f"] \arrow[urrr, "fgf"] \arrow[rr, "fg"', "\approx" description] & & X \arrow[ur, "f"'] &
\end{tikzcd}$$

Since (2) holds, we have $f \in \operatorname{Den} \mathcal{C}$. □

2.5.4 Functoriality

In this §2.5.4 we consider the functor $\operatorname{Ho} F$ induced on homotopy categories by an homotopical functor F between categories with split denominators; cf. Definition 68 below. Similarily for transformations. Furthermore, we establish (2-)functoriality properties of Ho.

For this §2.5.4, let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories with split denominators.

Definition 68.

(1) Let ${}_{\operatorname{Ho}}[\mathcal{C},\mathcal{D}]$ be the full subcategory of $[\mathcal{C},\mathcal{D}]$ with

$$\begin{array}{rcl} \operatorname{Ob}{}_{\operatorname{Ho}}[\mathcal{C},\mathcal{D}] & := & \{F \in \operatorname{Ob}[\mathcal{C},\mathcal{D}] : \mathrm{L}_{\mathcal{D}} \circ F \in \operatorname{Ob}{}_{(\sim)}[\mathcal{C},\operatorname{Ho}\mathcal{D}]\} \\ & \overset{\text{D56}}{=} & \{F \in \operatorname{Ob}[\mathcal{C},\mathcal{D}] : \text{for } f, g \in \operatorname{Mor}\mathcal{C} \text{ with } f \sim g, \text{ we have } Ff \sim Fg\} \\ & \overset{\text{P59}}{=} & \{F \in \operatorname{Ob}[\mathcal{C},\mathcal{D}] : \mathrm{L}_{\mathcal{D}} \circ F \in {}_{\operatorname{Loc}}[\mathcal{C},\operatorname{Ho}\mathcal{D}]\} \\ & \overset{\text{D58}}{=} & \{F \in \operatorname{Ob}[\mathcal{C},\mathcal{D}] : \text{for } f \in \operatorname{Den}\mathcal{C}, \text{ we have } [Ff] \in \operatorname{Iso}\operatorname{Ho}\mathcal{D}\}. \end{array}$$

The objects of ${}_{\operatorname{Ho}}[\mathcal{C},\mathcal{D}]$ are called *homotopical* functors; cf. also Definition 13.

(2) Let ${}_{\operatorname{Den}}[\mathcal{C},\mathcal{D}]$ be the full subcategory of $[\mathcal{C},\mathcal{D}]$ with

$$\operatorname{Ob}{}_{\operatorname{Den}}[\mathcal{C},\mathcal{D}] := \{F \in \operatorname{Ob}[\mathcal{C},\mathcal{D}] : F(\operatorname{Den}\mathcal{C}) \subseteq \operatorname{Den}\mathcal{D}\}.$$

The objects of ${}_{\operatorname{Den}}[\mathcal{C},\mathcal{D}]$ are called *denominatorial* functors.

Lemma 69. *Suppose given $F \in \operatorname{Ob}[\mathcal{C},\mathcal{D}]$. The following assertions* (1, 2) *hold.*

(1) *Suppose that $F \in \operatorname{Ob}{}_{\operatorname{Den}}[\mathcal{C},\mathcal{D}]$. Then we have $F \in \operatorname{Ob}{}_{\operatorname{Ho}}[\mathcal{C},\mathcal{D}]$.*

(2) *Suppose that $\mathcal{D}$ is saturated. Suppose that $F \in \operatorname{Ob}{}_{\operatorname{Ho}}[\mathcal{C},\mathcal{D}]$.*
Then we have $F \in \operatorname{Ob}{}_{\operatorname{Den}}[\mathcal{C},\mathcal{D}]$.

Proof. *Ad* (1). We have $(\mathrm{L}_{\mathcal{D}} \circ F)(\operatorname{Den}\mathcal{C}) \underset{\text{D68.(2)}}{\subseteq} \mathrm{L}_{\mathcal{D}}(\operatorname{Den}\mathcal{D}) \underset{\text{C60}}{\subseteq} \operatorname{Iso}\operatorname{Ho}\mathcal{D}$.

Ad (2). Suppose given $d \in \operatorname{Den}\mathcal{C}$. Then we have $(\mathrm{L}_{\mathcal{D}} \circ F)d \in \operatorname{Iso}\operatorname{Ho}\mathcal{D}$.
Since $\mathcal{D}$ is saturated, we obtain $Fd \in \operatorname{Den}\mathcal{D}$; cf. Definition 65. □

Corollary 70. Let $\mathcal{D}$ be saturated; cf. Definition 65. Then we have

$$\operatorname{Ob}{}_{\operatorname{Ho}}[\mathcal{C},\mathcal{D}] = \operatorname{Ob}{}_{\operatorname{Den}}[\mathcal{C},\mathcal{D}].$$

Definition 71 (and Lemma)**.** Let $F \in \operatorname{Ob}{}_{\operatorname{Ho}}[\mathcal{C},\mathcal{D}]$.
Let $\operatorname{Ho} F := \overline{(\mathrm{L}_{\mathcal{D}} \circ F)}$; cf. Definition 8.(1).
The following assertions (1, 2, 3) *hold.*

(1) *We have*

$$\begin{array}{ccc} \operatorname{Ho}\mathcal{C} & \xrightarrow{\operatorname{Ho} F} & \operatorname{Ho}\mathcal{D} \\ (X \xrightarrow{[f]} Y) & \mapsto & (FX \xrightarrow{[Ff]} FY). \end{array}$$

(2) *We have* $\mathrm{L}_{\mathcal{D}} \circ F = (\mathrm{Ho}\, F) \circ \mathrm{L}_{\mathcal{C}}$.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\mathrm{L}_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, "\mathrm{L}_{\mathcal{D}}"] \\
\mathrm{Ho}\,\mathcal{C} \arrow[r, "\mathrm{Ho}\,F"'] & \mathrm{Ho}\,\mathcal{D}
\end{tikzcd}$$

(3) *Suppose given* $\tilde{F} \in \mathrm{Ob}[\mathrm{Ho}\,\mathcal{C}, \mathrm{Ho}\,\mathcal{D}]$ *with* $\mathrm{L}_{\mathcal{D}} \circ F = \tilde{F} \circ \mathrm{L}_{\mathcal{C}}$. *Then we have* $\tilde{F} = \mathrm{Ho}\, F$.

Proof. This follows from Definition 14. □

Lemma 72. *Suppose given* $F \in \mathrm{Ob}\ {}_{\mathrm{Ho}}[\mathcal{C}, \mathcal{D}]$ *and* $G \in \mathrm{Ob}\ {}_{\mathrm{Ho}}[\mathcal{D}, \mathcal{E}]$.
The following assertions (1, 2) *hold.*

(1) *We have* $\mathrm{Ho}\, 1_{\mathcal{C}} = 1_{\mathrm{Ho}\,\mathcal{C}}$.

(2) *We have* $G \circ F \in {}_{\mathrm{Ho}}[\mathcal{C}, \mathcal{E}]$ *and* $\mathrm{Ho}(G \circ F) = \mathrm{Ho}\, G \circ \mathrm{Ho}\, F$.

Proof. This follows from Lemma 15. □

Definition 73 (and Lemma)**.** Suppose given $F \xrightarrow{\alpha} G$ in ${}_{\mathrm{Ho}}[\mathcal{C}, \mathcal{D}]$. Define

$$\mathrm{Ho}\,\alpha := \overline{(\mathrm{L}_{\mathcal{D}} * \alpha)} : \mathrm{Ho}\, F \longrightarrow \mathrm{Ho}\, G;$$

cf. Definition 10. *The following assertions* (1, 2, 3) *hold.*

(1) *We have* $(\mathrm{Ho}\,\alpha)_X = [\alpha_X]$ *for* $X \in \mathrm{Ob}\,\mathrm{Ho}\,\mathcal{C}$.

(2) *We have* $(\mathrm{Ho}\,\alpha) * \mathrm{L}_{\mathcal{C}} = \mathrm{L}_{\mathcal{D}} * \alpha$.

(3) *Supppose given* $\mathrm{Ho}\, F \xrightarrow{\tilde{\alpha}} \mathrm{Ho}\, G$ *with* $\tilde{\alpha} * \mathrm{L}_{\mathcal{C}} = \mathrm{L}_{\mathcal{D}} * \alpha$. *Then we have* $\tilde{\alpha} = \mathrm{Ho}\,\alpha$.

Proof. This follows from Definition 16. □

Lemma 74. *Suppose given* $F \xrightarrow{\alpha} F'$ *and* $F' \xrightarrow{\alpha'} F''$ *in* ${}_{\mathrm{Ho}}[\mathcal{C}, \mathcal{D}]$.
The following assertions (1, 2) *hold.*

(1) *We have* $\mathrm{Ho}\, 1_F = 1_{\mathrm{Ho}\, F}$.

(2) *We have* $\mathrm{Ho}(\alpha\alpha') = \mathrm{Ho}\,\alpha \cdot \mathrm{Ho}\,\alpha'$.

Proof. This follows from Lemma 17. □

Lemma 75. *Suppose given* $F \xrightarrow{\alpha} F'$ *in* ${}_{\mathrm{Ho}}[\mathcal{C}, \mathcal{D}]$ *and* $G \xrightarrow{\beta} G'$ *in* ${}_{\mathrm{Ho}}[\mathcal{D}, \mathcal{E}]$.
We have $\mathrm{Ho}(\beta * \alpha) = (\mathrm{Ho}\,\beta) * (\mathrm{Ho}\,\alpha)$.

Proof. This follows from Lemma 18. □

Lemma 76. *Suppose given* $F \overset{\alpha}{\to} F'$ *in* ${}_{\mathrm{Ho}}[\mathcal{C}, \mathcal{D}]$. *Suppose that* $\alpha = (\alpha_X)_{X \in \mathrm{Ob}\,\mathcal{C}}$ *with* $\alpha_X \in \mathrm{Den}\,\mathcal{D}$ *for* $X \in \mathrm{Ob}\,\mathcal{C}$.

Then $\mathrm{Ho}\,F \xrightarrow{\mathrm{Ho}\,\alpha} \mathrm{Ho}\,F'$ *is an isotransformation.*

Proof. Suppose given $X \in \mathrm{Ob}\,\mathrm{Ho}\,\mathcal{C}$. We have

$$(\mathrm{Ho}\,\alpha)_X \overset{\mathrm{D73}}{=} [\alpha_X] \overset{\mathrm{D56}}{=} \mathrm{L}_{\mathcal{D}}\,\alpha_X \in \mathrm{Iso}\,\mathrm{Ho}\,\mathcal{D};$$

cf. Corollary 60. □

2.5.5 Adjunctions between categories with split denominators

For this §2.5.5, let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories with split denominators.

Definition 77.

(1) Let $F \overset{\alpha}{\to} F'$ in $[\mathcal{C}, \mathcal{D}]$. We call α *denominatorial* if $\alpha_X \in \mathrm{Den}\,\mathcal{D}$ for $X \in \mathrm{Ob}\,\mathcal{C}$.

(2) We say that an adjunction $(F, G, \eta, \varepsilon)$ with $F \in \mathrm{Ob}[\mathcal{C}, \mathcal{D}]$ is *unit-denominatorial* if η is denominatorial.

(3) We say that an adjunction $(F, G, \eta, \varepsilon)$ with $F \in \mathrm{Ob}[\mathcal{C}, \mathcal{D}]$ is *counit-denominatorial* if ε is denominatorial.

(4) We say that an adjunction $(F, G, \eta, \varepsilon)$ with $F \in \mathrm{Ob}[\mathcal{C}, \mathcal{D}]$ is *denominatorial* if η and ε are denominatorial.

(5) Let $F \in \mathrm{Ob}[\mathcal{C}, \mathcal{D}]$. We call F a *unit-denominatorial left adjoint* if there exists a unit-denominatorial adjunction $(F, G, \eta, \varepsilon)$.

(6) Let $F \in \mathrm{Ob}[\mathcal{C}, \mathcal{D}]$. We call F a *counit-denominatorial left adjoint* if there exists a counit-denominatorial adjunction $(F, G, \eta, \varepsilon)$.

(7) Let $F \in \mathrm{Ob}[\mathcal{C}, \mathcal{D}]$. We call F a *denominatorial left adjoint* if there exists a denominatorial adjunction $(F, G, \eta, \varepsilon)$.

Lemma 78. *Suppose given a denominatorial adjunction* $(F, G, \eta, \varepsilon)$ *with* $F \in \mathrm{Ob}\;{}_{\mathrm{Ho}}[\mathcal{C}, \mathcal{D}]$ *and* $G \in \mathrm{Ob}\;{}_{\mathrm{Ho}}[\mathcal{D}, \mathcal{C}]$.

We have isotransformations

$$1_{\mathrm{Ho}\,\mathcal{C}} \xrightarrow[\sim]{\mathrm{Ho}\,\eta} (\mathrm{Ho}\,G) \circ (\mathrm{Ho}\,F) \qquad \textit{and} \qquad (\mathrm{Ho}\,F) \circ (\mathrm{Ho}\,G) \xrightarrow[\sim]{\mathrm{Ho}\,\varepsilon} 1_{\mathrm{Ho}\,\mathcal{D}}\,.$$

In particular, we have mutually inverse equivalences $\mathrm{Ho}\,F$ *and* $\mathrm{Ho}\,G$.

Proof. Since F and G are homotopical, we can consider

$$1_{\mathrm{Ho}\,\mathcal{C}} \xrightarrow{\mathrm{Ho}\,\eta} (\mathrm{Ho}\,G) \circ (\mathrm{Ho}\,F) \qquad \text{and} \qquad (\mathrm{Ho}\,F) \circ (\mathrm{Ho}\,G) \xrightarrow{\mathrm{Ho}\,\varepsilon} 1_{\mathrm{Ho}\,\mathcal{D}}\,.$$

Cf. Definitions 71 and 73, and Lemma 72.

Since η and ε are denominatorial, we have isotransformations $\mathrm{Ho}\,\eta$ and $\mathrm{Ho}\,\varepsilon$; cf. Lemma 76. □

Remark 79. *Suppose given an adjunction* $(F, G, \eta, \varepsilon)$ *with* $F \in \mathrm{Ob}[\mathcal{C}, \mathcal{D}]$.

The following assertions (1, 2) *hold.*

(1) *Suppose that* η *is denominatorial. Suppose given* $X \xrightarrow{c} Y$ *in* $\mathcal{C}$.
The following assertions (a, b) *are equivalent.*

(a) *We have* $c \in \mathrm{Den}\,\mathcal{C}$.

(b) *We have* $GFc \in \mathrm{Den}\,\mathcal{C}$.

(2) *Suppose that* ε *is denominatorial. Suppose given* $X \xrightarrow{d} Y$ *in* $\mathcal{D}$.
The following assertions (a, b) *are equivalent.*

(a) *We have* $d \in \mathrm{Den}\,\mathcal{D}$.

(b) *We have* $FGd \in \mathrm{Den}\,\mathcal{D}$.

Proof. *Ad* (1). We have the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, "c"] \arrow[d, "\eta_X"', "\wr"] & Y \arrow[d, "\wr", "\eta_Y"'] \\
GFX \arrow[r, "GFc"'] & GFY
\end{tikzcd}$$

Thus, the claim follows from $\mathbf{S}_{\mathrm{Den}}$.

Ad (2). This is dual to (1). □

Definition 80. We say that a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ *detects denominators* if the following assertion (∗) holds.

(∗) Suppose given $c \in \mathrm{Mor}\,\mathcal{C}$ with $Fc \in \mathrm{Den}\,\mathcal{D}$. Then we have $c \in \mathrm{Den}\,\mathcal{C}$.

Lemma 81. *Suppose given an adjunction* $(F, G, \eta, \varepsilon)$ *with* $F \in \mathrm{Ob}[\mathcal{C}, \mathcal{D}]$.

The following assertions (1, 2) *hold.*

(1) *Suppose that* η *is denominatorial. The following assertions* (a, b) *hold.*

(a) *Suppose that* G *detects denominators. Then* $F \in \mathrm{Ob}_{\,\mathrm{Den}}[\mathcal{C}, \mathcal{D}]$.

(b) *Suppose that* $G \in \mathrm{Ob}_{\,\mathrm{Den}}[\mathcal{D}, \mathcal{C}]$. *Then* F *detects denominators.*

(2) *Suppose that* ε *is denominatorial. The following assertions* (a, b) *hold.*

(a) *Suppose that* F *detects denominators. Then* $G \in \mathrm{Ob}_{\,\mathrm{Den}}[\mathcal{D}, \mathcal{C}]$.

(b) *Suppose that* $F \in \mathrm{Ob}_{\,\mathrm{Den}}[\mathcal{C}, \mathcal{D}]$. *Then* G *detects denominators.*

Proof. *Ad* (1). This follows from Remark 79.(1).

Ad (2). This follows from Remark 79.(2). □

Lemma 82. *Let* $(F, G, \eta, \varepsilon)$ *and* $(F, \tilde{G}, \tilde{\eta}, \tilde{\varepsilon})$ *be adjunctions with* $F \in \operatorname{Ob}[\mathcal{C}, \mathcal{D}]$.

The following assertions (1, 2) *hold.*

(1) *Suppose that* $\operatorname{Iso}\mathcal{C} \subseteq \operatorname{Den}\mathcal{C}$. *Suppose given* $X \in \operatorname{Ob}\mathcal{C}$. *Suppose that* $\eta_X \in \operatorname{Den}\mathcal{C}$. *Then we have* $\tilde{\eta}_X \in \operatorname{Den}\mathcal{C}$.

(2) *Suppose that* $\operatorname{Iso}\mathcal{D} \subseteq \operatorname{Den}\mathcal{D}$. *Suppose given* $Y \in \operatorname{Ob}\mathcal{D}$. *Suppose that* $\varepsilon_Y \in \operatorname{Den}\mathcal{D}$. *Then we have* $\tilde{\varepsilon}_Y \in \operatorname{Den}\mathcal{D}$.

Proof. *Ad* (1). By Lemma 37, we have the following commutative diagram in $\mathcal{C}$.

$$\begin{tikzcd}
X \arrow[r, "\eta_X" above, "\approx" description] \arrow[dr, "\tilde{\eta}_X" below left] & GFX \arrow[d, "\gamma_{FX}", "\wr" left] \\
 & \tilde{G}FX
\end{tikzcd}$$

Thus, we have $\tilde{\eta}_X \in \operatorname{Den}\mathcal{C}$; cf. $\mathbf{S}_{\mathrm{Den}}$.(1).

Ad (2). By Lemma 37, we have the following commutative diagram in $\mathcal{D}$.

$$\begin{tikzcd}
FGY \arrow[r, "\varepsilon_Y" above, "\approx" description] \arrow[d, "F\gamma_Y" left, "\wr" right] & Y \\
F\tilde{G}Y \arrow[ur, "\tilde{\varepsilon}_Y" below right] &
\end{tikzcd}$$

Thus, we have $\tilde{\varepsilon}_Y \in \operatorname{Den}\mathcal{D}$; cf. $\mathbf{S}_{\mathrm{Den}}$.(2). □

Remark 83. *The following assertions* (1, 2) *hold.*

(1) *Suppose that* $\operatorname{Iso}\mathcal{C} \subseteq \operatorname{Den}\mathcal{C}$. *Let* $F \in \operatorname{Ob}[\mathcal{C}, \mathcal{D}]$ *be a unit-denominatorial left adjoint. Suppose given an adjunction* $(F, \tilde{G}, \tilde{\eta}, \tilde{\varepsilon})$. *Then* $(F, \tilde{G}, \tilde{\eta}, \tilde{\varepsilon})$ *is unit-denominatorial.*

(2) *Suppose that* $\operatorname{Iso}\mathcal{D} \subseteq \operatorname{Den}\mathcal{D}$. *Let* $F \in \operatorname{Ob}[\mathcal{C}, \mathcal{D}]$ *be a counit-denominatorial left adjoint. Suppose given an adjunction* $(F, \tilde{G}, \tilde{\eta}, \tilde{\varepsilon})$. *Then* $(F, \tilde{G}, \tilde{\eta}, \tilde{\varepsilon})$ *is counit-denominatorial.*

Proof. *Ad* (1). This follows from Lemma 82.(1).

Ad (2). This follows from Lemma 82.(2). □

Lemma 84. *Suppose that* $\operatorname{Iso}\mathcal{D} \subseteq \operatorname{Den}\mathcal{D}$*;* cf. also Remark 66.

Suppose given a denominatorial left adjoint $F \in \operatorname{Ob}[\mathcal{C}, \mathcal{D}]$*;* cf. Definition 77.(7).

The following assertions (1, 2, 3) *are equivalent.*

(1) *The functor* F *detects denominators;* cf. Definition 80.

(2) *Suppose given an adjunction* $(F, G, \eta, \varepsilon)$. *Then we have* $G \in \operatorname{Ob}_{\mathrm{Den}}[\mathcal{D}, \mathcal{C}]$.

(3) *There exists a denominatorial adjunction* $(F, G, \eta, \varepsilon)$ *with* $G \in \operatorname{Ob}_{\mathrm{Den}}[\mathcal{D}, \mathcal{C}]$.

Proof. *Ad* (1) $\Rightarrow$ (2). Since $\operatorname{Iso}\mathcal{D} \subseteq \operatorname{Den}\mathcal{D}$, ε is denominatorial; cf. Remark 83.(2). Therefore, G is denominatorial; cf. Lemma 81.(2.a).

Ad (2) $\Rightarrow$ (3). Since F is a denominatorial left adjoint, there exists a denominatorial adjunction $(F, G, \eta, \varepsilon)$. Since (2) holds, we obtain $G \in \operatorname{Ob}{}_{\operatorname{Den}}[\mathcal{D}, \mathcal{C}]$.

Ad (3) $\Rightarrow$ (1). This follows from Lemma 81.(1, b). □

Proposition 85. *Suppose that* $\operatorname{Iso}\mathcal{D} \subseteq \operatorname{Den}\mathcal{D}$*;* cf. also Remark 66.

Suppose given a denominatorial adjunction $(F, G, \eta, \varepsilon)$ *such that* $F \in \operatorname{Ob}{}_{\operatorname{Ho}}[\mathcal{C}, \mathcal{D}]$ *detects denominators;* cf. Lemma 69 and Definition 80.

Then the following assertions (1, 2) *hold.*

(1) *We have* $G \in \operatorname{Ob}{}_{\operatorname{Den}}[\mathcal{D}, \mathcal{C}]$*;* cf. also Lemma 69.

(2) *We have isotransformations*

$$1_{\operatorname{Ho}\mathcal{C}} \xrightarrow[\sim]{\operatorname{Ho}\eta} (\operatorname{Ho}G) \circ (\operatorname{Ho}F) \qquad \textit{and} \qquad (\operatorname{Ho}F) \circ (\operatorname{Ho}G) \xrightarrow[\sim]{\operatorname{Ho}\varepsilon} 1_{\operatorname{Ho}\mathcal{D}} .$$

In particular, we have mutually inverse equivalences $\operatorname{Ho}F$ *and* $\operatorname{Ho}G$*.*

Proof. *Ad* (1). This follows from Lemma 81.(2.a).

Ad (2). Since (1) holds, this follows from Lemma 78; cf. Lemma 69. □

Corollary 86. *Suppose that* $\operatorname{Iso}\mathcal{C} \subseteq \operatorname{Den}\mathcal{C}$ *and* $\operatorname{Iso}\mathcal{D} \subseteq \operatorname{Den}\mathcal{D}$*;* cf. also Remark 66.

Suppose given mutually inverse equivalences $F \in \operatorname{Ob}{}_{\operatorname{Ho}}[\mathcal{C}, \mathcal{D}]$ *and* $G \in \operatorname{Ob}[\mathcal{D}, \mathcal{C}]$*.*

Suppose that F *detects denominators;* cf. Definition 80.

Then the following assertions (1, 2) *hold.*

(1) *We have* $G \in \operatorname{Ob}{}_{\operatorname{Den}}[\mathcal{D}, \mathcal{C}]$*;* cf. also Lemma 69.

(2) *We have isotransformations*

$$1_{\operatorname{Ho}\mathcal{C}} \xrightarrow[\sim]{\operatorname{Ho}\eta} (\operatorname{Ho}G) \circ (\operatorname{Ho}F) \qquad \textit{and} \qquad (\operatorname{Ho}F) \circ (\operatorname{Ho}G) \xrightarrow[\sim]{\operatorname{Ho}\varepsilon} 1_{\operatorname{Ho}\mathcal{D}} .$$

In particular, we have mutually inverse equivalences $\operatorname{Ho}F$ *and* $\operatorname{Ho}G$*.*

Proof. This follows from Proposition 85; cf. Lemma 36. □

Lemma 87. *Suppose given denominatorial adjunctions* $(F, G, \eta, \varepsilon)$ *and* $(F', G', \eta', \varepsilon')$ *with* $G \in \operatorname{Ob}{}_{\operatorname{Den}}[\mathcal{D}, \mathcal{C}]$ *and* $F' \in \operatorname{Ob}{}_{\operatorname{Den}}[\mathcal{D}, \mathcal{E}]$*.*

Then $(F' \circ F, G \circ G', \eta \cdot G\eta' F, F'\varepsilon G' \cdot \varepsilon')$ *is a denominatorial adjunction;* cf. also Lemma 38.

Proof. Suppose given $X \in \operatorname{Ob}\mathcal{C}$. We have to show that $(\eta \cdot G\eta' F)_X \in \operatorname{Den}\mathcal{C}$.

Since η and η' are denominatorial and $G \in \operatorname{Ob}{}_{\operatorname{Den}}[\mathcal{D}, \mathcal{C}]$, we have

$$(\eta \cdot G\eta' F)_X = \eta_X \cdot G\eta'_{FX} \in \operatorname{Den}\mathcal{C}.$$

Cf. $\mathbf{S}_{\mathrm{Den}}$.(1).

Suppose given $Z \in \mathrm{Ob}\,\mathcal{E}$. We have to show that $(F'\varepsilon G' \cdot \varepsilon')_Z \in \mathrm{Den}\,\mathcal{E}$.

Since ε and ε' are denominatorial and $F' \in \mathrm{Ob}\,{}_{\mathrm{Den}}[\mathcal{D}, \mathcal{E}]$, we have

$$(F'\varepsilon G' \cdot \varepsilon')_Z = F'\varepsilon_{G'Z} \cdot \varepsilon'_Z \in \mathrm{Den}\,\mathcal{E}.$$

Cf. $\mathbf{S}_{\mathrm{Den}}$.(1). □

2.6 Left-homotopy and right-homotopy

In this §2.6, we consider variants of the notion of homotopy which will turn out to be useful in §3.5; cf. Lemma 110.

For this §2.6, let $\mathcal{C}$ be a category with split denominators.

Definition 88 (and Lemma)**.** Let $X, Y \in \mathrm{Ob}\,\mathcal{C}$.

(1) We call $f_0,\, f_1 \in {}_{\mathcal{C}}(X, Y)$ *left-homotopic*, written $f_0 \overset{l}{\sim} f_1$, if there exists a commutative diagram as follows.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, "i_0", "\wr"'] & Y \arrow[r, equal] \arrow[d, equal] & Y \arrow[d, equal] \\
X \arrow[d, equal] & \hat{X} \arrow[l, "t"', "\approx"] \arrow[r, "\hat{f}"] & Y \arrow[r, equal] \arrow[d, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f_1"'] \arrow[u, "i_1", "\wr"'] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

This defines a reflexive and symmetric relation $(\overset{l}{\sim})$ *on* ${}_{\mathcal{C}}(X, Y)$.

(2) We call $f_0,\, f_1 \in {}_{\mathcal{C}}(X, Y)$ *right-homotopic*, written $f_0 \overset{r}{\sim} f_1$, if there exists a commutative diagram as follows.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, equal] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "\hat{f}"] \arrow[d, equal] & \hat{Y} \arrow[u, "p_0"', "\wr"] \arrow[d, "p_1"', "\wr"] & Y \arrow[l, "s", "\approx"'] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f_1"'] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

This defines a reflexive and symmetric relation $(\overset{r}{\sim})$ *on* ${}_{\mathcal{C}}(X, Y)$.

Proof. *Ad* (2). Suppose given $X \xrightarrow{f} Y$. We have the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[u, "\wr 1"'] \arrow[d, "\wr 1"] & Y \arrow[l, "1", "\approx"'] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f"'] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

Cf. Remark 44.(3). Thus, $(\overset{r}{\sim})$ is reflexive.

Symmetry of $(\overset{r}{\sim})$ follows from the symmetry of the defining diagram.

Ad (1). This is dual to (2). □

Remark 89. *Suppose given* $X \underset{f_1}{\overset{f_0}{\rightrightarrows}} Y$.

The following assertions (1, 2) *hold.*

(1) *Suppose that* $f_0 \overset{l}{\sim} f_1$. *We have* $f_0 \sim f_1$.

(2) *Suppose that* $f_0 \overset{r}{\sim} f_1$. *We have* $f_0 \sim f_1$.

Proof. *Ad* (2). This follows from Remark 44.(3); cf. Definitions 50 and 88.

Ad (1). This is dual to (2). □

Lemma 90. *Suppose given* $X \underset{f_1}{\overset{f_0}{\rightrightarrows}} Y$. *The following assertions* (1, 2) *hold.*

(1) *Suppose that* $f_0 \overset{l}{\sim} f_1$. *There exists a commutative diagram as follows.*

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, "i_0"', "\circ" marking] & Y \arrow[r, equal] \arrow[d, equal] & Y \arrow[d, equal] \\
X \arrow[d, equal] & \hat{X} \arrow[l, "t"'] \arrow[r, "\hat{f}"] & Y \arrow[r, equal] \arrow[d, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f_1"'] \arrow[u, "i_1 \wr"] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

(2) *Suppose that* $f_0 \overset{r}{\sim} f_1$. *There exists a commutative diagram as follows.*

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, equal] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "\hat{f}"] \arrow[d, equal] & \hat{Y} \arrow[u, "p_0"'] \arrow[d, "\wr p_1"'] & Y \arrow[l, "s"'] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f_1"'] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

Proof. Ad (2). There exists a commutative diagram as follows.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, equal] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "\hat{f}"] \arrow[d, equal] & \hat{Y} \arrow[u, "\wr p_0"'] \arrow[d, "\wr p_1"'] & Y \arrow[l, "s \atop \approx"'] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f_1"'] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

Cf. Definition 88. Moreover, there exists a commutative diagram as follows.

$$\begin{tikzcd}
\hat{Y} \arrow[rr, "p_0 \atop \approx"] \arrow[dr, "i"'] & & Y \\
& \bar{Y} \arrow[ur, "q"'] &
\end{tikzcd}$$

Additionally, there exists $\bar{Y} \xrightarrow{j} \hat{Y}$ with $ij = 1_{\hat{Y}}$; cf. $\mathbf{S}_{\mathrm{SDen}}$.(3).

There exists a commutative diagram as follows.

$$\begin{tikzcd}
Y \arrow[rr, "si \atop \approx"] \arrow[dr, "\sigma"'] & & \bar{Y} \\
& \tilde{Y} \arrow[ur, "\tau"'] &
\end{tikzcd}$$

Furthermore, there exists $\hat{Y} \overset{w}{\underset{\approx}{\dashrightarrow}} \tilde{Y}$ with $w\tau = 1_{\bar{Y}}$; cf. $\mathbf{S}_{\mathrm{TDen}}$.(3).

In consequence, we have the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, equal] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "\hat{f}iw"] \arrow[d, equal] & \bar{Y} \arrow[u, "\tau q"'] \arrow[d, "\wr \tau j p_1"'] & Y \arrow[l, "\sigma \atop \approx"'] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f_1"'] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

Ad (1). This is dual to (2). □

Lemma 91. *Suppose given $X, Y \in \operatorname{Ob}\mathcal{C}$. The following assertions* (1, 2) *hold.*

(1) *The relation $(\overset{l}{\sim})$ on ${}_{\mathcal{C}}(X, Y)$ is an equivalence relation.*

(2) *The relation $(\overset{r}{\sim})$ on ${}_{\mathcal{C}}(X, Y)$ is an equivalence relation.*

Proof. *Ad* (2). The relation $(\overset{r}{\sim})$ is reflexive and symmetric; cf. Definition 88.(2).

We show that $(\overset{r}{\sim})$ is transitive.

Suppose given f_0, f_1 and f_2 in ${}_{\mathcal{C}}(X, Y)$ with $f_0 \overset{r}{\sim} f_1$ and $f_1 \overset{r}{\sim} f_2$.

Since $(\overset{r}{\sim})$ is symmetric, there exists a commutative diagram as follows.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, equal] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "\hat{f}_{0,1}"] \arrow[d, equal] & \hat{Y}_{0,1} \arrow[u, "\wr\, p_0"'] \arrow[d, "p_1"] & Y \arrow[l, "s_{0,1}"', "\approx"] \arrow[d, equal] \\
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_1"'] \arrow[d, equal] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "\hat{f}_{1,2}"] \arrow[d, equal] & \hat{Y}_{1,2} \arrow[u, "\wr\, p_2"'] \arrow[d, "\wr\, p_3"] & Y \arrow[l, "s_{1,2}"', "\approx"] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f_2"'] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

Cf. Lemma 90.(2). There exist commutative diagrams as follows.

$$\begin{tikzcd}
X \arrow[drr, bend left, "\hat{f}_{0,1}"] \arrow[dr, dotted, "h"] \arrow[ddr, bend right, "\hat{f}_{1,2}"'] & & \\
& \hat{Y} \arrow[r, "v", "\approx"'] \arrow[d, "u"'] \arrow[dr, phantom, "\mathrm{W}"] & \hat{Y}_{0,1} \arrow[d, "p_1"] \\
& \hat{Y}_{1,2} \arrow[r, "\approx", "p_2"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[drr, bend left, "s_{0,1}", "\approx"'] \arrow[dr, dotted, "s", "\approx"'] \arrow[ddr, bend right, "s_{1,2}\,\wr"'] & & \\
& \hat{Y} \arrow[r, "v", "\approx"'] \arrow[d, "u"'] \arrow[dr, phantom, "\mathrm{W}"] & \hat{Y}_{0,1} \arrow[d, "p_1"] \\
& \hat{Y}_{1,2} \arrow[r, "\approx", "p_2"'] & Y
\end{tikzcd}$$

Cf. $\mathbf{S}_{\mathrm{TDen}}$.(4) and $\mathbf{S}_{\mathrm{Den}}$. In consequence, we have the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, equal] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "h"] \arrow[d, equal] & \hat{Y} \arrow[u, "\wr\, v p_0"'] \arrow[d, "\wr\, u p_3"] & Y \arrow[l, "s"', "\approx"] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f_2"'] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

Ad (1). This is dual to (2). $\square$

Chapter 3

Quasi-model-categories

3.1 FCQ-categories

In order to fix notation, we define FCQ-categories, containing *fibrations, cofibrations* and *quasi-isomorphisms*. They provide a framework in which a notion of homotopy is defined, as needed to introduce *quasi-pushouts* and *quasi-pullbacks*; cf. Definition 95, Definition 96.(5) and Definition 97.(5) below.

Definition 92. Let $\mathcal{C}$ be a category.

Let $\operatorname{Cof}\mathcal{C}$, $\operatorname{Fib}\mathcal{C}$ and $\operatorname{Qis}\mathcal{C}$ be subsets of $\operatorname{Mor}\mathcal{C}$.

The elements of $\operatorname{Cof}\mathcal{C}$ are called *cofibrations*. To indicate that $i \in \operatorname{Mor}\mathcal{C}$ is a cofibration, we often write

$$X \xrightarrow{i} Y.$$

The elements of $\operatorname{Fib}\mathcal{C}$ are called *fibrations*. To indicate that $p \in \operatorname{Mor}\mathcal{C}$ is a fibration, we often write

$$X \xrightarrow{p} Y.$$

The elements of $\operatorname{Qis}\mathcal{C}$ are called *quasi-isomorphisms*. To indicate that $w \in \operatorname{Mor}\mathcal{C}$ is a quasi-isomorphism, we often write

$$X \xrightarrow[\approx]{w} Y.$$

The elements of $\operatorname{Cof}\mathcal{C} \cap \operatorname{Qis}\mathcal{C}$ are called *acyclic* cofibrations. To indicate that $i \in \operatorname{Mor}\mathcal{C}$ is an acyclic cofibration, we often write

$$X \xrightarrow{i} Y.$$

The elements of $\operatorname{Fib}\mathcal{C} \cap \operatorname{Qis}\mathcal{C}$ are called *acyclic* fibrations. To indicate that $p \in \operatorname{Mor}\mathcal{C}$ is an acyclic fibration, we often write

$$X \xrightarrow{p} Y.$$

If the following axioms ($\mathbf{A}_{\mathrm{Cof}},\ \mathbf{A}_{\mathrm{Fib}},\ \mathbf{A}_{\mathrm{Qis}},\ \mathbf{A}_{\mathrm{Lift}},\ \mathbf{A}_{\mathrm{Fact}}$) hold, we call $(\mathcal{C}, \operatorname{Cof}\mathcal{C}, \operatorname{Fib}\mathcal{C}, \operatorname{Qis}\mathcal{C})$ an *FCQ-category*. We often refer to just $\mathcal{C}$ as an FCQ-category.

A$_{\mathrm{Cof}}$ The following assertions (1, 2) hold.

(1) We have $\operatorname{Iso}\mathcal{C} \subseteq \operatorname{Cof}\mathcal{C}$.

(2) Suppose given $X \xrightarrow{i} Y \xrightarrow{j} Z$. Then we have $X \xrightarrow{ij} Z$.

A$_{\mathrm{Fib}}$ The following assertions (1, 2) hold.

(1) We have $\operatorname{Iso}\mathcal{C} \subseteq \operatorname{Fib}\mathcal{C}$.

(2) Suppose given $X \xrightarrow{p} Y \xrightarrow{q} Z$. Then we have $X \xrightarrow{pq} Z$.

A$_{\mathrm{Qis}}$ The following assertions (1, 2) hold.

(1) We have $\operatorname{Iso}\mathcal{C} \subseteq \operatorname{Qis}\mathcal{C}$.

(2) Suppose given $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$. The following assertions (a, b, c) hold.
   (a) Suppose that $f, g \in \operatorname{Qis}\mathcal{C}$. Then we have $fg \in \operatorname{Qis}\mathcal{C}$.
   (b) Suppose that $f, fg \in \operatorname{Qis}\mathcal{C}$. Then we have $g \in \operatorname{Qis}\mathcal{C}$.
   (c) Suppose that $g, fg \in \operatorname{Qis}\mathcal{C}$. Then we have $f \in \operatorname{Qis}\mathcal{C}$.

A$_{\mathrm{Lift}}$ Suppose given $(i, p) \in \operatorname{Cof}\mathcal{C} \times \operatorname{Fib}\mathcal{C}$. The following assertions $(1, 2)$ hold.

(1) Suppose that i is acyclic. Then (i, p) is an extension pair.

(2) Suppose that p is acyclic. Then (i, p) is an extension pair.

Cf. Definition 21.

A$_{\mathrm{Fact}}$ Suppose given $A \xrightarrow{f} B$ in $\mathcal{C}$. The following assertions (1, 2) hold.

(1) There exists a commutative diagram as follows.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "i"'] & & B \\ & X \arrow[ur, "p"'] & \end{tikzcd}$$

(2) There exists a commutative diagram as follows.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "i"'] & & B \\ & X \arrow[ur, "p"'] & \end{tikzcd}$$

Remark 93. *Let $\mathcal{C}$ be an FCQ-category. Let $X \in \operatorname{Ob}\mathcal{C}$.*
The following assertions (1, 2) *hold.*

(1) *Let I and I' be initial objects in $\mathcal{C}$. The following assertions* (a, b) *are equivalent.*

   (a) *We have* $I \longrightarrow X$.

   (b) *We have* $I' \longrightarrow X$.

(2) *Let T and T' be terminal objects in $\mathcal{C}$. The following assertions* (a, b) *are equivalent.*

   (a) *We have* $X \mathrel{-\!\!\!+\!\!\!\twoheadrightarrow} T$ *.*

   (b) *We have* $X \mathrel{-\!\!\!+\!\!\!\twoheadrightarrow} T'$ *.*

Proof. *Ad* (1). By symmetry, it suffices to show that (a) implies (b).

Since I, I' are initial, we have $I' \cong I$. Thus, we have $I' \mathrel{-\!\!\circ\!\!\!\rightarrowtail} I$; cf. $\mathbf{A}_{\mathrm{Cof}}$.(1) and $\mathbf{A}_{\mathrm{Qis}}$.(1).

Therefore, we have $I' \mathrel{-\!\!\circ\!\!\!\rightarrowtail} I \mathrel{-\!\!\bullet\!\!\!\rightarrowtail} X$ and the claim follows from $\mathbf{A}_{\mathrm{Cof}}$.(2).

Ad (2). This is dual to (1). □

Definition 94 (and Lemma)**.** Let $\mathcal{D}$ be a subcategory of an FCQ-category $\mathcal{C}$. Let

$$\begin{array}{rcl} \mathrm{Cof}\,\mathcal{D} & := & \mathrm{Cof}\,\mathcal{M} \cap \mathrm{Mor}\,\mathcal{D} \\ \mathrm{Fib}\,\mathcal{D} & := & \mathrm{Fib}\,\mathcal{M} \cap \mathrm{Mor}\,\mathcal{D} \\ \mathrm{Qis}\,\mathcal{D} & := & \mathrm{Qis}\,\mathcal{M} \cap \mathrm{Mor}\,\mathcal{D}. \end{array}$$

The following assertions (1, 2) *hold.*

(1) *The properties* $\mathbf{A}_{\mathrm{Cof}}$ *,* $\mathbf{A}_{\mathrm{Fib}}$ *and* $\mathbf{A}_{\mathrm{Qis}}$ *hold in* $(\mathcal{D}, \mathrm{Cof}\,\mathcal{D}, \mathrm{Fib}\,\mathcal{D}, \mathrm{Qis}\,\mathcal{D})$.

(2) *Suppose that $\mathcal{D}$ is a full subcategory of $\mathcal{D}$. Then* $\mathbf{A}_{\mathrm{Lift}}$ *holds in* $(\mathcal{D}, \mathrm{Cof}\,\mathcal{D}, \mathrm{Fib}\,\mathcal{D}, \mathrm{Qis}\,\mathcal{D})$.

Proof. *Ad* (1). This follows since $\mathbf{A}_{\mathrm{Cof}}$, $\mathbf{A}_{\mathrm{Fib}}$ and $\mathbf{A}_{\mathrm{Qis}}$ hold in $\mathcal{C}$.

Ad (2). This follows since $\mathbf{A}_{\mathrm{Lift}}$ holds in $\mathcal{C}$ and since $\mathcal{D}$ is a full subcategory. □

Definition 95. Let $\mathcal{C}$ be an FCQ-category.

Let $X, Y \in \mathrm{Ob}\,\mathcal{C}$. Suppose given $f_0\,, f_1 \in {}_{\mathcal{C}}(X, Y)$.

We write $f_0{\sim}f_1$, if there exists a commutative diagram as follows.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, "i_0\,\wr"'] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X & \hat{X} \arrow[l, "t"', "\approx"] \arrow[r, "\hat{f}"] & \hat{Y} \arrow[u, "\wr\,p_0"'] \arrow[d, "\wr\,p_1"] & Y \arrow[l, "s"', "\approx"] \\
X \arrow[r, equal] \arrow[u, equal] & X \arrow[r, "f_1"'] \arrow[u, "i_1\,\wr"] & Y \arrow[r, equal] & Y \arrow[u, equal]
\end{tikzcd}$$

Once an FCQ-category is equipped with extra data that turn it into a so-called quasi-model-category, it will in particular have the structure of a category with split denominators; cf. Definitions 39, 100 and 108. So then $(\sim)$ will just be the homotopy relation as introduced in Definition 50. In particular, it will be a congruence on $\mathcal{C}$; cf. Definition 1 and Proposition 52.

3.2 Quasi-pushouts and quasi-pullbacks

In this §3.2, we introduce the notions of quasi-pushouts and quasi-pullbacks. They will serve as practical replacements for *actual* pushouts and pullbacks in our constructions on the homotopy categories of quasi-model-categories; e.g. in the construction of loop and suspension functor; cf. §3.6.3 and §3.7.3 below.

At the end of this §3.2, we recall traditional names for some properties of quasi-pushouts and quasi-pullbacks and explain our nomenclature; cf. Remark 99 below.

For this §3.2, let $\mathcal{C}$ be an FCQ-category; cf. Definition 92.

Definition 96. Let $\operatorname{QPO}\mathcal{C}$ be a set of commutative quadrangles in $\mathcal{C}$.

To indicate that a quadrangle is in $\operatorname{QPO}\mathcal{C}$, we often write

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B'. \end{tikzcd}$$

We call $\operatorname{QPO}\mathcal{C}$ a set of *quasi-pushouts*, if the following assertions (1–8) hold.

(1) Suppose given

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B'. \end{tikzcd}$$

Then we have $a, b \in \operatorname{Cof}\mathcal{C}$.

(2) Suppose given

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B'. \end{tikzcd}$$

Then we have $b \in \operatorname{Cof}\mathcal{C} \cap \operatorname{Qis}\mathcal{C}$.

(3) Suppose given

$$\begin{tikzcd} A \arrow[r, "f", "\approx"'] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B'. \end{tikzcd}$$

Then we have $f' \in \operatorname{Qis}\mathcal{C}$.

(4) Suppose given the following commutative diagram.

$$\begin{tikzcd}
A \arrow[r, "f", "\bullet" marking] \arrow[d, "a"', "\bullet" marking] & B \arrow[d, "b", "\bullet" marking] \\
A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

Then the following assertions (a, b) are equivalent.

(a) We have (A, B, A', B') in $\mathrm{QPO}\,\mathcal{C}$.

(b) We have (A, A', B, B') in $\mathrm{QPO}\,\mathcal{C}$.

(5) Suppose given the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"', "\bullet" marking] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b", "\bullet" marking] & \\
A' \arrow[r, "f'"'] & B' \arrow[dr, shift left, "u"] \arrow[dr, shift right, "v"'] & \\
 & & T
\end{tikzcd}$$

in which $f'u = f'v$ and $bu = bv$. Then we have $u \sim v$; cf. Definition 95.

(6) Suppose given

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"', "\bullet" marking] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[r, "g"] \arrow[d, "b", "\bullet" marking] \arrow[dr, phantom, "\mathrm{Q}"] & C \arrow[d, "c", "\bullet" marking] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C'.
\end{tikzcd}$$

Then (A, C, A', C') is in $\mathrm{QPO}\,\mathcal{C}$.

(7) Suppose given

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a_0"', "\bullet" marking] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b_0", "\bullet" marking] \\
A' \arrow[r, "f'"'] \arrow[d, "a_1"', "\bullet" marking] \arrow[dr, phantom, "\mathrm{Q}"] & B' \arrow[d, "b_1", "\bullet" marking] \\
A'' \arrow[r, "f''"'] & B''
\end{tikzcd}$$

Then (A, B, A'', B'') is in $\mathrm{QPO}\,\mathcal{C}$.

(8) Suppose given a commutative diagram as follows.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"', "\bullet" marking] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b", "\bullet" marking] \arrow[ddr, bend left, "y"] & & \\
A' \arrow[r, "f'"'] \arrow[drr, bend right, "x"'] & B' \arrow[drr, bend left, "u"] & & \\
 & & T \arrow[r, "s"', "|" marking] & S
\end{tikzcd}$$

Then there exists $B' \xrightarrow{v} T$ in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \arrow[ddr, bend left, "y"] & \\
A' \arrow[r, "f'"'] \arrow[ur, phantom, "\mathrm{Q}"] \arrow[drr, bend right, "x"'] & B' \arrow[dr, dotted, "v"] \arrow[drr, bend left, "u"] & \\
 & & T \arrow[r, "s"'] & S
\end{tikzcd}$$

Given a set of quasi-pushouts $\mathrm{QPO}\,\mathcal{C}$, an element of $\mathrm{QPO}\,\mathcal{C}$ is referred to as a *quasi-pushout* (with respect to $\mathrm{QPO}\,\mathcal{C}$).

Definition 97. Let $\mathrm{QPB}\,\mathcal{C}$ be a set of commutative quadrangles in $\mathcal{C}$.

To indicate that a quadrangle is in $\mathrm{QPB}\,\mathcal{C}$, we often write

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] & B'.
\end{tikzcd}$$

We call $\mathrm{QPB}\,\mathcal{C}$ a set of *quasi-pullbacks*, if the following assertions (1–8) hold.

(1) Suppose given

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] & B'.
\end{tikzcd}$$

Then we have $a, b \in \mathrm{Fib}\,\mathcal{C}$.

(2) Suppose given

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] & B'.
\end{tikzcd}$$

Then we have $a \in \mathrm{Fib}\,\mathcal{C} \cap \mathrm{Qis}\,\mathcal{C}$.

(3) Suppose given

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b"] \\
A' \arrow[r, "\approx", "f'"'] & B'.
\end{tikzcd}$$

Then we have $f \in \mathrm{Qis}\,\mathcal{C}$.

(4) Suppose given the following commutative diagram.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"', "\shortmid"{marking}] & B \arrow[d, "b", "\shortmid"{marking}] \\
A' \arrow[r, "f'"', "\shortmid"{marking}] & B'
\end{tikzcd}$$

Then the following assertions (a, b) are equivalent.

(a) We have (A, B, A', B') in $\mathrm{QPB}\,\mathcal{C}$.

(b) We have (A, A', B, B') in $\mathrm{QPB}\,\mathcal{C}$.

(5) Suppose given the diagram

$$\begin{tikzcd}
T \arrow[dr, shift left, "u"] \arrow[dr, shift right, "v"'] & & \\
& A \arrow[r, "f"] \arrow[d, "a"', "\shortmid"{marking}] \arrow[dr, phantom, "\mathrm{Q}", very near start] & B \arrow[d, "b", "\shortmid"{marking}] \\
& A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

in which $ua = va$ and $uf = vf$. Then we have $u \sim v$; cf. Definition 95.

(6) Suppose given

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"', "\shortmid"{marking}] \arrow[dr, phantom, "\mathrm{Q}", very near start] & B \arrow[r, "g"] \arrow[d, "b"'] \arrow[dr, phantom, "\mathrm{Q}", very near start] & C \arrow[d, "c", "\shortmid"{marking}] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C'
\end{tikzcd}$$

Then (A, C, A', C') is in $\mathrm{QPB}\,\mathcal{C}$.

(7) Suppose given

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a_0"', "\shortmid"{marking}] \arrow[dr, phantom, "\mathrm{Q}", very near start] & B \arrow[d, "b_0", "\shortmid"{marking}] \\
A' \arrow[r, "f'"] \arrow[d, "a_1"', "\shortmid"{marking}] \arrow[dr, phantom, "\mathrm{Q}", very near start] & B' \arrow[d, "b_1", "\shortmid"{marking}] \\
A'' \arrow[r, "f''"'] & B''
\end{tikzcd}$$

Then (A, B, A'', B'') is in $\mathrm{QPB}\,\mathcal{C}$.

(8) Suppose given a commutative diagram as follows.

$$\begin{tikzcd}
S \arrow[r, "s", "\bullet"{marking}] \arrow[drr, bend right, "u"'] & T \arrow[drr, bend left, "x"] \arrow[ddr, bend right, "y"'] & & \\
& & A \arrow[r, "f"] \arrow[d, "a"', "\shortmid"{marking}] \arrow[dr, phantom, "\mathrm{Q}", very near start] & B \arrow[d, "b", "\shortmid"{marking}] \\
& & A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

Then there exists $T \xrightarrow{v} A$ in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
S \arrow[r, "s"] \arrow[drr, bend right, "u"'] & T \arrow[dr, dotted, "v"] \arrow[drr, bend left, "x"] \arrow[ddr, bend right, "y"'] & & \\
& & A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b"] \\
& & A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

Given a set of quasi-pullbacks $\mathrm{QPB}\,\mathcal{C}$, an element of $\mathrm{QPB}\,\mathcal{C}$ is referred to as a *quasi-pullback* (with respect to $\mathrm{QPB}\,\mathcal{C}$).

Remark 98. *The following assertions* (1, 2) *hold.*

(1) *Let* $\mathrm{QPO}\,\mathcal{C}$ *be a set of quasi-pushouts in* $\mathcal{C}$. *Suppose given*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] & B'.
\end{tikzcd}$$

Then we have $f' \in \mathrm{Cof}\,\mathcal{C}$.

(2) *Let* $\mathrm{QPB}\,\mathcal{C}$ *be a set of quasi-pullbacks in* $\mathcal{C}$. *Suppose given*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] & B'.
\end{tikzcd}$$

Then we have $f \in \mathrm{Fib}\,\mathcal{C}$.

Proof. Ad (1). This follows from Definition 96.(1,4).

Ad (2). This follows from Definition 97.(1,4). □

Remark 99.

- We often refer to property (2) in Definitions 96 and 97 as the *axiom of incision* or just as *incision*.
- We often refer to property (3) in Definitions 96 and 97 as the *axiom of excision* or just as *excision*.
- In the context of quasi-model-categories, property (8) in Definition 96 implies that a quasi-pushout is in particular a weak pushout, justifying the notion quasi-*pushout*; cf. Definition 24 and Lemma 104.(1) below.

  Dually, in the context of quasi-model-categories, property (8) in Definition 97 implies that a quasi-pullback is in particular a weak pullback, justifying the notion quasi-*pullback*; cf. Definition 25 and Lemma 104.(2) below.

3.3 Axioms for quasi-model-categories

Definition 100.

Let $\mathcal{C}$ be an FCQ-category that has initial and terminal objects; cf. Definition 92. Choose an initial object ¡ and a terminal object ! in $\mathcal{C}$.

Let $\operatorname{QPO}\mathcal{C}$ be a set of quasi-pushouts in $\mathcal{C}$; cf. Definition 96.

Let $\operatorname{QPB}\mathcal{C}$ be a set of quasi-pullbacks in $\mathcal{C}$; cf. Definition 97.

If the following axioms $\mathbf{Q}_{\mathrm{Cof}}$, $\mathbf{Q}_{\mathrm{Fib}}$ and $\mathbf{Q}_{\mathrm{Braid}}$ hold, we call

$$(\mathcal{C}, \operatorname{Cof}\mathcal{C}, \operatorname{Fib}\mathcal{C}, \operatorname{Qis}\mathcal{C}, \operatorname{QPO}\mathcal{C}, \operatorname{QPB}\mathcal{C})$$

a *quasi-model-category*. We often refer to just $\mathcal{C}$ as a quasi-model-category.

> So altogether, a quasi-model-category is a category $\mathcal{C}$ with initial object ¡ and terminal object !, together with subsets $\operatorname{Cof}\mathcal{C} \subseteq \operatorname{Mor}\mathcal{C}$ of cofibrations, $\operatorname{Fib}\mathcal{C} \subseteq \operatorname{Mor}\mathcal{C}$ of fibrations and $\operatorname{Qis}\mathcal{C} \subseteq \operatorname{Mor}\mathcal{C}$ of quasi-isomorphisms, and sets of quasi-pushouts $\operatorname{QPO}\mathcal{C}$ and quasi-pullbacks $\operatorname{QPB}\mathcal{C}$ such that the axioms $\mathbf{A}_{\mathrm{Cof}}$, $\mathbf{A}_{\mathrm{Fib}}$, $\mathbf{A}_{\mathrm{Qis}}$, $\mathbf{A}_{\mathrm{Lift}}$, $\mathbf{A}_{\mathrm{Fact}}$, $\mathbf{Q}_{\mathrm{Cof}}$, $\mathbf{Q}_{\mathrm{Fib}}$ and $\mathbf{Q}_{\mathrm{Braid}}$ hold, where the properties of quasi-pushouts are given in Definition 96 and the properties of quasi-pullbacks are given in Definition 97.

$\mathbf{Q}_{\mathrm{Cof}}$ The following assertions (1, 2) hold.

(1) Suppose given $A \in \operatorname{Ob}\mathcal{C}$. Then we have $¡ \xrightarrow{\bullet} A$. Cf. also Remark 93.(1).

(2) Suppose given $A' \xleftarrow{\bullet\, a} A \xrightarrow{f} B$ in $\mathcal{C}$. There exists a quasi-pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a\,\bullet"'] & B \arrow[d, "\bullet\, b"] \\ A' \arrow[r, "f'"'] & B' \arrow[ul, phantom, "\mathrm{Q}", near start] \end{tikzcd}$$

cf. Definition 96.

$\mathbf{Q}_{\mathrm{Fib}}$ The following assertions (1, 2) hold.

(1) Suppose given $A \in \operatorname{Ob}\mathcal{C}$. We have $A \xrightarrow{+} !$. Cf. also Remark 93.(2).

(2) Suppose given $A' \xrightarrow{f'} B' \xleftarrow{+\, b} B$ in $\mathcal{C}$. There exists a quasi-pullback

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a\,+"'] \arrow[dr, phantom, "\mathrm{Q}", near start] & B \arrow[d, "+\, b"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

cf. Definition 97.

$\mathbf{Q}_{\mathrm{Braid}}$ Suppose given a commutative diagram as follows.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b"] \arrow[rrdd, bend left, "u"] \arrow[rddd, bend left, "v"] & & \\
A' \arrow[r, "f'"'] \arrow[rrrd, bend right, "u'"] \arrow[rrdd, bend right, "v'"'] & B' & & \\
& & X \arrow[r, "g"] \arrow[d, "x"'] \arrow[dr, phantom, "\mathrm{Q}"] & Y \arrow[d, "y"] \\
& & X' \arrow[r, "g'"'] & Y'
\end{tikzcd}$$

There exists $B' \xrightarrow{w} X$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b"] \arrow[rrdd, bend left, "u"] \arrow[rddd, bend left, "v"] & & \\
A' \arrow[r, "f'"'] \arrow[rrrd, bend right, "u'"] \arrow[rrdd, bend right, "v'"'] & B' \arrow[dr, dotted, "w"] & & \\
& & X \arrow[r, "g"] \arrow[d, "x"'] \arrow[dr, phantom, "\mathrm{Q}"] & Y \arrow[d, "y"] \\
& & X' \arrow[r, "g'"'] & Y'
\end{tikzcd}$$

Remark 101. Let $\mathcal{M}$ be a weakly pointed model category; cf. Definitions 172 and 174.(1). Then $\mathcal{M}_{\mathrm{bif}}$ is a quasi-model-category with the quasi-model-structure described in Theorem 193, as shown in loc. cit.

Remark 102. Let $\mathcal{C}$ be a quasi-model-category.

Let

$$\begin{array}{rcl}
\mathrm{Cof}\,\mathcal{C}^\circ & := & \{f^\circ : f \in \mathrm{Fib}\,\mathcal{C}\} \\
\mathrm{Fib}\,\mathcal{C}^\circ & := & \{f^\circ : f \in \mathrm{Cof}\,\mathcal{C}\} \\
\mathrm{Qis}\,\mathcal{C}^\circ & := & \{f^\circ : f \in \mathrm{Qis}\,\mathcal{C}\}.
\end{array}$$

Let

$$\mathrm{QPO}\,\mathcal{C}^\circ \;:=\; \left\{ \begin{tikzcd} A \arrow[r, "f^\circ"] \arrow[d, "a^\circ"'] & B \arrow[d, "b^\circ"] \\ A' \arrow[r, "(f')^\circ"'] & B' \end{tikzcd} \;:\; \begin{tikzcd} A & B \arrow[l, "f"'] \\ A' \arrow[u, "a"] & B' \arrow[l, "f'"] \arrow[u, "b"'] \end{tikzcd} \in \mathrm{QPB}\,\mathcal{C} \right\}$$

$$\mathrm{QPB}\,\mathcal{C}^\circ \;:=\; \left\{ \begin{tikzcd} A \arrow[r, "f^\circ"] \arrow[d, "a^\circ"'] & B \arrow[d, "b^\circ"] \\ A' \arrow[r, "(f')^\circ"'] & B' \end{tikzcd} \;:\; \begin{tikzcd} A & B \arrow[l, "f"'] \\ A' \arrow[u, "a"] & B' \arrow[l, "f'"] \arrow[u, "b"'] \end{tikzcd} \in \mathrm{QPO}\,\mathcal{C} \right\}.$$

Then

$$(\mathcal{C}^\circ, \operatorname{Cof}\mathcal{C}^\circ, \operatorname{Fib}\mathcal{C}^\circ, \operatorname{Qis}\mathcal{C}^\circ, \operatorname{QPO}\mathcal{C}^\circ, \operatorname{QPB}\mathcal{C}^\circ)$$

is a quasi-model-category.

Remark 103. *Let $\mathcal{C}$ be a quasi-model-category. Suppose given $X \xrightarrow{f} Y$ in $\mathcal{C}$. The following assertions* (1, 2) *hold.*

(1) *Suppose that (f, p) is an extension pair for all acyclic fibrations p. Then f is a retract of a cofibration.*

(2) *Suppose that (f, p) is an extension pair for all fibrations p. Then f is a retract of an acyclic cofibration.*

Proof. *Ad* (1). Since $\mathbf{A}_{\mathrm{Fact}}.(2)$ holds in $\mathcal{C}$, there exists a commutative diagram as follows.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "i"'] & & B \\ & X \arrow[ur, "p"'] & \end{tikzcd}$$

By assumption, (f, p) is an extension pair. Thus, $f = ip$ is a retract of i; cf. Lemma 23.(1).

Ad (2). Since $\mathbf{A}_{\mathrm{Fact}}.(1)$ holds in $\mathcal{C}$, there exists a commutative diagram as follows.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "i"'] & & B \\ & X \arrow[ur, "p"'] & \end{tikzcd}$$

By assumption, (f, p) is an extension pair. Thus, $f = ip$ is a retract of i; cf. Lemma 23.(1).

□

3.4 Elementary properties

For this §3.4 let $\mathcal{C}$ be a quasi-model-category.

3.4.1 Brown factorisation

Lemma 104. *The following assertions* (1, 2) *hold.*

(1) *Suppose given*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B'. \end{tikzcd}$$

Then (A, B, A', B') is a weak pushout; cf. Definition 24.

(2) *Suppose given*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] & B'.
\end{tikzcd}$$

Then (A, B, A', B') is a weak pullback; cf. Definition 25.

Proof. *Ad* (1). Suppose given $A' \xrightarrow{x} T$ and $B \xrightarrow{y} T$ with $ax = fy$. We have to show that there exists $B' \xrightarrow{v} T$ with $f'v = x$ and $bv = y$.

We have a commutative diagram as follows.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b"] \arrow[ddr, bend left, "y"] & & \\
A' \arrow[r, "f'"'] \arrow[drr, bend right, "x"'] & B' \arrow[drr, bend left] & & \\
& & T \arrow[r] & !
\end{tikzcd}$$

Cf. $\mathbf{Q}_{\mathrm{Fib}}$.(1). Thus, there exists $B' \xrightarrow{v} T$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b"] \arrow[ddr, bend left, "y"] & & \\
A' \arrow[r, "f'"'] \arrow[drr, bend right, "x"'] & B' \arrow[drr, bend left] \arrow[dr, dotted, "v"] & & \\
& & T \arrow[r] & !
\end{tikzcd}$$

Cf. Definition 96.(8). In particular, we have $f'v = x$ and $bv = y$.

Ad (2). This is dual to (1). □

The following lemma and its proof are essentially due to K. Brown [**2,** p. 421]. He established it in the context of *categories of fibrant objects*.

Lemma 105 (Brown factorisation)**.** *Suppose given $X \xrightarrow{f} Y$.*

The following assertions (1, 2) *hold.*

(1) *There exists a diagram*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \arrow[dl, bend left, "k"] \\
& Z \arrow[ur, "q"] &
\end{tikzcd}$$

such that $jq = f$ and $kq = 1_Y$.

(2) *There exists a diagram*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\
& Z \arrow[ur, "q"'] \arrow[ul, bend left, "k"] &
\end{tikzcd}$$

such that $jq = f$ *and* $jk = 1_X$.

Proof. *Ad* (1). We have a commutative diagram as follows.

$$\begin{tikzcd}
! \arrow[r] \arrow[d] & X \arrow[d, "x"] \arrow[ddr, bend left, "f"] & \\
Y \arrow[r, "y"'] \arrow[drr, bend right, "1"'] \arrow[ur, phantom, "\mathrm{Q}"] & M \arrow[dr, dotted, "g"] & \\
& & Y
\end{tikzcd}$$

Cf. $\mathbf{Q}_{\text{Cof}}$, Remark 98.(1) and Lemma 104.(1).

Furthermore, there exists a commutative diagram as follows.

$$\begin{tikzcd}
M \arrow[rr, "g"] \arrow[dr, "a"'] & & Y \\
& Z \arrow[ur, "s"'] &
\end{tikzcd}$$

Cf. $\mathbf{A}_{\text{Fact}}$.(2). In consequence, we have the following diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "xa"'] & & Y \arrow[dl, bend left, "ya"] \\
& Z \arrow[ur, "s"'] &
\end{tikzcd}$$

with $xas = f$ and $yas = 1_Y$; cf. $\mathbf{A}_{\text{Cof}}$.(1) and $\mathbf{A}_{\text{Qis}}$.(2).

Ad (2). This is dual to (1). □

3.4.2 Split denominator structure

Remark 106. *The following assertions* (1, 2) *hold.*

(1) *We have* $\operatorname{Qis}\mathcal{C} \cap \operatorname{Cof}\mathcal{C} \subseteq \operatorname{Coret}\mathcal{C}$.

(2) *We have* $\operatorname{Qis}\mathcal{C} \cap \operatorname{Fib}\mathcal{C} \subseteq \operatorname{Ret}\mathcal{C}$.

Proof. *Ad* (1). Suppose given $A \xrightarrow{i} B$. We have a commutative diagram as follows.

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "i"'] & A \arrow[d] \\
B \arrow[r] \arrow[ur, dotted, "q"] & !
\end{tikzcd}$$

Cf. $\mathbf{A}_{\text{Lift}}$.(1) and $\mathbf{Q}_{\text{Fib}}$.(1). Thus, $i \in \operatorname{Coret}\mathcal{C}$.

Ad (2). This is dual to (1). $\square$

Remark 107. *Suppose given $d \in \operatorname{Mor}\mathcal{C}$. The following assertions* (1, 2) *are equivalent.*

(1) *We have $d \in \operatorname{Qis}\mathcal{C}$.*

(2) *There exists $(i, p) \in (\operatorname{Cof}\mathcal{C} \cap \operatorname{Qis}\mathcal{C}) \times (\operatorname{Fib}\mathcal{C} \cap \operatorname{Qis}\mathcal{C})$ with $q = ip$.*

Proof. *Ad* (1) $\Rightarrow$ (2). By $\mathbf{A}_{\text{Fact}}$.(1), there exists $(i, p) \in (\operatorname{Cof}\mathcal{C} \cap \operatorname{Qis}\mathcal{C}) \times \operatorname{Fib}\mathcal{C}$ with $q = ip$. By $\mathbf{A}_{\text{Qis}}$.(2.b), we have $p \in (\operatorname{Fib}\mathcal{C} \cap \operatorname{Qis}\mathcal{C})$.

Ad (2) $\Rightarrow$ (1). This follows from $\mathbf{A}_{\text{Qis}}$.(2.a). $\square$

Definition 108 (and Lemma)**.** Recall that $\mathcal{C}$ is a quasi-model-category; cf. Definition 100.

Define

$$\begin{array}{rcl} \operatorname{SDen}\mathcal{C} & := & \operatorname{Cof}\mathcal{C} \cap \operatorname{Qis}\mathcal{C} \\ \operatorname{TDen}\mathcal{C} & := & \operatorname{Fib}\mathcal{C} \cap \operatorname{Qis}\mathcal{C}. \end{array}$$

Then $(\mathcal{C}, \operatorname{SDen}\mathcal{C}, \operatorname{TDen}\mathcal{C})$ is a category with split denominators; cf. Definition 39.

Furthermore, we have

$$\operatorname{Den}\mathcal{C} = \operatorname{Qis}\mathcal{C}.$$

In the following we will use this split denominator structure on $\mathcal{C}$ without further notice. In particular, we will make use of $\mathrm{L}_{\mathcal{C}} : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$; cf. §2.5.

Proof. By Remark 107, we have $\operatorname{Den}\mathcal{C} = \operatorname{Qis}\mathcal{C}$; cf. Definition 39.

Furthermore, we have

$$\begin{array}{lll} \mathbf{S}_{\text{SDen}}.(1) & \text{by} & \mathbf{A}_{\text{Cof}}.(2) \text{ and } \mathbf{A}_{\text{Qis}}.(2.\text{a}), \\ \mathbf{S}_{\text{SDen}}.(2) & \text{by} & \mathbf{A}_{\text{Cof}}.(1) \text{ and } \mathbf{A}_{\text{Qis}}.(1), \\ \mathbf{S}_{\text{SDen}}.(3) & \text{by} & \text{Remark 106.(1), and} \\ \mathbf{S}_{\text{SDen}}.(4) & \text{by} & \mathbf{Q}_{\text{Cof}}.(2)\text{; cf. Definition 96.(2).} \end{array}$$

Dually, we have

$$\begin{array}{lll} \mathbf{S}_{\text{TDen}}.(1) & \text{by} & \mathbf{A}_{\text{Fib}}.(2) \text{ and } \mathbf{A}_{\text{Qis}}.(2.\text{a}), \\ \mathbf{S}_{\text{TDen}}.(2) & \text{by} & \mathbf{A}_{\text{Fib}}.(1) \text{ and } \mathbf{A}_{\text{Qis}}.(1), \\ \mathbf{S}_{\text{TDen}}.(3) & \text{by} & \text{Remark 106.(2), and} \\ \mathbf{S}_{\text{TDen}}.(4) & \text{by} & \mathbf{Q}_{\text{Fib}}.(2)\text{; cf. Definition 97.(2).} \end{array}$$

Finally, we have $\mathbf{S}_{\text{Den}}$ by $\mathbf{A}_{\text{Qis}}$.(2). $\square$

3.4.3 Uniqueness properties of quasi-pushouts

Lemma 109. *Suppose given*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"', "\bullet" marking] & B \arrow[d, "b", "\bullet" marking] \\ A' \arrow[r, "f'"'] \arrow[ur, phantom, "\mathrm{Q}"] & B' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"', "\bullet" marking] & B \arrow[d, "\hat{b}", "\bullet" marking] \\ A' \arrow[r, "\hat{f}"'] \arrow[ur, phantom, "\mathrm{Q}"] & \hat{B}. \end{tikzcd}$$

The following assertions (1, 2, 3) *hold.*

(1) *There exists a commutative diagrams as follows.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"', "\bullet" marking] & B \arrow[d, "b", "\bullet" marking] \arrow[ddr, bend left, "\hat{b}", "\bullet" marking] & \\ A' \arrow[r, "f'"'] \arrow[ur, phantom, "\mathrm{Q}"] \arrow[drr, bend right, "\hat{f}"'] & B' \arrow[dr, dotted, "u"] & \\ & & \hat{B} \end{tikzcd}$$

(2) *Suppose given commutative diagrams*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"', "\bullet" marking] & B \arrow[d, "b", "\bullet" marking] \arrow[ddr, bend left, "\hat{b}", "\bullet" marking] & \\ A' \arrow[r, "f'"'] \arrow[ur, phantom, "\mathrm{Q}"] \arrow[drr, bend right, "\hat{f}"'] & B' \arrow[dr, "u"] & \\ & & \hat{B} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"', "\bullet" marking] & B \arrow[d, "\hat{b}", "\bullet" marking] \arrow[ddr, bend left, "b", "\bullet" marking] & \\ A' \arrow[r, "\hat{f}"'] \arrow[ur, phantom, "\mathrm{Q}"] \arrow[drr, bend right, "f'"'] & \hat{B} \arrow[dr, "v"] & \\ & & B'. \end{tikzcd}$$

Then we have $uv \sim 1_{B'}$ *and* $vu \sim 1_{\hat{B}}$.

(3) *Suppose given a commutative diagram as follows.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"', "\bullet" marking] & B \arrow[d, "b", "\bullet" marking] \arrow[ddr, bend left, "\hat{b}", "\bullet" marking] & \\ A' \arrow[r, "f'"'] \arrow[ur, phantom, "\mathrm{Q}"] \arrow[drr, bend right, "\hat{f}"'] & B' \arrow[dr, "u"] & \\ & & \hat{B} \end{tikzcd}$$

Then we have $\mathrm{L}_{\mathcal{C}}\, u \in \operatorname{Iso} \operatorname{Ho} \mathcal{C}$. *If* $\mathcal{C}$ *is saturated, then we have* $u \in \operatorname{Qis} \mathcal{C}$.

Proof. Ad (1). This follows from Lemma 104.(1).

Ad (2). We have to show that $uv \sim 1_{B'}$ and $vu \sim 1_{\hat{B}}$. By symmetry, it suffices to show that $uv \sim 1_{B'}$.

Since we have the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b"] & \\
A' \arrow[r, "f'"'] & B' \arrow[dr, shift left, "1"] \arrow[dr, shift right, "uv"'] & \\
& & B'
\end{tikzcd}$$

in which $f'uv = f' = f' \cdot 1_{B'}$ and $b \cdot 1_{B'} = b = buv$, we obtain $uv \sim 1_{B'}$; cf. Definition 96.(5).

Ad (3). By (1, 2), we have $\mathrm{L}\, u \in \operatorname{Iso} \operatorname{Ho} \mathcal{C}$.

If $\mathcal{C}$ is saturated, this implies $u \in \operatorname{Qis} \mathcal{C}$; cf. Definition 65. $\square$

3.5 Hirschhorn replacement

For this §3.5, let $\mathcal{C}$ be a quasi-model-category.

3.5.1 Homotopy in quasi-model-categories

Lemma 110. *Suppose given* $X \overset{f_0}{\underset{f_1}{\rightrightarrows}} Y$. *The following assertions* (1, 2, 3) *are equivalent.*

(1) *We have* $f_0 \sim f_1$.

(2) *We have* $f_0 \overset{l}{\sim} f_1$.

(3) *We have* $f_0 \overset{r}{\sim} f_1$.

This proof is essentially due to Quillen [**12,** Lem. I.1.5].

Proof. We have (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1); cf. Remark 89.

Ad (1) $\Rightarrow$ (3). Suppose that $f_0 \sim f_1$. There exists a commutative diagram as follows.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, "i_0"'] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[d, equal] & \hat{X} \arrow[l, "t"'] \arrow[r, "\hat{f}"] & \hat{Y} \arrow[u, "\wr\, p_0"'] \arrow[d, "p_1"] & Y \arrow[l, "s"'] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f_1"'] \arrow[u, "i_1\, \wr"] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

Cf. Lemma 49.(3′). We have a commutative diagram as follows.

$$\begin{tikzcd}
\hat{Y} \arrow[dr, dotted, "\bar{p}"] \arrow[drr, bend left, "p_0"] \arrow[ddr, bend right, "p_1"'] & & \\
& \bar{Y} \arrow[r, "c_0"] \arrow[d, "c_1"'] \arrow[dr, phantom, "\mathrm{Q}"] & Y \arrow[d] \\
& Y \arrow[r] & !
\end{tikzcd}$$

Cf. $\mathbf{Q}_{\mathrm{Fib}}$.(2), Remark 98.(2) and Lemma 104.(2).

Furthermore, we have a commutative diagram as follows.

$$\begin{tikzcd}
\hat{Y} \arrow[rr, "\bar{p}"] \arrow[dr, "j"'] & & \bar{Y} \\
& Y' \arrow[ur, "q"'] &
\end{tikzcd}$$

Cf. $\mathbf{A}_{\mathrm{Fact}}$.(1). Thus, we have the following commutative diagrams.

$$\begin{tikzcd}
\hat{Y} \arrow[rr, "p_0"] \arrow[dr, "j"'] & & Y \\
& Y' \arrow[ur, "qc_0"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\hat{Y} \arrow[rr, "p_1"] \arrow[dr, "j"'] & & Y \\
& Y' \arrow[ur, "qc_1"'] &
\end{tikzcd}$$

Cf. $\mathbf{A}_{\mathrm{Fib}}$.(2). Therefore, we obtain qc_0, $qc_1 \in \mathrm{Qis}\,\mathcal{C} \cap \mathrm{Fib}\,\mathcal{C}$; cf. $\mathbf{A}_{\mathrm{Qis}}$.(2.b).

Moreover, we have a commutative diagram as follows.

$$\begin{tikzcd}
X \arrow[r, "i_0"] \arrow[drr, bend right, "f_0 s\bar{p}"'] & \hat{X} \arrow[dr, dotted, "v"] \arrow[drr, bend left, "tf_0"] \arrow[ddr, bend right, "\hat{f}p_0"'] & & \\
& & \bar{Y} \arrow[r, "c_0"] \arrow[d, "c_1"'] \arrow[dr, phantom, "\mathrm{Q}"] & Y \arrow[d] \\
& & Y \arrow[r] & !
\end{tikzcd}$$

Cf. Definition 97.(8). Additionally, we have the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, "f_0 sj"] \arrow[d, "i_0"'] & Y' \arrow[d, "q"] \\
\hat{X} \arrow[r, "v"'] \arrow[ur, dotted, "w"] & B'
\end{tikzcd}$$

Cf. $\mathbf{A}_{\mathrm{Lift}}$.(1). Since the following diagram commutes, we have $f_0 \overset{r}{\sim} i_1 \hat{f} p_0$.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, equal] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "i_1 w"] \arrow[d, equal] & Y' \arrow[u, "qc_0"'] \arrow[d, "qc_1"] & Y \arrow[l, "sj"'] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "i_1 \hat{f} p_0"'] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

Since the following diagram commutes, we have $i_1 \hat{f} p_0 \overset{r}{\sim} f_1$.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "i_1 \hat{f} p_0"] \arrow[d, equal] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "i_1 \hat{f}"] \arrow[d, equal] & \hat{Y} \arrow[u, "\wr p_0"'] \arrow[d, "p_1"] & Y \arrow[l, "s"'] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f_1"'] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

Altogether, we have $f_0 \overset{r}{\sim} i_1 \hat{f} p_0 \overset{r}{\sim} f_1$. Thus, we have $f_0 \overset{r}{\sim} f_1$; cf. Lemma 91.(2).

Ad (1) $\Rightarrow$ (2). This is dual to (1) $\Rightarrow$ (3). $\square$

Lemma 111. *Suppose given* $X \underset{f_1}{\overset{f_0}{\rightrightarrows}} Y$ *with* $f_0 \sim f_1$ *. The following assertions* (1, 2) *hold.*

(1) *Suppose given a commutative diagram as follows.*

$$\begin{tikzcd}
¡ \arrow[r] \arrow[d] & X \arrow[d, "\iota_1"] \\
X \arrow[r, "\iota_0"'] \arrow[ur, phantom, "\mathrm{Q}"] & X'
\end{tikzcd}$$

Cf. $\mathbf{Q}_{\mathrm{Cof}}$.(1,2) and Remark 98.(1).

There exists $X' \xrightarrow{j} \tilde{X}$ *and a commutative diagram as follows.*

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, "\iota_0 j"'] & Y \arrow[r, equal] \arrow[d, equal] & Y \arrow[d, equal] \\
X \arrow[d, equal] & \tilde{X} \arrow[l, "t"'] \arrow[r, "\tilde{f}"] & Y \arrow[r, equal] \arrow[d, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f_1"'] \arrow[u, "\iota_1 j"] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

(2) *Suppose given a commutative diagram as follows.*

$$\begin{tikzcd}
Y' \arrow[r, "\pi_0"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\mathrm{Q}"] & Y \arrow[d] \\
Y \arrow[r] & !
\end{tikzcd}$$

Cf. $\mathbf{Q}_{\mathrm{Fib}}$.(1,2) and Remark 98.(2).

There exist $\tilde{Y} \xrightarrow{q} Y'$ and a commutative diagram as follows.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, equal] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "\tilde{f}"] \arrow[d, equal] & \tilde{Y} \arrow[u, "q\pi_0"'] \arrow[d, "q\pi_1"] & Y \arrow[l, "s"'] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f_1"'] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

Proof. *Ad* (2). By Lemmas 110 and 90.(2), there exists a commutative diagram as follows.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, equal] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "\hat{f}"] \arrow[d, equal] & \hat{Y} \arrow[u, "p_0"'] \arrow[d, "\wr\, p_1"] & Y \arrow[l, "s"'] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f_1"'] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

By Lemma 105.(1), there exists a diagram

$$\begin{tikzcd}
X \arrow[rr, "\hat{f}"] \arrow[dr, "\bar{f}"'] & & \hat{Y} \arrow[dl, bend left, "v"] \\
& \bar{Y} \arrow[ur, "u"] &
\end{tikzcd}$$

such that $\bar{f}u = \hat{f}$ and $vu = 1_{\hat{Y}}$. There exists a commutative diagram as follows.

$$\begin{tikzcd}
\bar{Y} \arrow[dr, dotted, "p"] \arrow[drr, bend left, "up_0"] \arrow[ddr, bend right, "up_1"'] & & \\
& Y' \arrow[r, "\pi_0"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\mathrm{Q}"] & Y \arrow[d] \\
& Y \arrow[r] & !
\end{tikzcd}$$

Cf. Lemma 104.(2). By $\mathbf{A}_{\mathrm{Fact}}$.(1), there exists a commutative diagram as follows.

$$\begin{tikzcd}
\bar{Y} \arrow[rr, "p"] \arrow[dr, "j"'] & & Y' \\
& \tilde{Y} \arrow[ur, "q"'] &
\end{tikzcd}$$

In consequence, we have a commutative diagram as follows.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] \arrow[d, equal] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "\bar{f}j"] \arrow[d, equal] & \tilde{Y} \arrow[u, "q\pi_0"'] \arrow[d, "q\pi_1"] & Y \arrow[l, "svj"'] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, "f_1"'] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

Cf. $\mathbf{A}_{\mathrm{Fib}}$.(1), $\mathbf{A}_{\mathrm{Cof}}$.(1) and $\mathbf{A}_{\mathrm{Qis}}$.

Ad (1). This is dual to (2). ☐

3.5.2 The replacement lemma

Proposition 112 (Hirschhorn replacement lemma, [**9,** Cor. 7.3.12])**.**
The following assertions (1, 2) *hold.*

(1) *Suppose given*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & X \\
B \arrow[ur, "g"'] &
\end{tikzcd}$$

such that $ig \sim f$*. There exists* $B \xrightarrow{g'} X$ *with* $g \sim g'$ *and* $ig' = f$*.*

(2) *Suppose given*

$$\begin{tikzcd}
 & X \arrow[d, "p"] \\
A \arrow[ur, "f"] \arrow[r, "g"'] & Y
\end{tikzcd}$$

such that $fp \sim g$*. There exists* $A \xrightarrow{f'} X$ *with* $f \sim f'$ *and* $f'p = g$*.*

Proof. *Ad* (1). Since $ig \sim f$, there exists a commutative diagram as follows.

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] & A \arrow[r, "i"] \arrow[d, equal] & B \arrow[r, "g"] & X \arrow[r, equal] & X \arrow[d, equal] \\
A \arrow[r, equal] \arrow[d, equal] & A \arrow[rr, "h"] \arrow[d, equal] & & \hat{X} \arrow[u, "p"'] \arrow[d, "\approx q"] & X \arrow[l, "s"', "\approx"] \arrow[d, equal] \\
A \arrow[r, equal] & A \arrow[rr, "f"'] & & X \arrow[r, equal] & X
\end{tikzcd}$$

Cf. Lemma 110 and Lemma 90.(2).

Furthermore, we have a commutative diagram as follows.

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "i"'] & \hat{X} \arrow[d, "p"] \\
B \arrow[r, "g"'] \arrow[ur, dotted, "m"] & X
\end{tikzcd}$$

Cf. $\mathbf{A}_{\mathrm{Lift}}$.(2). Let $B \xrightarrow{g' := mq} X$. Then we have $ig' = f$.

Since the following diagram commutes, we have $g \sim g'$; cf. Lemma 110.

$$\begin{tikzcd}
B \arrow[r, equal] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, equal] & X \arrow[r, equal] & X \arrow[d, equal] \\
B \arrow[r, equal] \arrow[d, equal] & B \arrow[r, "m"] \arrow[d, equal] & \hat{X} \arrow[u, "p"'] \arrow[d, "\approx q"] & Y \arrow[l, "s"', "\approx"] \arrow[d, equal] \\
B \arrow[r, equal] & B \arrow[r, "g'"'] & X \arrow[r, equal] & X
\end{tikzcd}$$

Ad (2). This is dual to (1). $\square$

Corollary 113. *Suppose given*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] \arrow[ur, phantom, "\mathrm{Q}"] & B'.
\end{tikzcd}$$

The following assertion $(*)$ *holds.*

$(*)$ *Suppose given* $A' \xrightarrow{x} T \xleftarrow{y} B$ *with* $fy \sim ax$.
Then there exists $B' \xrightarrow{u} T$ *with* $f'u \sim x$ *and* $bu = y$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \arrow[ddr, bend left, "y"] & \\
A' \arrow[r, "f'"'] \arrow[ur, phantom, "\mathrm{Q}"] \arrow[drr, bend right, "x"'] & B' \arrow[dr, dotted, "u"] & \\
& & T
\end{tikzcd}$$

In particular,

$$\begin{tikzcd}
A \arrow[r, "{[f]}"] \arrow[d, "{[a]}"'] & B \arrow[d, "{[b]}"] \\
A' \arrow[r, "{[f']}"'] & B'
\end{tikzcd}$$

is a weak pushout in $\mathrm{Ho}\,\mathcal{C}$.

Proof. There exists $A' \xrightarrow{x'} T$ with $x \sim x'$ and $ax' = fy$; cf. Proposition 112.(1). Moreover, there exists a commutative diagram as follows.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b"] \arrow[ddr, bend left, "y"] & \\
A' \arrow[r, "f'"'] \arrow[drr, bend right, "x'"'] & B' \arrow[dr, dotted, "u"] & \\
& & T
\end{tikzcd}$$

Cf. Lemma 104.(1). Additionally, we have $f'u = x' \sim x$. □

Corollary 114. *Suppose given*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

The following assertion $(*)$ *holds.*

$(*)$ *Suppose given* $A' \xleftarrow{x} T \xrightarrow{y} B$ *with* $yb \sim xf'$.
Then there exists $T \xrightarrow{u} A$ *with* $uf \sim y$ *and* $ua = x$.

$$\begin{tikzcd}
T \arrow[dr, dotted, "u"] \arrow[drr, bend left, "y"] \arrow[ddr, bend right, "x"'] & & \\
& A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b"] \\
& A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

In particular,

$$\begin{tikzcd}
A \arrow[r, "{[f]}"] \arrow[d, "{[a]}"'] & B \arrow[d, "{[b]}"] \\
A' \arrow[r, "{[f']}"'] & B'
\end{tikzcd}$$

is a weak pullback in $\mathrm{Ho}\,\mathcal{C}$.

Proof. This is dual to Corollary 113. □

3.6 The suspension functor

In this §3.6, we establish the suspension functor $\Sigma_{\mathcal{C}} : \mathrm{Ho}\,\mathcal{C} \to \mathrm{Ho}\,\mathcal{C}$ on the homotopy category of a quasi-model-category $\mathcal{C}$; cf. Definition 141 below.

3.6.1 Brown-Heath-Kamps-Gunnarsson gluing

3.6.1.1 H-pushouts

3.6.1.1.1 The general case

For this §3.6.1.1.1, let $\mathcal{C}$ be a quasi-model-category.

Definition 115. A commutative quadrangle

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"', "\bullet" marking] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

is called an *H-pushout*, if the following assertion $(*)$ holds.

$(*)$ Suppose given a commutatitive diagram as follows.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dd, "a"', "\bullet" marking] & & B \arrow[dl, "\hat{b}", "\bullet" marking] \arrow[dd, "b"] \\ & \hat{B} \arrow[dr, "u"] \arrow[ul, phantom, "\mathrm{Q}"] & \\ A' \arrow[ur, "\hat{f}"'] \arrow[rr, "f'"'] & & B' \end{tikzcd}$$

Then we have $\mathrm{L}_{\mathcal{C}}\, u \in \mathrm{Iso}\,\mathrm{Ho}\,\mathcal{C}$.

To indicate that (A, B, A', B') is an H-pushout, we often write

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"', "\bullet" marking] \arrow[dr, phantom, "\mathrm{H}"] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B'. \end{tikzcd}$$

Remark 116. *Suppose given the following commutative quadrangle.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"', "\bullet" marking] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

The following assertions (1, 2) *are equivalent.*

(1) *The commutative quadrangle* (A, B, A', B') *is an H-pushout;* cf. Definition 115.

(2) *There exists a commutative diagram*

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dd, "a"'] & & B \arrow[dl, "\hat{b}"] \arrow[dd, "b"] \\
& {}^{\mathrm{Q}}\hat{B} \arrow[dr, "u"] & \\
A' \arrow[ur, "\hat{f}"'] \arrow[rr, "f'"'] & & B'
\end{tikzcd}$$

with $\mathrm{L}_{\mathcal{C}}\, u \in \mathrm{Iso}\,\mathrm{Ho}\,\mathcal{C}$.

Proof. *Ad* (1) ⇒ (2). We have a commutative diagram as follows.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dd, "a"'] & & B \arrow[dl, "\hat{b}"] \arrow[dd, "b"] \\
& {}^{\mathrm{Q}}\hat{B} \arrow[dr, dotted, "u"] & \\
A' \arrow[ur, "\hat{f}"'] \arrow[rr, "f'"'] & & B'
\end{tikzcd}$$

Cf. $\mathbf{Q}_{\mathrm{Cof}}$.(2) and Lemma 104.(1). Since (1) holds, we have $\mathrm{L}\, u \in \mathrm{Iso}\,\mathrm{Ho}\,\mathcal{C}$.

Ad (2) ⇒ (1). Suppose given the following commutative diagram.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dd, "a"'] & & B \arrow[dl, "\tilde{b}"] \arrow[dd, "b"] \\
& {}^{\mathrm{Q}}\tilde{B} \arrow[dr, "\tilde{u}"] & \\
A' \arrow[ur, "\tilde{f}"'] \arrow[rr, "f'"'] & & B'
\end{tikzcd}$$

We have to show that $\mathrm{L}\,\tilde{u} \in \mathrm{Iso}\,\mathrm{Ho}\,\mathcal{C}$; cf. Definition 115.

By Lemma 109, we have a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "\hat{b}"] \arrow[ddr, bend left, "\tilde{b}"] & \\
A' \arrow[r, "\hat{f}"'] \arrow[drr, bend right, "\tilde{f}"'] & {}^{\mathrm{Q}}\hat{B} \arrow[dr, "v"] & \\
& & {}^{\mathrm{Q}}\tilde{B}
\end{tikzcd}$$

with $\mathrm{L}\, v \in \mathrm{Iso}\,\mathrm{Ho}\,\mathcal{C}$.

Since we have $\hat{f}\cdot u = f' = \hat{f}\cdot v\tilde{u}$ and $\hat{b}\cdot u = b = \hat{b}\cdot v\tilde{u}$, we obtain $u \sim v\tilde{u}$; cf. Definition 96.(5).

Thus, we have $\mathrm{L}\, u = \mathrm{L}(v\tilde{u}) = \mathrm{L}\, v \cdot \mathrm{L}\,\tilde{u}$ with $\mathrm{L}\, u, \mathrm{L}\, v \in \mathrm{Iso}\,\mathrm{Ho}\,\mathcal{C}$. Therefore, $\mathrm{L}\,\tilde{u} \in \mathrm{Iso}\,\mathrm{Ho}\,\mathcal{C}$. □

Remark 117. *Suppose given a quasi-pushout* (A, B, A', B').
Then (A, B, A', B') *is an H-pushout.*

Proof. This follows from Remark 116 using $u = 1_{B'}$. ☐

Remark 118. *Suppose given*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{H}"] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B'. \end{tikzcd}$$

Then we have $\mathrm{L}_{\mathcal{C}}\, b \in \mathrm{Iso}\,\mathrm{Ho}\,\mathcal{C}$.

Proof. There exists a commutative diagram

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dd, "a"'] & & B \arrow[dl, "\hat{b}"] \arrow[dd, "b"] \\ & \hat{B} \arrow[dr, "u"] \arrow[ul, phantom, "\mathrm{Q}"] & \\ A' \arrow[ur, "\hat{f}"'] \arrow[rr, "f'"'] & & B' \end{tikzcd}$$

where $\mathrm{L}\, u \in \mathrm{Iso}\,\mathrm{Ho}\,\mathcal{C}$; cf. Remark 116 and Definition 96.(2). Thus, we have $\mathrm{L}\, b \in \mathrm{Iso}\,\mathrm{Ho}\,\mathcal{C}$. ☐

Remark 119. *Suppose given the following commutative diagram.*

$$\begin{tikzcd} A \arrow[r, "f", "\approx"'] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B'. \end{tikzcd}$$

The following assertions (1, 2) *are equivalent.*

(1) *We have* $\mathrm{L}_{\mathcal{C}}\, f' \in \mathrm{Iso}\,\mathrm{Ho}\,\mathcal{C}$.

(2) *The commutative quadrangle* (A, B, A', B') *is an H-pushout.*

Proof. There exists a commutative diagram as follows.

$$\begin{tikzcd} A \arrow[rr, "f", "\approx"'] \arrow[dd, "a"'] & & B \arrow[dl, "\hat{b}"] \arrow[dd, "b"] \\ & \hat{B} \arrow[dr, dotted, "u"] \arrow[ul, phantom, "\mathrm{Q}"] & \\ A' \arrow[ur, "\approx", "\hat{f}"'] \arrow[rr, "f'"'] & & B' \end{tikzcd}$$

Cf. $\mathbf{Q}_{\mathrm{Cof}}$.(2), Definition 96.(3) and Lemma 104.(1). In particular, we have $L\hat{f} \in \operatorname{Iso}\operatorname{Ho}\mathcal{C}$; cf. Corollary 60.

Ad (1) $\Rightarrow$ (2). Since $\mathrm{L}\, f'$ is in $\operatorname{Iso}\operatorname{Ho}\mathcal{C}$, so is $\mathrm{L}\, u$. Thus, (A, B, A', B') is an H-pushout; cf. Remark 116.

Ad (2) $\Rightarrow$ (1). Since (A, B, A', B') is an H-pushout, we have $\mathrm{L}\, u \in \operatorname{Iso}\operatorname{Ho}\mathcal{C}$. In consequence, we have $\mathrm{L}\, f' \in \operatorname{Iso}\operatorname{Ho}\mathcal{C}$. □

Lemma 120. *Suppose given the following commutative diagram.*

$$\begin{tikzcd}
A \arrow[r, "f", "\bullet" marking] \arrow[d, "a_0"', "\bullet" marking] \arrow[dd, bend right=60, "a"', "\bullet" marking] \arrow[dr, phantom, "\mathrm{H}" near end] & B \arrow[d, "b_0"] \arrow[dd, bend left=60, "b", "\mathrm{H}"' near end] \\
A' \arrow[r, "f'"'] \arrow[d, "a_1 \wr"'] & B' \arrow[d, "b_1"'] \\
A'' \arrow[r, "f''"'] & B''
\end{tikzcd}$$

Then we have $\mathrm{L}_{\mathcal{C}}\, b_1 \in \operatorname{Iso}\operatorname{Ho}\mathcal{C}$.

Proof. We have a commutative diagram

$$\begin{tikzcd}
A \arrow[rr, "f", "\bullet" marking] \arrow[dd, "a_0"', "\bullet" marking] \arrow[dr, phantom, "\mathrm{Q}" near end] & & B \arrow[dl, "\hat{b}", "\bullet" marking] \arrow[dd, "b_0"] \\
& \hat{B} \arrow[dr, "u"] \arrow[dd, "\wr\, \tilde{b}" near end] & \\
A' \arrow[ur, "\hat{f}"', "\bullet" marking] \arrow[rr, "f'"' near start] \arrow[dd, "a_1 \wr"'] \arrow[dr, phantom, "\mathrm{Q}" near end] & & B' \arrow[dd, "b_1"] \\
& \tilde{B} \arrow[dr, dotted, "v"] & \\
A'' \arrow[ur, "\tilde{f}"', "\bullet" marking] \arrow[rr, "f''"'] & & B''
\end{tikzcd}$$

where $\mathrm{L}\, u \in \operatorname{Iso}\operatorname{Ho}\mathcal{C}$; cf. Remarks 116 and 98.(1), $\mathbf{Q}_{\mathrm{Cof}}$.(2), Definition 96.(3) and Lemma 104.(1).

Thus, we have an H-pushout (A, B, A'', B'') and a quasi-pushout $(A, B, A'', \tilde{B})$; cf. Definition 96.(7). Therefore, we obtain $\mathrm{L}\, v \in \operatorname{Iso}\operatorname{Ho}\mathcal{C}$; cf. Definition 115.

In consequence, we have $\mathrm{L}\, b_1 \in \operatorname{Iso}\operatorname{Ho}\mathcal{C}$; cf. also Corollary 60. □

Remark 121. *Suppose given*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"', "\bullet" marking] \arrow[dr, phantom, "\mathrm{H}" near end] & B \arrow[r, "g", "\approx" marking] \arrow[d, "b", "\bullet" marking] \arrow[dr, phantom, "\mathrm{H}" near end] & C \arrow[d, "c"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C'.
\end{tikzcd}$$

Then (A, C, A', C') is an H-pushout.

Proof. There exists a commutative diagram

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dd, "a"'] & & B \arrow[rr, "g", "\approx"'] \arrow[dl, "\hat{b}"] \arrow[dd, "b"] & & C \arrow[dl, "\hat{c}"'] \arrow[dd, "c"] \\
& {}^{\mathrm{Q}}\hat{B} \arrow[rr, "\approx", "\hat{g}"'] \arrow[dr, "u"] & & {}^{\mathrm{Q}}\hat{C} \arrow[dr, dotted, "v"] & \\
A' \arrow[ur, "\hat{f}"] \arrow[rr, "f'"'] & & B' \arrow[rr, "g'"'] & & C'
\end{tikzcd}$$

where $\mathrm{L}\, u \in \operatorname{Iso}\operatorname{Ho}\mathcal{C}$; cf. Remark 116, $\mathbf{Q}_{\mathrm{Cof}}$.(2), Definition 96.(3) and Lemma 104.(1).

Since $(A, C, A', \hat{C})$ is a quasi-pushout, it suffices to show that $\mathrm{L}\, v \in \operatorname{Iso}\operatorname{Ho}\mathcal{C}$; cf. Definition 96.(6) and Remark 116.

We have $\mathrm{L}\, u \cdot \mathrm{L}\, g' = \mathrm{L}\, \hat{g} \cdot \mathrm{L}\, v$ with $\mathrm{L}\, u$, $\mathrm{L}\, g'$, $\mathrm{L}\, \hat{g} \in \operatorname{Iso}\operatorname{Ho}\mathcal{C}$; cf. Corollary 60 and Remark 119.

Thus, we obtain $\mathrm{L}\, v \in \operatorname{Iso}\operatorname{Ho}\mathcal{C}$. □

Remark 122. *Suppose given the following commutative diagram.*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[rr, bend left, "h"] & B \arrow[r, "g", "\approx"'] \arrow[d, "b"] \arrow[dl, phantom, "\mathrm{H}"] & C \arrow[d, "c"] \\
A' \arrow[r, "f'"'] \arrow[rr, bend right, "h'"'] & B' \arrow[r, "g'", "\mathrm{H}"'] & C'
\end{tikzcd}$$

Then (B, C, B', C') is an H-pushout.

Proof. There exists a commutative diagram

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dd, "a"'] & & B \arrow[rr, "g", "\approx"'] \arrow[dl, "\hat{b}"] \arrow[dd, "b"] & & C \arrow[dl, "\hat{c}"'] \arrow[dd, "c"] \\
& {}^{\mathrm{Q}}\hat{B} \arrow[rr, "\approx", "\hat{g}"'] \arrow[dr, "u"] & & {}^{\mathrm{Q}}\hat{C} \arrow[dr, dotted, "v"] & \\
A' \arrow[ur, "\hat{f}"] \arrow[rr, "f'"'] & & B' \arrow[rr, "g'"'] & & C'
\end{tikzcd}$$

where $\mathrm{L}\, u \in \operatorname{Iso}\operatorname{Ho}\mathcal{C}$; cf. Remark 116, $\mathbf{Q}_{\mathrm{Cof}}$.(2), Definition 96.(3) and Lemma 104.(1).

By Remark 119, it suffices to show that $\mathrm{L}\, g' \in \operatorname{Iso}\operatorname{Ho}\mathcal{C}$. Thus, it suffices to show that $\mathrm{L}\, v \in \operatorname{Iso}\operatorname{Ho}\mathcal{C}$; cf. also Corollary 60.

Since we have an H-pushout (A, C, A', C') and a quasi-pushout $(A, C, A', \hat{C})$, we have $\mathrm{L}\, v \in \operatorname{Iso}\operatorname{Ho}\mathcal{C}$; cf. Definition 115 and Definition 96.(6). □

Lemma 123. *Suppose given*

$$\begin{tikzcd}
A \arrow[r, dashed, "f", "\approx"'] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[d, "c"] \\
A' \arrow[r, dashed, "\approx", "f'"'] & B' \arrow[r, "g'"'] \arrow[ur, phantom, "\mathrm{H}"] & C'.
\end{tikzcd}$$

Then (A, C, A', C') is an H-pushout.

Proof. There exists a commutative diagram as follows

$$\begin{tikzcd}
A \arrow[rr, "f", "\approx"'] \arrow[ddd, "a"'] & & B \arrow[r, "i"] \arrow[rrr, bend left, "g"] \arrow[ddl, "\hat{b}"] \arrow[ddd, "b"] & X \arrow[rr, "p"] \arrow[d, "\hat{x}"] \arrow[ddr, "\bar{x}"] & & C \arrow[ddd, "c"] \\
 & & & \overset{\mathrm{Q}}{\hat{X}} \arrow[dr, dotted, "v"', "\mathrm{Q}"] & & \\
 & \overset{\mathrm{Q}}{\hat{B}} \arrow[urr, "\hat{i}"'] \arrow[dr, dotted, "u", "\approx"'] & & & \bar{X} \arrow[dr, dotted, "w"] & \\
A' \arrow[ur, "\approx", "\hat{f}"'] \arrow[rr, "\approx", "f'"'] & & B' \arrow[urr, "\bar{i}"'] \arrow[rrr, "g'"'] & & & C'
\end{tikzcd}$$

Cf. $\mathbf{Q}_{\text{Cof}}$.(2), Definition 96.(3), Lemma 104.(1), $\mathbf{A}_{\text{Fact}}$.(2), Remark 98.(1) and $\mathbf{A}_{\text{Qis}}$.(2.b).

Since $(A, X, A', \hat{X})$ is an H-pushout, it suffices to show that $(X, C, \hat{X}, C')$ is an H-pushout; cf. Definition 96.(6), Remark 117 and Remark 121.

Thus, it suffices to show that $\mathrm{L}(vw) \in \operatorname{Iso} \operatorname{Ho} \mathcal{C}$; cf. Remark 119.

We show that $\mathrm{L}\, v \in \operatorname{Iso} \operatorname{Ho} \mathcal{C}$ and $\mathrm{L}\, w \in \operatorname{Iso} \operatorname{Ho} \mathcal{C}$.

Since $(B, X, B', \bar{X})$ and $(B, X, \hat{B}, \hat{X})$ are H-pushouts, we have $\mathrm{L}\, v \in \operatorname{Iso} \operatorname{Ho} \mathcal{C}$; cf. Remark 117 and Lemma 120.

Since (B, C, B', C') and $(B, X, B', \bar{X})$ are H-pushouts, so is $(X, C, \bar{X}, C')$; cf. Remarks 117 and 122. Thus, $\mathrm{L}\, w \in \operatorname{Iso} \operatorname{Ho} \mathcal{C}$; cf. Remark 119. □

Lemma 124. *Suppose given the following commutative diagram.*

$$\begin{tikzcd}
A \arrow[r, dashed, "f", "\approx"'] \arrow[d, "a"'] \arrow[rr, bend left, "h"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[d, "c"] \\
A' \arrow[r, dashed, "\approx", "f'"'] \arrow[rr, bend right, "h'"'] & B' \arrow[r, "g'", "\mathrm{H}"'] & C'
\end{tikzcd}$$

Then (B, C, B', C') is an H-pushout.

Proof. There exists a commutative diagram as follows.

$$\begin{tikzcd}
A \arrow[r, "f", "\approx"'] \arrow[dd, "a"'] & B \arrow[r, "i"] \arrow[dd, "b"] \arrow[rr, bend left, "g"] & X \arrow[r, "p"] \arrow[d, "\hat{x}"] & C \arrow[dd, "c"] \\
& & {}^{\mathrm{Q}}\hat{X} \arrow[dr, dotted, "v"] & \\
A' \arrow[r, "\approx", "f'"'] & B' \arrow[ur, "\hat{i}"'] \arrow[rr, "g'"'] & & C'
\end{tikzcd}$$

Cf. $\mathbf{A}_{\mathrm{Fact}}$.(2), $\mathbf{Q}_{\mathrm{Cof}}$.(2), Remark 98.(1) and Lemma 104.(1).

Since $(B, X, B', \hat{X})$ is an H-pushout, so is $(A, X, A', \hat{X})$; cf. Remark 117 and Lemma 123.

Since (A, C, A', C') and $(A, X, A', \hat{X})$ are H-pushouts, so is $(X, C, \hat{X}, C')$; cf. Remark 122.

Since $(B, X, B', \hat{X})$ and $(X, C, \hat{X}, C')$ are H-pushouts, so is (B, C, B', C'); cf. Remark 121.

□

3.6.1.1.2 The saturated case

For this §3.6.1.1.2, let $\mathcal{C}$ be a saturated quasi-model-category.

Remark 125. *Suppose given the following commutative quadrangle.*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

The following assertions (1, 2, 3) *are equivalent.*

(1) *The commutative quadrangle (A, B, A', B') is an H-pushout;* cf. Definition 115.

(2) *Suppose given a commutative diagram as follows.*

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dd, "a"'] & & B \arrow[dl, "\hat{b}"] \arrow[dd, "b"] \\
& {}^{\mathrm{Q}}\hat{B} \arrow[dr, "u"] & \\
A' \arrow[ur, "\hat{f}"'] \arrow[rr, "f'"'] & & B'
\end{tikzcd}$$

Then we have $u \in \operatorname{Qis} \mathcal{C}$.

(3) *There exists a commutative diagram as follows.*

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dd, "a"'] & & B \arrow[dl, "\hat{b}"] \arrow[dd, "b"] \\
& {}^{\mathrm{Q}}\hat{B} \arrow[dr, "u", "\approx"'] & \\
A' \arrow[ur, "\hat{f}"'] \arrow[rr, "f'"'] & & B'
\end{tikzcd}$$

Proof. *Ad* (1) ⇒ (2). Since (A, B, A', B') is an H-pushout, we have $\mathrm{L}\, u \in \operatorname{Iso} \operatorname{Ho} \mathcal{C}$. Thus, we have $u \in \operatorname{Qis} \mathcal{C}$; cf. Definition 65.

Ad (2) ⇒ (3). We have a commutative diagram as follows.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dd, "a"'] & & B \arrow[dl, "\hat{b}"] \arrow[dd, "b"] \\
& {}^{\mathrm{Q}}\hat{B} \arrow[dr, dotted, "u"] & \\
A' \arrow[ur, "\hat{f}"] \arrow[rr, "f'"'] & & B'
\end{tikzcd}$$

Cf. $\mathbf{Q}_{\mathrm{Cof}}$.(2) and Lemma 104.(1). Since (2) holds, we have $u \in \operatorname{Qis} \mathcal{C}$.

Ad (3) ⇒ (1). By Corollary 60, we have $\mathrm{L}\, u \in \operatorname{Iso} \operatorname{Ho} \mathcal{C}$.

Thus, (A, B, A', B') is an H-pushout; cf. Remark 116. $\square$

Lemma 126. *Suppose given*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{H}"] & B \arrow[r, "g"] \arrow[d, "b"] \arrow[dr, phantom, "\mathrm{H}"] & C \arrow[d, "c"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C'.
\end{tikzcd}$$

Then (A, C, A', C') is an H-pushout.

Proof. There exists a commutative diagram as follows.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[ddd, "a"'] & & B \arrow[r, "i"] \arrow[rrr, bend left, "g"] \arrow[ddl, "\hat{b}"] \arrow[ddd, "b"] & X \arrow[rr, "p"] \arrow[d, "\hat{x}"'] \arrow[ddr, "\bar{x}"] & & C \arrow[ddd, "c"] \\
& & & {}^{\mathrm{Q}}\hat{X} \arrow[dr, dotted, "v"'] \arrow[dr, phantom, "\mathrm{Q}"] & & \\
& {}^{\mathrm{Q}}\hat{B} \arrow[urr, "\hat{i}"'] \arrow[dr, "u", "\sim"'] & & & \bar{X} \arrow[dr, dotted, "w"] & \\
A' \arrow[ur, "\hat{f}"] \arrow[rr, "f'"'] & & B' \arrow[urr, "\bar{i}"'] \arrow[rrr, "g'"'] & & & C'
\end{tikzcd}$$

Cf. Remark 125, $\mathbf{A}_{\mathrm{Fact}}$.(2), $\mathbf{Q}_{\mathrm{Cof}}$.(2), Remark 98.(1) and Lemma 104.(1).

Since $(A, X, A', \hat{X})$ is an H-pushout, it suffices to show that $(X, C, \hat{X}, C')$ is an H-pushout; cf. Definition 96.(6), Remark 117 and Remark 121.

Thus, it suffices to show that $\mathrm{L}(vw) \in \operatorname{Iso} \operatorname{Ho} \mathcal{C}$; cf. Remark 119.

We show that $\mathrm{L}\, v \in \operatorname{Iso} \operatorname{Ho} \mathcal{C}$ and $\mathrm{L}\, w \in \operatorname{Iso} \operatorname{Ho} \mathcal{C}$.

Since $(B, X, B', \bar{X})$ and $(B, X, \hat{B}, \hat{X})$ are H-pushouts, we have $\mathrm{L}\, v \in \operatorname{Iso} \operatorname{Ho} \mathcal{C}$; cf. Remark 117 and Lemma 120.

Since (B, C, B', C') and $(B, X, B', \bar{X})$ are H-pushouts, so is $(X, C, \bar{X}, C')$; cf. Remarks 117 and 122. Thus, $\mathrm{L}\, w \in \operatorname{Iso} \operatorname{Ho} \mathcal{C}$; cf. Remark 119. $\square$

Lemma 127. *Suppose given the following commutative diagram.*

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend left, "h"] \arrow[d, "a"', "\bullet" marking] & B \arrow[r, "g"] \arrow[d, "b", "\bullet" marking] & C \arrow[d, "c"] \\
A' \arrow[r, "f'"'] \arrow[rr, bend right, "h'"'] \arrow[ur, phantom, "\mathrm{H}"] & B' \arrow[r, "g'"] \arrow[r, phantom, shift right=3, "\mathrm{H}"] & C'
\end{tikzcd}
$$

Then (B, C, B', C') *is an H-pushout.*

Proof. There exists a commutative diagram as follows.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dd, "a"', "\bullet" marking] & B \arrow[r, "i", "\bullet" marking] \arrow[rr, bend left, "g"] \arrow[dd, "b", "\bullet" marking] & X \arrow[r, "p"] \arrow[d, "\hat{x}", "\bullet" marking] & C \arrow[dd, "c"] \\
 & & \hat{X} \arrow[dr, dotted, "v"] \arrow[ul, phantom, "\mathrm{Q}"] & \\
A' \arrow[r, "f'"'] \arrow[uur, phantom, "\mathrm{H}"] & B' \arrow[ur, "\hat{i}"', "\bullet" marking] \arrow[rr, "g'"'] & & C'
\end{tikzcd}
$$

Cf. $\mathbf{A}_{\mathrm{Fact}}$.(2), $\mathbf{Q}_{\mathrm{Cof}}$.(2), Remark 98.(1) and Lemma 104.(1).

Since (A, B, A', B') and $(B, X, B', \hat{X})$ are H-pushouts, so is $(A, X, A', \hat{X})$; cf. Remark 117 and Lemma 126.

Since (A, C, A', C') and $(A, X, A', \hat{X})$ are H-pushouts, so is $(X, C, \hat{X}, C')$; cf. Remark 122.

Since $(B, X, B', \hat{X})$ and $(X, C, \hat{X}, C')$ are H-pushouts, so is (B, C, B', C'); cf. Remark 121. □

Lemma 128. *Suppose given*

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a_0"', "\bullet" marking] & B \arrow[d, "b_0"] \\
A' \arrow[r, "f'"] \arrow[d, "a_1"', "\bullet" marking] \arrow[ur, phantom, "\mathrm{H}"] & B' \arrow[d, "b_1"] \\
A'' \arrow[r, "f''"'] \arrow[ur, phantom, "\mathrm{H}"] & B''.
\end{tikzcd}
$$

Then (A, B, A'', B'') *is an H-pushout.*

Proof. We have a commutative diagram as follows.

$$
\begin{tikzcd}
A \arrow[rrr, "f"] \arrow[dd, "a_0"'] & & & B \arrow[dl, "\hat b"] \arrow[dd, "b_0"] \\
& {}^{\mathrm{Q}}\hat B \arrow[ddr, "u \approx"] \arrow[dd, "\tilde b"] & & \\
A' \arrow[ur, "\hat f"] \arrow[rrr, "f'"'] \arrow[ddd, "a_1"'] & & & B' \arrow[ddl, "\bar b"] \arrow[ddd, "b_1"] \\
& {}^{\mathrm{Q}}\tilde B \arrow[dr, "v \approx"'] & & \\
& & {}^{\mathrm{Q}}\bar B \arrow[dr, dotted, "w \approx"] & \\
A'' \arrow[uur, "\tilde f"] \arrow[urr, "\bar f"'] \arrow[rrr, "f''"'] & & & B''
\end{tikzcd}
$$

Cf. Remark 125, $\mathbf{Q}_{\mathrm{Cof}}$.(2) and Definition 96.(3). Since $(A', \hat{B}, A'', \tilde{B})$ and $(\hat{B}, B', \tilde{B}, \bar{B})$ are quasi-pushouts, so is $(A', B', A'', \bar{B})$; cf. Definition 96.(6). Since $(A', B', A'', \bar{B})$ is a quasi-pushout and (A', B', A'', B'') is an H-pushout, there exists $w \in \operatorname{Qis} \mathcal{C}$ making the diagram above commutative; cf. Lemma 104.(1) and Remark 125.

Since $(A, B, A'', \tilde{B})$ is a quasi-pushout and $vw \in \operatorname{Qis} \mathcal{C}$, (A, B, A'', B'') is an H-pushout; cf. Definition 96.(7), $\mathbf{A}_{\mathrm{Qis}}$.(2.a) and Remark 125. $\square$

Lemma 129. *Suppose given the following commutative diagram.*

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a_0"'] \arrow[dd, bend right=50, "a"'] & B \arrow[d, "b_0"] \arrow[dd, bend left=50, "b"] \\
A' \arrow[r, "f'"'] \arrow[d, "a_1"'] & {}^{\mathrm{H}}B' \arrow[d, "b_1"'] \\
A'' \arrow[r, "f''"'] & B''
\end{tikzcd}
$$

Then (A', B', A'', B'') is an H-pushout.

Proof. We have a commutative diagram as follows.

$$
\begin{tikzcd}
A \arrow[rrr, "f"] \arrow[dd, "a_0"'] & & & B \arrow[dll, "\hat b"] \arrow[dd, "b_0"] \\
& {}^{\mathrm{Q}}\hat{B} \arrow[drr, "u", "\approx"'] \arrow[dd, "\tilde b"] & & \\
A' \arrow[ur, "\hat f"'] \arrow[rrr, "f'"'] \arrow[ddd, "a_1"'] & & & B' \arrow[ddl, "\bar b"] \arrow[ddd, "b_1"] \\
& {}^{\mathrm{Q}}\tilde{B} \arrow[dr, "v"', "\approx"] & & \\
& & {}^{\mathrm{Q}}\bar{B} \arrow[dr, dotted, "w"] & \\
A'' \arrow[uur, "\tilde f"] \arrow[urr, "\bar f"'] \arrow[rrr, "f''"'] & & & B''
\end{tikzcd}
$$

Cf. Remark 125, $\mathbf{Q}_{\mathrm{Cof}}$.(2) and Definition 96.(3). Lemma 104.(1). Since $(A', \hat{B}, A'', \tilde{B})$ and $(\hat{B}, B', \tilde{B}, \bar{B})$ are quasi-pushouts, so is $(A', B', A'', \bar{B})$; cf. Definition 96.(6). Thus, there exists $w \in \operatorname{Mor} \mathcal{C}$ making the diagram above commutative; cf. Lemma 104.(1).

It suffices to show that $w \in \operatorname{Qis} \mathcal{C}$; cf. Remark 125. Therefore, it suffices to show that $vw \in \operatorname{Qis} \mathcal{C}$; cf. $\mathbf{A}_{\mathrm{Qis}}$.(2.b).

Since $(A, B, B'', \tilde{B}) \in \operatorname{QPO} \mathcal{C}$ and (A, B, A'', B'') is an H-pushout, we have $vw \in \operatorname{Qis} \mathcal{C}$; cf. Definition 96.(7) and Remark 125. □

3.6.1.2 The gluing lemma

> The following proposition is inspired by ideas of R. Brown [**3,** 7.5.7], R. Brown and Heath [**4,** Thm. 1.2], and Kamps [**10,** Satz 8.2]. The author learned about these ideas from Gunnarsson [**8,** Lem. 7.4] and Thomas [**17,** Prop. 3.50]. Cf. also [**7,** Lem. II.8.8]. Moreover, Waldhausen's axiom Weq 2 [**18,** p. 326] is an analogue of these ideas.

For this §3.6.1.2, let $\mathcal{C}$ be quasi-model-category.

Proposition 130. *Suppose given the following commutative diagram.*

$$
\begin{tikzcd}
& A_0' \arrow[rr, "f_0'"] \arrow[dd, "u'", "\approx"'] & & B_0' \arrow[dd, "v'"] \\
A_0 \arrow[ur, "a_0"] \arrow[rr, "f_0"'] \arrow[dd, "u"', "\approx"] \arrow[urrr, phantom, "\mathrm{H}"] & & B_0 \arrow[ur, "b_0"'] \arrow[dd, "v", "\approx"'] & \\
& A_1' \arrow[rr, "f_1'"] & & B_1' \\
A_1 \arrow[ur, "a_1"] \arrow[rr, "f_1"'] \arrow[urrr, phantom, "\mathrm{H}"] & & B_1 \arrow[ur, "b_1"'] &
\end{tikzcd}
$$

Then we have $\mathrm{L}\, v' \in \mathrm{Iso}\,\mathrm{Ho}\,\mathcal{C}$.

Proof. Since (A_1, A_1', B_1, B_1') is an H-pushout, so is (A_0, B_1, A_0', B_1'); cf. Lemma 123.

Since (A_0, B_1, A_0', B_1') and (A_0, B_0, A_0', B_0') are H-pushouts, so is (B_0, B_1, B_0', B_1'); cf. Remark 122.

Thus, $\mathrm{L}\, v' \in \mathrm{Iso}\,\mathrm{Ho}\,\mathcal{C}$; cf. Remark 119. □

3.6.2 Acyclic objects

For this §3.6.2, let $\mathcal{C}$ be a quasi-model-category.

3.6.2.1 Definition and elementary properties

Definition 131. Let $\mathrm{Ac}\,\mathcal{C}$ be the full subcategory of $\mathcal{C}$ with

$$\mathrm{Ob}\,\mathrm{Ac}\,\mathcal{C} := \{ A \in \mathrm{Ob}\,\mathcal{C} :\ A \longrightarrow\, ! \,\}.$$

The elements of $\mathrm{Ob}\,\mathrm{Ac}\,\mathcal{C}$ are called *acyclic* objects.

Remark 132. Let $X \in \mathrm{Ob}\,\mathcal{C}$ be a terminal object. Then $X \in \mathrm{Ob}\,\mathrm{Ac}\,\mathcal{C}$.

Lemma 133. *Suppose given* $A, B \in \mathrm{Ob}\,\mathrm{Ac}\,\mathcal{C}$. *The following assertions* (1, 2, 3) *hold.*

(1) *There exists* $A \xrightarrow[\approx]{d} B$.

(2) *Suppose given* $A \xrightarrow{f} B$. *Then we have* $A \xrightarrow[\approx]{f} B$.

(3) *Suppose given* $A \underset{g}{\overset{f}{\rightrightarrows}} B$. *Then we have* $f \sim g$.

Proof. Ad (1). By Remark 106.(2), there exists a commutative diagram as follows.

$$
\begin{tikzcd}
! \arrow[rr, equal] \arrow[dr, "b"', "\approx"] & & ! \\
& B \arrow[ur] &
\end{tikzcd}
$$

Cf. $\mathbf{A}_{\mathrm{Qis}}$. Thus, we have $A \longrightarrow ! \overset{b}{\underset{\approx}{\longrightarrow}} B$; cf. $\mathbf{A}_{\mathrm{Qis}}$.(2.a).

Ad (2). We have the following commutative diagram.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & ! \\ B \arrow[ur] & \end{tikzcd}$$

Thus, we have $f \in \operatorname{Qis}\mathcal{C}$; cf. $\mathbf{A}_{\mathrm{Qis}}$.(2.c).

Ad (3). Since ! is terminal, we have $ft = gt$ with $B \xrightarrow{t} !$. Thus, we obtain

$$\mathrm{L}\, f \cdot \mathrm{L}\, t = \mathrm{L}(ft) = \mathrm{L}(gt) = \mathrm{L}\, g \cdot \mathrm{L}\, t.$$

Since $\mathrm{L}\, t \in \operatorname{Iso}\operatorname{Ho}\mathcal{C}$, we have $\mathrm{L}\, f = \mathrm{L}\, g$; cf. Corollary 60. Thus, we obtain $f \sim g$. □

Remark 134. *Suppose given $X \in \operatorname{Ob}\mathcal{C}$. There exists $X \xrightarrow{a} A$ with $A \in \operatorname{Ob}\operatorname{Ac}\mathcal{C}$.*

Proof. By $\mathbf{A}_{\mathrm{Fact}}$.(2), there exists a commutative diagram as follows.

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "a"'] & & ! \\ & A \arrow[ur] & \end{tikzcd}$$

□

Remark 135. *Suppose given*

$$\begin{tikzcd} A & X \arrow[l, "a"'] \arrow[r, "b"] \arrow[d, "f"] & B \\ C & Y \arrow[l, "c"] \arrow[r, "d"'] & D \end{tikzcd}$$

with C and D in $\operatorname{Ob}\operatorname{Ac}\mathcal{C}$.

There exist α and β in $\operatorname{Mor}\mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[d, "\alpha"'] & X \arrow[l, "a"'] \arrow[r, "b"] \arrow[d, "f"] & B \arrow[d, "\beta"] \\ C & Y \arrow[l, "c"] \arrow[r, "d"'] & D \end{tikzcd}$$

Proof. There exist commutative diagrams as follows.

$$\begin{tikzcd} X \arrow[r, "fc"] \arrow[d, "a"'] & C \arrow[d] \\ A \arrow[r] \arrow[ur, dashed, "\alpha"] & ! \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "fd"] \arrow[d, "b"'] & D \arrow[d] \\ B \arrow[r] \arrow[ur, dashed, "\beta"] & ! \end{tikzcd}$$

Cf. $\mathbf{A}_{\mathrm{Lift}}$.(2). □

3.6.2.2 The key lemma

Lemma 136. *The following assertions* (1, 2) *hold.*

(1) *Suppose given commutative diagrams*

$$
\begin{tikzcd}
& A \arrow[rr, "\tilde b", "\bullet"{marking}] \arrow[dd, "\alpha"'] \arrow[dr, phantom, "\mathrm{Q}" near end] & & \tilde X \arrow[dd, "f'"] \\
X \arrow[ur, "a", "\bullet"{marking}] \arrow[rr, "b"', "\bullet"{marking}, crossing over] \arrow[dd, "f"'] & & B \arrow[ur, "\tilde a"', "\bullet"{marking}] \arrow[dd, "\beta' \wr"' near start, crossing over] & \\
& C \arrow[rr, "\tilde d"' near start] & & \tilde Y \\
Y \arrow[ur, "c"] \arrow[rr, "d"'] & & D \arrow[ur, "\tilde c"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \arrow[rr, "\tilde b", "\bullet"{marking}] \arrow[dd, "\alpha"'] \arrow[dr, phantom, "\mathrm{Q}" near end] & & \tilde X \arrow[dd, "f''"] \\
X \arrow[ur, "a", "\bullet"{marking}] \arrow[rr, "b"', "\bullet"{marking}, crossing over] \arrow[dd, "f"'] & & B \arrow[ur, "\tilde a"', "\bullet"{marking}] \arrow[dd, "\beta'' \wr"' near start, crossing over] & \\
& C \arrow[rr, "\tilde d"' near start] & & \tilde Y \\
Y \arrow[ur, "c"] \arrow[rr, "d"'] & & D \arrow[ur, "\tilde c"'] &
\end{tikzcd}
$$

with B *and* D *in* $\operatorname{Ob}\operatorname{Ac}\mathcal{C}$. *Then we have* $f' \sim f''$.

(2) *Suppose given commutative diagrams*

$$
\begin{tikzcd}
& A \arrow[rr, "\tilde b", "\bullet"{marking}] \arrow[dd, "\alpha' \wr"' near start] \arrow[dr, phantom, "\mathrm{Q}" near end] & & \tilde X \arrow[dd, "f'"] \\
X \arrow[ur, "a", "\bullet"{marking}] \arrow[rr, "b"', "\bullet"{marking}, crossing over] \arrow[dd, "f"'] & & B \arrow[ur, "\tilde a"', "\bullet"{marking}] \arrow[dd, "\beta' \wr"' near start, crossing over] & \\
& C \arrow[rr, "\tilde d"' near start] & & \tilde Y \\
Y \arrow[ur, "c"] \arrow[rr, "d"'] & & D \arrow[ur, "\tilde c"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \arrow[rr, "\tilde b", "\bullet"{marking}] \arrow[dd, "\alpha'' \wr"' near start] \arrow[dr, phantom, "\mathrm{Q}" near end] & & \tilde X \arrow[dd, "f''"] \\
X \arrow[ur, "a", "\bullet"{marking}] \arrow[rr, "b"', "\bullet"{marking}, crossing over] \arrow[dd, "f"'] & & B \arrow[ur, "\tilde a"', "\bullet"{marking}] \arrow[dd, "\beta'' \wr"' near start, crossing over] & \\
& C \arrow[rr, "\tilde d"' near start] & & \tilde Y \\
Y \arrow[ur, "c"] \arrow[rr, "d"'] & & D \arrow[ur, "\tilde c"'] &
\end{tikzcd}
$$

with A, B, C *and* D *in* $\operatorname{Ob}\operatorname{Ac}\mathcal{C}$. *Then we have* $f' \sim f''$.

Proof. *Ad* (1). There exist commutative diagrams as follows.

$$
\begin{tikzcd}
X \arrow[r, "b", "\bullet"{marking}] \arrow[d, "b"', "\bullet"{marking}] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b_0", "\bullet"{marking}] \arrow[ddr, bend left, "\beta'"] & \\
B \arrow[r, "b_1"', "\bullet"{marking}] \arrow[drr, bend right, "\beta''"'] & \bar X \arrow[dr, dotted, "\bar\beta"] & \\
& & D
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "b", "\bullet"{marking}] \arrow[d, "b"', "\bullet"{marking}] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b_0", "\bullet"{marking}] \arrow[ddr, bend left, "\beta''"] & \\
B \arrow[r, "b_1"', "\bullet"{marking}] \arrow[drr, bend right, "\beta''"'] & \bar X \arrow[dr, dotted, "\bar{\bar\beta}"] & \\
& & D
\end{tikzcd}
$$

Cf. Lemma 104.(1). We choose $\bar X \xrightarrow{\bar b} \bar B \longrightarrow !$; cf. Remark 134.

By $\mathbf{A}_{\mathrm{Lift}}$.(2), there exist commutative diagrams as follows.

$$
\begin{tikzcd}
\bar X \arrow[r, "\bar\beta"] \arrow[d, "\bar b"', "\bullet"{marking}] & D \arrow[d, "\square"{marking}] \\
\bar B \arrow[ur, dotted, "\bar{\mathfrak{b}}"] \arrow[r, "\square"{marking}] & !
\end{tikzcd}
\qquad
\begin{tikzcd}
\bar X \arrow[r, "\bar{\bar\beta}"] \arrow[d, "\bar b"', "\bullet"{marking}] & D \arrow[d, "\square"{marking}] \\
\bar B \arrow[ur, dotted, "\bar{\bar{\mathfrak{b}}}"] \arrow[r, "\square"{marking}] & !
\end{tikzcd}
$$

There exist commutative diagrams as follows.

$$
\begin{tikzcd}
X \arrow[r, "bb_0\bar{b}"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & \bar{B} \arrow[d, "\hat{a}"] \arrow[ddr, bend left, "\bar{\mathfrak{b}}\tilde{c}"] & \\
A \arrow[r, "\hat{b}"'] \arrow[drr, bend right, "\alpha\tilde{d}"'] & \hat{X} \arrow[dr, dotted, "\bar{f}"] & \\
& & \tilde{Y}
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "bb_0\bar{b}"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & \bar{B} \arrow[d, "\hat{a}"] \arrow[ddr, bend left, "\bar{\bar{\mathfrak{b}}}\tilde{c}"] & \\
A \arrow[r, "\hat{b}"'] \arrow[drr, bend right, "\alpha\tilde{d}"'] & \hat{X} \arrow[dr, dotted, "\bar{\bar{f}}"] & \\
& & \tilde{Y}
\end{tikzcd}
$$

Cf. Lemma 104.(1). Furthermore, we have a commutative diagram as follows.

$$
\begin{tikzcd}
X \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "\tilde{a}"] \arrow[ddr, bend left, "b_0\bar{b}\hat{a}"] & \\
A \arrow[r, "\tilde{b}"'] \arrow[drr, bend right, "\hat{b}"'] & \tilde{X} \arrow[dr, dotted, "w"] & \\
& & \hat{X}
\end{tikzcd}
$$

We have diagrams

$$
\begin{tikzcd}
X \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "\tilde{a}"] & \\
A \arrow[r, "\tilde{b}"'] & \tilde{X} \arrow[dr, shift left, "f'"] \arrow[dr, shift right, "w\bar{f}"'] & \\
& & \tilde{Y}
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "\tilde{a}"] & \\
A \arrow[r, "\tilde{b}"'] & \tilde{X} \arrow[dr, shift left, "f''"] \arrow[dr, shift right, "w\bar{\bar{f}}"'] & \\
& & \tilde{Y}
\end{tikzcd}
$$

where $\tilde{a}f' = \tilde{a}w\bar{f}$, where $\tilde{b}f' = \tilde{b}w\bar{f}$, where $\tilde{a}f'' = \tilde{a}w\bar{\bar{f}}$ and where $\tilde{b}f'' = \tilde{b}w\bar{\bar{f}}$.

Thus, we obtain $f' \sim w\bar{f}$ and $f'' \sim w\bar{\bar{f}}$; cf. Definition 96.(5). Therefore, it suffices to show that $\bar{f} \sim \bar{\bar{f}}$.

We have the following commutative cuboid.

$$
\begin{tikzcd}
& A \arrow[rr, "\tilde{b}"] \arrow[dd, equal] \arrow[drr, phantom, "\mathrm{Q}"] & & \tilde{X} \arrow[dd, dotted, "v"] \\
X \arrow[ur, "a"] \arrow[rr, "b"'] \arrow[dd, equal] & & B \arrow[ur, "\tilde{a}"'] \arrow[dd, "b_1\bar{b}"] & \\
& A \arrow[rr, "\hat{b}"] \arrow[drr, phantom, "\mathrm{Q}"] & & \hat{X} \\
X \arrow[ur, "a"] \arrow[rr, "bb_0\bar{b}"'] & & \bar{B} \arrow[ur, "\hat{a}"'] &
\end{tikzcd}
$$

Cf. Lemma 133.(2), $\mathbf{A}_{\mathrm{Cof}}$.(2), $\mathbf{A}_{\mathrm{Qis}}$.(1) and Lemma 104.(1). By Proposition 130, we obtain that $\mathrm{L}\,v \in \operatorname{Iso}\operatorname{Ho}\mathcal{C}$; cf. Remark 117. Thus, it suffices to show that $v\bar{f} \sim v\bar{\bar{f}}$.

We have the diagram

$$\begin{tikzcd}
X \arrow[r, "\bullet" marking, "b"] \arrow[d, "\bullet" marking, "a"'] & B \arrow[d, "\bullet" marking, "\tilde{a}"] \\
A \arrow[r, "\bullet" marking, "\tilde{b}"'] \arrow[ur, phantom, "\mathrm{Q}"] & \tilde{X} \arrow[dr, shift left, "v\bar{f}"] \arrow[dr, shift right, "v\bar{\bar{f}}"'] & \\
& & \tilde{Y}
\end{tikzcd}$$

in which $\tilde{b} \cdot v\bar{f} = \alpha\tilde{d} = \tilde{b} \cdot v\bar{\bar{f}}$ and $\tilde{a} \cdot v\bar{f} = \beta''\tilde{c} = \tilde{a} \cdot v\bar{\bar{f}}$. Thus, we obtain $v\bar{f} \sim v\bar{\bar{f}}$; cf. Definition 96.(5).

Ad (2). We have a commutative diagram as follows.

$$\begin{tikzcd}
X \arrow[r, "\bullet" marking, "a"] \arrow[d, "\bullet" marking, "b"'] & A \arrow[d, "\bullet" marking, "\tilde{b}"] \arrow[ddr, bend left, "\alpha'\tilde{d}"] & \\
B \arrow[r, "\bullet" marking, "\tilde{a}"'] \arrow[ur, phantom, "\mathrm{Q}"] \arrow[drr, bend right, "\beta''\tilde{c}"'] & \tilde{X} \arrow[dr, dotted, "\tilde{f}"] & \\
& & \tilde{Y}
\end{tikzcd}$$

Cf. Lemma 104.(1).

Since B and D are acyclic and the following diagrams commute, we obtain $f' \sim \tilde{f}$ by (1).

$$\begin{tikzcd}
& A \arrow[rr, "\bullet" marking, "\tilde{b}"] \arrow[dd, "\alpha' \wr"'] & & \tilde{X} \arrow[dd, "f'"] \\
X \arrow[ur, "\bullet" marking, "a"] \arrow[rr, "\bullet" marking, "b"'] \arrow[dd, "f"'] & & B \arrow[ur, "\bullet" marking, "\tilde{a}"'] \arrow[dd, "\beta' \wr"'] \arrow[u, phantom, "\mathrm{Q}"] & \\
& C \arrow[rr, "\tilde{d}"'] & & \tilde{Y} \\
Y \arrow[ur, "c"] \arrow[rr, "d"'] & & D \arrow[ur, "\tilde{c}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \arrow[rr, "\bullet" marking, "\tilde{b}"] \arrow[dd, "\alpha' \wr"'] & & \tilde{X} \arrow[dd, "\tilde{f}"] \\
X \arrow[ur, "\bullet" marking, "a"] \arrow[rr, "\bullet" marking, "b"'] \arrow[dd, "f"'] & & B \arrow[ur, "\bullet" marking, "\tilde{a}"'] \arrow[dd, "\beta'' \wr"'] \arrow[u, phantom, "\mathrm{Q}"] & \\
& C \arrow[rr, "\tilde{d}"'] & & \tilde{Y} \\
Y \arrow[ur, "\bullet" marking, "c"] \arrow[rr, "d"'] & & D \arrow[ur, "\tilde{c}"'] &
\end{tikzcd}$$

Since A and C are acyclic and the following diagrams commute, we obtain $\tilde{f} \sim f''$ by (1).

$$\begin{tikzcd}
& A \arrow[rr, "\bullet" marking, "\tilde{b}"] \arrow[dd, "\alpha' \wr"'] & & \tilde{X} \arrow[dd, "\tilde{f}"] \\
X \arrow[ur, "\bullet" marking, "a"] \arrow[rr, "\bullet" marking, "b"'] \arrow[dd, "f"'] & & B \arrow[ur, "\bullet" marking, "\tilde{a}"'] \arrow[dd, "\beta'' \wr"'] \arrow[u, phantom, "\mathrm{Q}"] & \\
& C \arrow[rr, "\tilde{d}"'] & & \tilde{Y} \\
Y \arrow[ur, "\bullet" marking, "c"] \arrow[rr, "d"'] & & D \arrow[ur, "\tilde{c}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \arrow[rr, "\bullet" marking, "\tilde{b}"] \arrow[dd, "\alpha'' \wr"'] & & \tilde{X} \arrow[dd, "f''"] \\
X \arrow[ur, "\bullet" marking, "a"] \arrow[rr, "\bullet" marking, "b"'] \arrow[dd, "f"'] & & B \arrow[ur, "\bullet" marking, "\tilde{a}"'] \arrow[dd, "\beta'' \wr"'] \arrow[u, phantom, "\mathrm{Q}"] & \\
& C \arrow[rr, "\tilde{d}"'] & & \tilde{Y} \\
Y \arrow[ur, "\bullet" marking, "c"] \arrow[rr, "d"'] & & D \arrow[ur, "\tilde{c}"'] &
\end{tikzcd}$$

Cf. Definition 96.(4). In consequence, we obtain $f' \sim f''$; cf. Definition 50. $\square$

3.6.2.3 A variant of the key lemma

In this §3.6.2.3, we give a variant of our key lemma, Lemma 136. This variant is an aside and is not used in the sequel.

Lemma 137. *The following assertions* (1, 2) *hold.*

(1) *Suppose given commutative diagrams*

$$\begin{tikzcd}
& A \arrow[rr, "\tilde b", "\bullet" marking] \arrow[dd, "\alpha"'] & & \tilde X \arrow[dd, "f'"] \\
X \arrow[ur, "a", "\bullet" marking] \arrow[rr, "b"', "\bullet" marking] \arrow[dd, "f"'] \arrow[urrr, phantom, "\mathrm{Q}"] & & B \arrow[ur, "\tilde a"', "\bullet" marking] \arrow[dd, "\beta'"] & \\
& C \arrow[rr, "\tilde d"'] & & \tilde Y \\
Y \arrow[ur, "c", "\bullet" marking] \arrow[rr, "d"'] \arrow[urrr, phantom, "\mathrm{Q}"] & & D \arrow[ur, "\tilde c"', "\bullet" marking] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \arrow[rr, "\tilde b", "\bullet" marking] \arrow[dd, "\alpha"'] & & \tilde X \arrow[dd, "f''"] \\
X \arrow[ur, "a", "\bullet" marking] \arrow[rr, "b"', "\bullet" marking] \arrow[dd, "f"'] \arrow[urrr, phantom, "\mathrm{Q}"] & & B \arrow[ur, "\tilde a"', "\bullet" marking] \arrow[dd, "\beta''"] & \\
& C \arrow[rr, "\tilde d"'] & & \tilde Y \\
Y \arrow[ur, "c", "\bullet" marking] \arrow[rr, "d"'] \arrow[urrr, phantom, "\mathrm{Q}"] & & D \arrow[ur, "\tilde c"', "\bullet" marking] &
\end{tikzcd}$$

where D in $\mathrm{Ob}\,\mathrm{Ac}\,\mathcal{C}$. *Then we have $f' \sim f''$.*

(2) *Suppose given commutative diagrams*

$$\begin{tikzcd}
& A \arrow[rr, "\tilde b", "\bullet" marking] \arrow[dd, "\alpha'"'] & & \tilde X \arrow[dd, "f'"] \\
X \arrow[ur, "a", "\bullet" marking] \arrow[rr, "b"', "\bullet" marking] \arrow[dd, "f"'] \arrow[urrr, phantom, "\mathrm{Q}"] & & B \arrow[ur, "\tilde a"', "\bullet" marking] \arrow[dd, "\beta'"] & \\
& C \arrow[rr, "\tilde d"', "\bullet" marking] & & \tilde Y \\
Y \arrow[ur, "c", "\bullet" marking] \arrow[rr, "d"', "\bullet" marking] \arrow[urrr, phantom, "\mathrm{Q}"] & & D \arrow[ur, "\tilde c"', "\bullet" marking] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \arrow[rr, "\tilde b", "\bullet" marking] \arrow[dd, "\alpha''"'] & & \tilde X \arrow[dd, "f''"] \\
X \arrow[ur, "a", "\bullet" marking] \arrow[rr, "b"', "\bullet" marking] \arrow[dd, "f"'] \arrow[urrr, phantom, "\mathrm{Q}"] & & B \arrow[ur, "\tilde a"', "\bullet" marking] \arrow[dd, "\beta''"] & \\
& C \arrow[rr, "\tilde d"', "\bullet" marking] & & \tilde Y \\
Y \arrow[ur, "c", "\bullet" marking] \arrow[rr, "d"', "\bullet" marking] \arrow[urrr, phantom, "\mathrm{Q}"] & & D \arrow[ur, "\tilde c"', "\bullet" marking] &
\end{tikzcd}$$

where C and D in $\mathrm{Ob}\,\mathrm{Ac}\,\mathcal{C}$. *Then we have $f' \sim f''$.*

Proof. *Ad* (1). We have

$$\begin{tikzcd}
 & A \arrow[rr, "\tilde b"] \arrow[dd, "\alpha"'] & & \tilde X \arrow[dd, shift right, "f'"'] \arrow[dd, shift left, "f''"] \\
X \arrow[ur, "a"] \arrow[rr, "b"'] \arrow[dd, "f"'] & & B \arrow[ur, "\tilde a"'] \arrow[dd, shift right, "\beta'"'] \arrow[dd, shift left, "\beta''"] & \\
 & C \arrow[rr, "\tilde d"'] \arrow[dd, equal] & & \tilde Y \arrow[dd, dotted, "v"] \\
Y \arrow[ur, "c"] \arrow[rr, "d"'] \arrow[dd, equal] & & D \arrow[dd, "p"] & \\
 & C \arrow[rr, "\tilde e"] & & Z \\
Y \arrow[ur, "c"] \arrow[rr, "e"'] & & ! \arrow[ur, "z"'] &
\end{tikzcd}$$

such that the lower cuboid is commutative; cf. $\mathbf{Q}_{\mathrm{Cof}}$.(2) and Lemma 104.(1).

Since

$$\tilde b \cdot f'v = \alpha \tilde d v = \tilde b \cdot f'v \quad \text{and} \quad \tilde a \cdot f'v = \beta' pz = \beta'' pz = \tilde a \cdot f''v,$$

we obtain $f'v \sim f''v$; cf. Definition 96.(5). Thus, it suffices to show that $\mathrm{L}\, v \in \mathrm{Iso}\,\mathrm{Ho}\,\mathcal{C}$.

We obtain $\mathrm{L}\, v \in \mathrm{Iso}\,\mathrm{Ho}\,\mathcal{C}$ by applying Proposition 130 to the lower cuboid; cf. Remark 117.

Ad (2). We have a commutative diagram as follows.

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "b"'] & A \arrow[d, "\tilde b"] \arrow[ddr, bend left, "\alpha' \tilde d"] & \\
B \arrow[r, "\tilde a"'] \arrow[drr, bend right, "\beta'' \tilde c"'] & \tilde X \arrow[dr, dotted, "\tilde f"] \arrow[ul, phantom, "\mathrm{Q}"] & \\
 & & \tilde Y
\end{tikzcd}$$

Cf. Lemma 104.(1).

Since D in $\mathrm{Ob}\,\mathrm{Ac}\,\mathcal{C}$ and the following diagrams commute, we obtain $f' \sim \tilde f$ by (1).

$$\begin{tikzcd}
 & A \arrow[rr, "\tilde b"] \arrow[dd, "\alpha'"'] & & \tilde X \arrow[dd, "f'"] \\
X \arrow[ur, "a"] \arrow[rr, "b"'] \arrow[dd, "f"'] & & B \arrow[ur, "\tilde a"'] \arrow[dd, "\beta'"'] & \\
 & C \arrow[rr, "\tilde d"'] & & \tilde Y \\
Y \arrow[ur, "c"] \arrow[rr, "d"'] & & D \arrow[ur, "\tilde c"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[rr, "\tilde b"] \arrow[dd, "\alpha'"'] & & \tilde X \arrow[dd, "\tilde f"] \\
X \arrow[ur, "a"] \arrow[rr, "b"'] \arrow[dd, "f"'] & & B \arrow[ur, "\tilde a"'] \arrow[dd, "\beta''"'] & \\
 & C \arrow[rr, "\tilde d"'] & & \tilde Y \\
Y \arrow[ur, "c"] \arrow[rr, "d"'] & & D \arrow[ur, "\tilde c"'] &
\end{tikzcd}$$

Since C in $\operatorname{Ob}\operatorname{Ac}\mathcal{C}$ and the following diagrams commute, we obtain $\tilde{f} \sim f''$ by (1).

$$\begin{tikzcd}
 & A \arrow[rr, "\tilde{b}"] \arrow[dd, "\alpha'"'] & & \tilde{X} \arrow[dd, "\tilde{f}"] \\
X \arrow[ur, "a"] \arrow[rr, "b"', crossing over] \arrow[dd, "f"'] & & B \arrow[ur, "\tilde{a}"'] \arrow[ul, phantom, ""] & \\
 & C \arrow[rr, "\tilde{d}"'] & & \tilde{Y} \\
Y \arrow[ur, "c"] \arrow[rr, "d"'] & & D \arrow[ur, "\tilde{c}"'] \arrow[from=uu, "\beta''", crossing over] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[rr, "\tilde{b}"] \arrow[dd, "\alpha''"'] & & \tilde{X} \arrow[dd, "f''"] \\
X \arrow[ur, "a"] \arrow[rr, "b"', crossing over] \arrow[dd, "f"'] & & B \arrow[ur, "\tilde{a}"'] & \\
 & C \arrow[rr, "\tilde{d}"'] & & \tilde{Y} \\
Y \arrow[ur, "c"] \arrow[rr, "d"'] & & D \arrow[ur, "\tilde{c}"'] \arrow[from=uu, "\beta''", crossing over] &
\end{tikzcd}$$

Cf. Definition 96.(4). In consequence, we obtain $f' \sim f''$; cf. Definition 50. □

Remark 138. The key lemma, Lemma 136, and its variant Lemma 137 are related as follows.

- Suppose that A and B in Lemma 137.(2) are acyclic. Then the conclusion of Lemma 137.(2) also follows from Lemma 136.(2); cf. Lemma 133.(2).
- Suppose that $c, \tilde{c}, d, \tilde{d} \in \operatorname{Cof}\mathcal{C}$ and that $(Y, D, C, \tilde{Y})$ is in $\operatorname{QPO}\mathcal{C}$ in Lemma 136.(2). Then the conclusion of Lemma 136.(2) also follows from Lemma 137.(2).

3.6.3 Construction of the suspension

For this §3.6.3, let $\mathcal{C}$ be a quasi-model-category.

Definition 139 (and Lemma)**.**

(1) For $X \in \operatorname{Ob}\mathcal{C}$, we choose

$$\begin{tikzcd}
X \arrow[r, "b_X"] \arrow[d, "a_X"'] & B_X \arrow[d, "c_X"] \\
A_X \arrow[r, "d_X"'] \arrow[ur, phantom, "\mathrm{Q}"] & \mathcal{S}X
\end{tikzcd}$$

with A_X and B_X in $\operatorname{Ob}\operatorname{Ac}\mathcal{C}$; cf. Remark 134, $\mathbf{Q}_{\mathrm{Cof}}$.(2) and Remark 98.(1).

(2) For $X \xrightarrow{f} Y$, we choose $A_X \xrightarrow[\approx]{\alpha_f} A_Y$, $B_X \xrightarrow[\approx]{\beta_f} B_Y$ and $\mathcal{S}X \xrightarrow{\mathcal{S}f} \mathcal{S}Y$ such that the following diagram commutes.

$$\begin{tikzcd}
 & A_X \arrow[rr, "d_X"] \arrow[dd, "\alpha_f \wr"'] & & \mathcal{S}X \arrow[dd, "\mathcal{S}f"] \\
X \arrow[ur, "a_X"] \arrow[rr, "b_X"', crossing over] \arrow[dd, "f"'] & & B_X \arrow[ur, "c_X"'] & \\
 & A_Y \arrow[rr, "d_Y"'] & & \mathcal{S}Y \\
Y \arrow[ur, "a_Y"] \arrow[rr, "b_Y"'] & & B_Y \arrow[ur, "c_Y"'] \arrow[from=uu, "\beta_f \wr", crossing over] &
\end{tikzcd}$$

Cf. Remark 135, Lemma 133.(2) and Lemma 104.(1).

This defines the functor

$$
\begin{array}{rcl}
\mathcal{C} & \xrightarrow{\mathbf{S}_{\mathcal{C}}} & \operatorname{Ho}\mathcal{C} \\
(X \xrightarrow{f} Y) & \mapsto & (\mathbf{S}_{\mathcal{C}}X \xrightarrow{\mathbf{S}_{\mathcal{C}}f} \mathbf{S}_{\mathcal{C}}Y) := (\mathcal{S}X \xrightarrow{[\mathcal{S}f]} \mathcal{S}Y).
\end{array}
$$

If unambiguous, we often write $\mathbf{S} := \mathbf{S}_{\mathcal{C}}$.

Proof. First we show that $\mathbf{S}1_X = 1_{\mathbf{S}X}$ for $X \in \operatorname{Ob}\mathcal{C}$.

Suppose given $X \in \operatorname{Ob}\mathcal{C}$. We need to show that $\mathcal{S}1_X \sim 1_{\mathcal{S}X}$.

For brevity, we write $\alpha_X := \alpha_{1_X}$ and $\beta_X := \beta_{1_X}$.

We have commutative diagrams

$$
\begin{tikzcd}
& A_X \arrow[rr, "d_X"] \arrow[rr, phantom, "\mathrm{Q}"', shift right=2ex] \arrow[dd, "\alpha_X\,\wr"'] & & \mathcal{S}X \arrow[dd, "\mathcal{S}1_X"] \\
X \arrow[ur, "a_X"] \arrow[rr, "b_X"'] \arrow[dd, "1_X"'] & & B_X \arrow[ur, "c_X"'] \arrow[dd, "\beta_X\,\wr"'] & \\
& A_X \arrow[rr, "d_X"'] \arrow[rr, phantom, "\mathrm{Q}"', shift right=2ex] & & \mathcal{S}X \\
X \arrow[ur, "a_X"] \arrow[rr, "b_X"'] & & B_X \arrow[ur, "c_X"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A_X \arrow[rr, "d_X"] \arrow[rr, phantom, "\mathrm{Q}"', shift right=2ex] \arrow[dd, "\wr\,1_{A_X}"] & & \mathcal{S}X \arrow[dd, "1_{\mathcal{S}X}"] \\
X \arrow[ur, "a_X"] \arrow[rr, "b_X"'] \arrow[dd, "1_X"'] & & B_X \arrow[ur, "c_X"'] \arrow[dd, "1_{B_X}\,\wr"'] & \\
& A_X \arrow[rr, "d_X"'] \arrow[rr, phantom, "\mathrm{Q}"', shift right=2ex] & & \mathcal{S}X \\
X \arrow[ur, "a_X"] \arrow[rr, "b_X"'] & & B_X \arrow[ur, "c_X"'] &
\end{tikzcd}
$$

with A_X and B_X in $\operatorname{Ob}\operatorname{Ac}\mathcal{C}$. Thus, we obtain $\mathcal{S}1_X \sim 1_{\mathcal{S}X}$ by Lemma 136.(2).

Now we show that $\mathbf{S}(fg) = \mathbf{S}f \cdot \mathbf{S}g$ for $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$.

We need to show that $\mathcal{S}(fg) \sim \mathcal{S}f \cdot \mathcal{S}g$. We have commuative diagrams

$$
\begin{tikzcd}
& A_X \arrow[rr, "d_X"] \arrow[rr, phantom, "\mathrm{Q}"', shift right=2ex] \arrow[dd, "\alpha_{fg}\,\wr"'] & & \mathcal{S}X \arrow[dd, "\mathcal{S}(fg)"] \\
X \arrow[ur, "a_X"] \arrow[rr, "b_X"'] \arrow[dd, "fg"'] & & B_X \arrow[ur, "c_X"'] \arrow[dd, "\beta_{fg}\,\wr"'] & \\
& A_Z \arrow[rr, "d_Z"'] \arrow[rr, phantom, "\mathrm{Q}"', shift right=2ex] & & \mathcal{S}Z \\
Z \arrow[ur, "a_Z"] \arrow[rr, "b_Z"'] & & B_Z \arrow[ur, "c_Z"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A_X \arrow[rr, "d_X"] \arrow[rr, phantom, "\mathrm{Q}"', shift right=2ex] \arrow[dd, "\wr\,\alpha_f\alpha g"] & & \mathcal{S}X \arrow[dd, "\mathcal{S}f\cdot\mathcal{S}g"] \\
X \arrow[ur, "a_X"] \arrow[rr, "b_X"'] \arrow[dd, "fg"'] & & B_X \arrow[ur, "c_X"'] \arrow[dd, "\beta_f\beta_g\,\wr"'] & \\
& A_Z \arrow[rr, "d_Z"'] \arrow[rr, phantom, "\mathrm{Q}"', shift right=2ex] & & \mathcal{S}Z \\
Z \arrow[ur, "a_Z"] \arrow[rr, "b_Z"'] & & B_Z \arrow[ur, "c_Z"'] &
\end{tikzcd}
$$

with A_X, B_X, A_Z and B_Z in $\operatorname{Ob}\operatorname{Ac}\mathcal{C}$. Thus, we obtain $\mathcal{S}(fg) \sim \mathcal{S}f \cdot \mathcal{S}g$; cf. Lemma 136.(2). $\square$

Lemma 140. *We have* $\mathbf{S}_{\mathcal{C}} \in \operatorname{Ob}{}_{(\sim)}[\mathcal{C}, \operatorname{Ho}\mathcal{C}]$.

Proof. By Proposition 59, it suffices to show that $\mathbf{S}(\mathrm{Qis}\,\mathcal{C}) \subseteq \mathrm{Iso}\,\mathrm{Ho}\,\mathcal{C}$; cf. Definition 108.

Suppose given $X \overset{f}{\underset{\approx}{\to}} Y$. We have the following commutative diagram.

$$
\begin{tikzcd}
& A_X \arrow[rr, "d_X"] \arrow[dd, "\alpha_f \wr"'] & & \mathcal{S}X \arrow[dd, "\mathcal{S}f"] \\
X \arrow[ur, "a_X"] \arrow[rr, "b_X"'] \arrow[dd, "f \wr"'] & & B_X \arrow[ur, "c_X"'] \arrow[dd, "\beta_f \wr"'] \arrow[u, phantom, "\mathrm{Q}"] & \\
& A_Y \arrow[rr, "d_Y"'] & & \mathcal{S}Y \\
Y \arrow[ur, "a_Y"] \arrow[rr, "b_Y"'] & & B_Y \arrow[ur, "c_Y"'] \arrow[u, phantom, "\mathrm{Q}"] &
\end{tikzcd}
$$

Cf. Definition 139.(2). By Proposition 130, we obtain

$$\mathbf{S}f = [\mathcal{S}f] = \mathrm{L}(\mathcal{S}f) \in \mathrm{Iso}\,\mathrm{Ho}\,\mathcal{C}.$$

Cf. Remark 117. $\square$

Definition 141 (and Lemma)**.** Let $\Sigma_{\mathcal{C}} := \overline{\mathbf{S}_{\mathcal{C}}}$; cf. Definition 8.
We have the functor

$$
\begin{array}{ccc}
\mathrm{Ho}\,\mathcal{C} & \xrightarrow{\Sigma_{\mathcal{C}}} & \mathrm{Ho}\,\mathcal{C} \\
(X \xrightarrow{[f]} Y) & \mapsto & (\Sigma_{\mathcal{C}}X \xrightarrow{\Sigma_{\mathcal{C}}[f]} \Sigma_{\mathcal{C}}Y) = (\mathbf{S}_{\mathcal{C}}X \xrightarrow{\mathbf{S}_{\mathcal{C}}f} \mathbf{S}_{\mathcal{C}}Y).
\end{array}
$$

We call $\Sigma_{\mathcal{C}}$ the *suspension functor* of $\mathcal{C}$. If unambiguous, we often write $\Sigma := \Sigma_{\mathcal{C}}$.

Proof. This follows from Lemma 140; cf. Definition 8. $\square$

We aim to show that the construction of the suspension functor is essentially independent of the choices made.

Lemma 142. *Suppose given a choice of a commutative diagram*

$$
\begin{tikzcd}
X \arrow[r, "\tilde{b}_X"] \arrow[d, "\tilde{a}_X"'] & \tilde{B}_X \arrow[d, "\tilde{c}_X"] \\
\tilde{A}_X \arrow[r, "\tilde{d}_X"'] \arrow[ur, phantom, "\mathrm{Q}"] & \tilde{\mathcal{S}}X
\end{tikzcd}
$$

with A_X and B_X in $\mathrm{Ob}\,\mathrm{Ac}\,\mathcal{C}$ *for $X \in \mathrm{Ob}\,\mathcal{C}$.*

Suppose given a choice of a commutative diagram

$$\begin{tikzcd}
 & \tilde{A}_X \arrow[rr, "\tilde{d}_X"] \arrow[dd, "\tilde{\alpha}_f \wr"'] & & \tilde{\mathcal{S}}X \arrow[dd, "\tilde{\mathcal{S}}f"] \\
X \arrow[ur, "\tilde{a}_X"] \arrow[rr, "\tilde{b}_X"'] \arrow[dd, "f"'] & & \tilde{B}_X \arrow[ur, "\tilde{c}_X"'] \arrow[dd, "\tilde{\beta}_f \wr"' near start] \arrow[u, phantom, "\mathrm{Q}" near end] & \\
 & \tilde{A}_Y \arrow[rr, "\tilde{d}_Y"' near start] & & \tilde{\mathcal{S}}Y \\
Y \arrow[ur, "\tilde{a}_Y"] \arrow[rr, "\tilde{b}_Y"'] & & \tilde{B}_Y \arrow[ur, "\tilde{c}_Y"'] \arrow[u, phantom, "\mathrm{Q}" near end] &
\end{tikzcd}$$

for $X \xrightarrow{f} Y$ *in* $\mathcal{C}$.

This defines the functor

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\tilde{\mathbf{S}}} & \operatorname{Ho}\mathcal{C} \\
(X \xrightarrow{f} Y) & \mapsto & (\tilde{\mathcal{S}}X \xrightarrow{[\tilde{\mathcal{S}}f]} \tilde{\mathcal{S}}Y).
\end{array}$$

Cf. Definition 139.

For $X \in \operatorname{Ob}\mathcal{C}$, *we choose* $A_X \xrightarrow[\approx]{\mathfrak{a}_X} \tilde{A}_X$, $B_X \xrightarrow[\approx]{\mathfrak{b}_X} \tilde{B}_X$ *and* $\mathcal{S}X \xrightarrow{\mathfrak{d}_X} \tilde{\mathcal{S}}X$ *such that the following diagram commutes.*

$$\begin{tikzcd}
 & A_X \arrow[rr, "d_X"] \arrow[dd, "\mathfrak{a}_X \wr"'] & & \mathcal{S}X \arrow[dd, "\mathfrak{d}_X"] \\
X \arrow[ur, "a_X"] \arrow[rr, "b_X"'] \arrow[dd, "1_X"'] & & B_X \arrow[ur, "c_X"'] \arrow[dd, "\mathfrak{b}_X \wr"' near start] \arrow[u, phantom, "\mathrm{Q}" near end] & \\
 & \tilde{A}_X \arrow[rr, "\tilde{d}_X"' near start] & & \tilde{\mathcal{S}}X \\
X \arrow[ur, "\tilde{a}_X"] \arrow[rr, "\tilde{b}_X"'] & & \tilde{B}_X \arrow[ur, "\tilde{c}_X"'] \arrow[u, phantom, "\mathrm{Q}" near end] &
\end{tikzcd}$$

Cf. Remark 135, Lemma 133.(2) and Lemma 104.(1).

Let $\gamma := (\gamma_X)_{X \in \operatorname{Ob}\mathcal{C}}$, *where* $\gamma_X := [\mathfrak{d}_X]$ *for* $X \in \operatorname{Ob}\mathcal{C}$.

We have $\tilde{\mathbf{S}} \in \operatorname{Ob}{}_{(\sim)}[\mathcal{C}, \operatorname{Ho}\mathcal{C}]$; cf. Lemma 140. *Let* $\tilde{\Sigma} := \overline{\overline{\tilde{\mathbf{S}}}}$; cf. Definition 8.

The following assertions (1, 2) *hold.*

(1) *We have the isotransformation* $\mathbf{S} \xrightarrow[\sim]{\gamma} \tilde{\mathbf{S}}$.

(2) *We have the isotransformation* $\Sigma \xrightarrow[\sim]{\bar{\gamma}} \tilde{\Sigma}$; cf. Definition 10.

Proof. *Ad* (1). By Proposition 130, we have

$$\gamma_X = [\mathfrak{d}_X] = \mathrm{L}\,\mathfrak{d}_X \in \operatorname{Iso}\operatorname{Ho}\mathcal{C}$$

for $X \in \operatorname{Ob}\mathcal{C}$; cf. Remark 117.

We show that γ is natural. Suppose given $X \xrightarrow{f} Y$ in $\mathcal{C}$. We need to show that

$$\mathfrak{d}_X \cdot \tilde{\mathcal{S}}f \quad \sim \quad \mathcal{S}f \cdot \mathfrak{d}_Y\,.$$

We have commutative diagrams

$$\begin{tikzcd}
& A_X \arrow[rr, "d_X", "\bullet" marking] \arrow[dd, "\wr\, \mathfrak{a}_X\tilde{\alpha}_f"'] & & \mathcal{S}X \arrow[dd, "\mathfrak{d}_X\cdot\tilde{\mathcal{S}}f"] \arrow[dl, phantom, "\mathrm{Q}", near start] \\
X \arrow[ur, "a_X", "\bullet" marking] \arrow[rr, "b_X"', "\bullet" marking, near start] \arrow[dd, "f"'] & & B_X \arrow[ur, "c_X"', "\bullet" marking] \arrow[dd, "\mathfrak{b}_X\tilde{\beta}_f\,\wr"', near start] & \\
& \tilde{A}_Y \arrow[rr, "\tilde{d}_Y"', "\bullet" marking, near start] & & \tilde{\mathcal{S}}Y \arrow[dl, phantom, "\mathrm{Q}", near start] \\
Y \arrow[ur, "\tilde{a}_Y", "\bullet" marking] \arrow[rr, "\tilde{b}_Y"', "\bullet" marking] & & \tilde{B}_Y \arrow[ur, "\tilde{c}_Y"', "\bullet" marking] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A_X \arrow[rr, "d_X", "\bullet" marking] \arrow[dd, "\wr\, \alpha_f\mathfrak{a}_Y"'] & & \mathcal{S}X \arrow[dd, "\mathcal{S}f\cdot\mathfrak{d}_Y"] \arrow[dl, phantom, "\mathrm{Q}", near start] \\
X \arrow[ur, "a_X", "\bullet" marking] \arrow[rr, "b_X"', "\bullet" marking, near start] \arrow[dd, "f"'] & & B_X \arrow[ur, "c_X"', "\bullet" marking] \arrow[dd, "\beta_f\mathfrak{b}_Y\,\wr"', near start] & \\
& \tilde{A}_Y \arrow[rr, "\tilde{d}_Y"', "\bullet" marking, near start] & & \tilde{\mathcal{S}}Y \arrow[dl, phantom, "\mathrm{Q}", near start] \\
Y \arrow[ur, "\tilde{a}_Y", "\bullet" marking] \arrow[rr, "\tilde{b}_Y"', "\bullet" marking] & & \tilde{B}_Y \arrow[ur, "\tilde{c}_Y"', "\bullet" marking] &
\end{tikzcd}$$

where A_X, B_X, $\tilde{A}_Y$ and $\tilde{B}_Y$ in $\operatorname{Ob}\operatorname{Ac}\mathcal{C}$.

Thus, we obtain $\mathfrak{d}_X \cdot \tilde{\mathcal{S}}f \sim \mathcal{S}f \cdot \mathfrak{d}_Y$ by Lemma 136.(2).

Ad (2). This follows from (1) and Remark 11. $\square$

3.7 The loop functor

In this §3.7, we establish the loop functor $\Omega_{\mathcal{C}} : \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{C}$ on the homotopy category of a quasi-model-category $\mathcal{C}$; cf. Definition 153 below.

3.7.1 Brown-Heath-Kamps-Gunnarson cogluing

For this §3.7.1, let $\mathcal{C}$ be a quasi-model-category.

Definition 143. A commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b", "|" marking] \\
A' \arrow[r, "f'"'] & B'.
\end{tikzcd}$$

is called an *H-pullback*, if the following assertion $(*)$ holds.

(∗) Suppose given a commutative diagram as follows.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "u"] \arrow[dd, "a"'] & & B \arrow[dd, "b", "\shortmid" marking] \\
& \hat{A} \arrow[ur, "\hat{f}"'] \arrow[dl, "\hat{a}"', "\shortmid" marking] \arrow[r, phantom, "\mathrm{Q}"] & {} \\
A' \arrow[rr, "f'"'] & & B'
\end{tikzcd}$$

Then we have $\mathrm{L}_{\mathcal{C}}\, u \in \operatorname{Iso} \operatorname{Ho} \mathcal{C}$.

To indicate that (A, B, A', B') is an H-pullback, we often write

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{H}"] & B \arrow[d, "b", "\shortmid" marking] \\
A' \arrow[r, "f'"'] & B'.
\end{tikzcd}$$

Proposition 144. Recall that $\mathcal{C}$ is a quasi-model-category.

Suppose given the following commutive diagram.

$$\begin{tikzcd}
& A'_0 \arrow[rr, "f'_0"] \arrow[dd, "\wr\, u'" near start] & & B'_0 \arrow[dd, "\wr\, v'"] \\
A_0 \arrow[ur, "a_0"] \arrow[rr, "f_0"', "\mathrm{H}"] \arrow[dd, "u"'] & & B_0 \arrow[ur, "b_0"', "\shortmid" marking] \arrow[dd, "\wr\, v" near start] & \\
& A'_1 \arrow[rr, "f'_1" near start] & & B'_1 \\
A_1 \arrow[ur, "a_1", "\shortmid" marking] \arrow[rr, "f_1"', "\mathrm{H}"] & & B_1 \arrow[ur, "b_1"', "\shortmid" marking] &
\end{tikzcd}$$

Then we have $\mathrm{L}\, u \in \operatorname{Iso} \operatorname{Ho} \mathcal{C}$.

Proof. This is dual to Proposition 130. □

3.7.2 Coacyclic objects

For this §3.7.2, let $\mathcal{C}$ be a quasi-model-category.

Definition 145. Let $\operatorname{Coac} \mathcal{C}$ be the full subcategory of $\mathcal{C}$ with

$$\operatorname{Ob} \operatorname{Coac} \mathcal{C} := \{ C \in \operatorname{Ob} \mathcal{C} :\ \text{¡} \xrightarrow{\circ} C \ \}.$$

The elements of $\operatorname{Ob} \operatorname{Coac} \mathcal{C}$ are called *coacyclic* objects.

Remark 146. Let X be an initial object in $\mathcal{C}$. Then $X \in \operatorname{Ob} \operatorname{Coac} \mathcal{C}$.

Lemma 147. *Suppose given $A, B \in \operatorname{Ob} \operatorname{Coac} \mathcal{C}$. The following assertions* (1, 2, 3) *hold.*

(1) *There exists* $A \xrightarrow[\approx]{d} B$.

(2) *Suppose given* $A \xrightarrow{f} B$. *Then we have* $A \xrightarrow[\approx]{f} B$.

(3) *Suppose given* $A \underset{g}{\overset{f}{\rightrightarrows}} B$. *Then we have* $f \sim g$.

Proof. This is dual to Lemma 133. $\square$

Remark 148. *Suppose given $X \in \operatorname{Ob} \mathcal{C}$. There exists* $\begin{tikzcd} C \arrow[r, "c", "\shortmid" marking] & X \end{tikzcd}$ *with $C \in \operatorname{Ob} \operatorname{Coac} \mathcal{C}$.*

Proof. This is dual to Remark 134. $\square$

Remark 149. *Suppose given*

$$\begin{tikzcd}
A \arrow[r, "a"] & X \arrow[d, "f"] & B \arrow[l, "b"'] \\
C \arrow[r, "c"', "\shortmid" marking] & Y & D \arrow[l, "d", "\shortmid" marking]
\end{tikzcd}$$

with A and B in $\operatorname{Ob} \operatorname{Coac} \mathcal{C}$.

There exist α and β in $\operatorname{Mor} \mathcal{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\alpha"'] & X \arrow[d, "f"] & B \arrow[l, "b"'] \arrow[d, "\beta"] \\
C \arrow[r, "c"', "\shortmid" marking] & Y & D \arrow[l, "d", "\shortmid" marking]
\end{tikzcd}$$

Proof. This is dual to Remark 135. $\square$

Lemma 150. *Suppose given commutative diagrams*

$$\begin{tikzcd}
& A \arrow[rr, "\tilde{b}"] \arrow[dd, "\alpha' \wr"' near start] & & \tilde{X} \arrow[dd, "\tilde{f}"] \\
X \arrow[ur, "a"] \arrow[rr, "b"' near start] \arrow[dd, "f'"'] & & B \arrow[ur, "\tilde{a}"'] \arrow[dd, "\beta' \wr"' near start] & \\
& C \arrow[rr, "\tilde{d}"' near start, "\shortmid" marking] & & \tilde{Y} \\
Y \arrow[ur, "c", "\shortmid" marking] \arrow[rr, "d"', "\shortmid" marking] \arrow[r, phantom, "\mathrm{Q}"] & & D \arrow[ur, "\tilde{c}"', "\shortmid" marking] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \arrow[rr, "\tilde{b}"] \arrow[dd, "\alpha'' \wr"' near start] & & \tilde{X} \arrow[dd, "\tilde{f}"] \\
X \arrow[ur, "a"] \arrow[rr, "b"' near start] \arrow[dd, "f''"'] & & B \arrow[ur, "\tilde{a}"'] \arrow[dd, "\beta'' \wr"' near start] & \\
& C \arrow[rr, "\tilde{d}"' near start, "\shortmid" marking] & & \tilde{Y} \\
Y \arrow[ur, "c", "\shortmid" marking] \arrow[rr, "d"', "\shortmid" marking] \arrow[r, phantom, "\mathrm{Q}"] & & D \arrow[ur, "\tilde{c}"', "\shortmid" marking] &
\end{tikzcd}$$

where A, B, C and D in $\operatorname{Ob} \operatorname{Coac} \mathcal{C}$. Then we have $f' \sim f''$.

Proof. This is dual to Lemma 136.(2) $\square$

3.7.3 Construction of the loop functor

For this §3.7.3, let $\mathcal{C}$ be a quasi-model-category.

Definition 151 (and Lemma)**.**

(1) For $X \in \operatorname{Ob}\mathcal{C}$, we choose

$$\begin{tikzcd} \mathcal{O}X \arrow[r, "l_X", "|" marking] \arrow[d, "m_X"', "|" marking] \arrow[dr, phantom, "\mathrm{Q}"] & K_X \arrow[d, "k_X", "|" marking] \\ J_X \arrow[r, "j_X"', "|" marking] & X \end{tikzcd}$$

with J_X and K_X in $\operatorname{Ob}\operatorname{Coac}\mathcal{C}$; cf. Remark 148, $\mathbf{Q}_{\mathrm{Fib}}$.(2) and Remark 98.(2).

(2) For $X \xrightarrow{f} Y$, we choose $J_X \overset{\iota_f}{\underset{\approx}{\dashrightarrow}} J_Y$, $K_X \overset{\kappa_f}{\underset{\approx}{\dashrightarrow}} K_Y$ and $\mathcal{O}X \xrightarrow{\mathcal{O}f} \mathcal{O}Y$ such that the following diagram commutes.

$$\begin{tikzcd} & J_X \arrow[rr, "j_X", "|" marking] \arrow[dd, "\iota_f", "\wr"'] & & X \arrow[dd, "f"] \\ \mathcal{O}X \arrow[ur, "m_X", "|" marking] \arrow[rr, "l_X"', "|" marking] \arrow[dd, "\mathcal{O}f"'] \arrow[r, phantom, "\mathrm{Q}"] & & K_X \arrow[ur, "k_X"', "|" marking] \arrow[dd, "\kappa_f"', "\wr"] & \\ & J_Y \arrow[rr, "j_Y", "|" marking] & & Y \\ \mathcal{O}Y \arrow[ur, "m_Y", "|" marking] \arrow[rr, "l_Y"', "|" marking] \arrow[r, phantom, "\mathrm{Q}"] & & K_Y \arrow[ur, "k_Y"', "|" marking] & \end{tikzcd}$$

Cf. Remark 149, Lemma 147.(2) and Lemma 104.(2).

This defines the functor

$$\begin{array}{rcl} \mathcal{C} & \xrightarrow{\mathbf{O}_{\mathcal{C}}} & \operatorname{Ho}\mathcal{C} \\ (X \xrightarrow{f} Y) & \mapsto & (\mathbf{O}_{\mathcal{C}}X \xrightarrow{\mathbf{O}_{\mathcal{C}}f} \mathbf{O}_{\mathcal{C}}Y) := (\mathcal{O}X \xrightarrow{[\mathcal{O}f]} \mathcal{O}Y). \end{array}$$

If unambiguous, we often write $\mathbf{O} := \mathbf{O}_{\mathcal{C}}$.

Proof. This is dual to Definition 139. □

Lemma 152. *We have* $\mathbf{O}_{\mathcal{C}} \in \operatorname{Ob}\,_{(\sim)}[\mathcal{C}, \operatorname{Ho}\mathcal{C}]$ *.*

Proof. This is dual to Lemma 140. □

Definition 153 (and Lemma)**.** Let $\Omega_{\mathcal{C}} := \overline{\mathbf{O}_{\mathcal{C}}}$; cf. Definition 8. *We have*

$$\begin{array}{rcl} \operatorname{Ho}\mathcal{C} & \xrightarrow{\Omega_{\mathcal{C}}} & \operatorname{Ho}\mathcal{C} \\ (X \xrightarrow{[f]} Y) & \mapsto & (\Omega_{\mathcal{C}}X \xrightarrow{\Omega_{\mathcal{C}}[f]} \Omega_{\mathcal{C}}Y) = (\mathbf{O}_{\mathcal{C}}X \xrightarrow{\mathbf{O}_{\mathcal{C}}f} \mathbf{O}_{\mathcal{C}}Y). \end{array}$$

We call $\Omega_{\mathcal{C}}$ the *loop functor* of $\mathcal{C}$. If unambiguous, we often write $\Omega := \Omega_{\mathcal{C}}$.

Proof. This is dual to Definition 141. $\square$

Lemma 154. *Suppose given a choice of a commutative diagram*

$$\begin{tikzcd}
\tilde{\mathcal{O}}X \arrow[r, "\tilde{l}_X", "\shortmid"{marking}] \arrow[d, "\tilde{m}_X"', "\shortmid"{marking}] \arrow[dr, phantom, "\mathrm{Q}"] & \tilde{K}_X \arrow[d, "\tilde{k}_X", "\shortmid"{marking}] \\
\tilde{J}_X \arrow[r, "\tilde{j}_X"', "\shortmid"{marking}] & X
\end{tikzcd}$$

with $\tilde{J}_X$ and $\tilde{K}_X$ in $\operatorname{Ob}\operatorname{Coac}\mathcal{C}$ for $X \in \operatorname{Ob}\mathcal{C}$.

Suppose given a choice of a commutative diagram

$$\begin{tikzcd}
& \tilde{J}_X \arrow[rr, "\tilde{j}_X", "\shortmid"{marking}] \arrow[dd, "\wr\, \tilde{\imath}_f"] & & X \arrow[dd, "f"] \\
\tilde{\mathcal{O}}X \arrow[ur, "\tilde{m}_X", "\shortmid"{marking}] \arrow[rr, "\tilde{l}_X"', "\shortmid"{marking}, "\mathrm{Q}"] \arrow[dd, "\tilde{\mathcal{O}}f"'] & & \tilde{K}_X \arrow[ur, "\tilde{k}_X"', "\shortmid"{marking}] \arrow[dd, "\tilde{\kappa}_f\, \wr"' near start] & \\
& \tilde{J}_Y \arrow[rr, "\tilde{j}_Y"' near start, "\shortmid"{marking}] & & Y \\
\tilde{\mathcal{O}}Y \arrow[ur, "\tilde{m}_Y", "\shortmid"{marking}] \arrow[rr, "\tilde{l}_Y"', "\shortmid"{marking}, "\mathrm{Q}"] & & \tilde{K}_Y \arrow[ur, "\tilde{k}_Y"', "\shortmid"{marking}] &
\end{tikzcd}$$

for $X \xrightarrow{f} Y$ in $\mathcal{C}$.

This defines the functor

$$\begin{array}{rcl}
\mathcal{C} & \xrightarrow{\tilde{\mathbf{O}}} & \operatorname{Ho}\mathcal{C} \\
(X \xrightarrow{f} Y) & \mapsto & (\tilde{\mathbf{O}}X \xrightarrow{\tilde{\mathbf{O}}f} \tilde{\mathbf{O}}Y) := (\tilde{\mathcal{O}}X \xrightarrow{[\tilde{\mathcal{O}}f]} \tilde{\mathcal{O}}Y).
\end{array}$$

Cf. Definition 151.

For $X \in \operatorname{Ob}\mathcal{C}$, we choose $J_X \overset{\mathfrak{j}_X}{\underset{\approx}{\dashrightarrow}} \tilde{J}_X$, $K_X \overset{\mathfrak{k}_X}{\underset{\approx}{\dashrightarrow}} \tilde{K}_X$ and $\mathcal{O}X \xrightarrow{\mathfrak{p}_X} \tilde{\mathcal{O}}X$ such that the following diagram commutes.

$$\begin{tikzcd}
& J_X \arrow[rr, "j_X", "\shortmid"{marking}] \arrow[dd, "\wr\, \mathfrak{j}_X"] & & X \arrow[dd, "1_X"] \\
\mathcal{O}X \arrow[ur, "m_X", "\shortmid"{marking}] \arrow[rr, "l_X"', "\shortmid"{marking}, "\mathrm{Q}"] \arrow[dd, "\mathfrak{p}_X"'] & & K_X \arrow[ur, "k_X"', "\shortmid"{marking}] \arrow[dd, "\mathfrak{k}_X\, \wr"' near start] & \\
& \tilde{J}_X \arrow[rr, "\tilde{j}_X"' near start, "\shortmid"{marking}] & & X \\
\tilde{\mathcal{O}}X \arrow[ur, "\tilde{m}_X", "\shortmid"{marking}] \arrow[rr, "\tilde{l}_X"', "\shortmid"{marking}, "\mathrm{Q}"] & & \tilde{K}_X \arrow[ur, "\tilde{k}_X"', "\shortmid"{marking}] &
\end{tikzcd}$$

Cf. Remark 149, Lemma 147.(2) and Lemma 104.(2).

Let $\varphi := (\varphi_X)_{X \in \operatorname{Ob} \mathcal{C}}$ *, where* $\varphi_X := [\mathfrak{p}_X]$ *for* $X \in \operatorname{Ob} \mathcal{C}$*.*

We have $\tilde{\mathbf{O}} \in \operatorname{Ob}\, _{(\sim)}[\mathcal{C}, \operatorname{Ho} \mathcal{C}]$*;* cf. Lemma 152. *Let* $\tilde{\Omega} := \overline{\tilde{\mathbf{O}}}$; cf. Definition 8.

The following assertions (1, 2) *hold.*

(1) *We have the isotransformation* $\mathbf{O} \xrightarrow[\sim]{\varphi} \tilde{\mathbf{O}}$.

(2) *We have the isotransformation* $\Omega \xrightarrow[\sim]{\bar{\varphi}} \tilde{\Omega}$; cf. Definition 10.

Proof. This is dual to Lemma 142. $\square$

3.8 Adjunction of loop and suspension functor

3.8.1 Pointedness

Lemma 155. *Suppose given a quasi-model-category* $\mathcal{C}$*.*

The following assertions (1, 2, 3) *are equivalent.*

(1) *There exists an isomorphism* $\text{¡} \xrightarrow[\sim]{d} !$.

(2) *There exists a quasi-isomorphism* $\text{¡} \overset{d}{\underset{\approx}{\dashrightarrow}} !$.

(3) *We have* $\operatorname{Ac} \mathcal{C} = \operatorname{Coac} \mathcal{C}$*.*

Proof. *Ad* (1) $\Rightarrow$ (2). We have $d \in \operatorname{Qis} \mathcal{C}$; cf. $\mathbf{A}_{\mathrm{Qis}}$.(1).

Ad (2) $\Rightarrow$ (1). We show that $d \in \operatorname{Iso} \mathcal{C}$. There exists a commutative diagram as follows.

$$\begin{tikzcd}
\text{¡} \arrow[rr, "d", "\approx"'] \arrow[dr, "i"'] & & ! \\
& X \arrow[ur, "p"'] &
\end{tikzcd}$$

Cf. $\mathbf{A}_{\mathrm{Fact}}$.(1) and $\mathbf{A}_{\mathrm{Qis}}$.(2.a).

There exist $! \overset{q}{\underset{\approx}{\dashrightarrow}} X \overset{j}{\underset{\approx}{\dashrightarrow}} \text{¡}$ with $ij = 1_{\text{¡}}$ and $qp = 1_!$; cf. Remark 106 and $\mathbf{A}_{\mathrm{Qis}}$.

We have $\text{¡} \overset{dqj}{\underset{\approx}{\dashrightarrow}} \text{¡}$. Since ¡ is initial, we obtain $dqj = 1_{\text{¡}}$.

We have $! \overset{qjd}{\underset{\approx}{\dashrightarrow}} !$. Since ! is terminal, we obtain $qjd = 1_!$.

Ad (2) $\Rightarrow$ (3). By duality, it suffices to show that $\operatorname{Coac} \mathcal{C} \subseteq \operatorname{Ac} \mathcal{C}$.

Suppose given $A \in \operatorname{Ob} \operatorname{Coac} \mathcal{C}$. Since (2) holds, we have a commutative diagram as follows.

$$\begin{tikzcd}
\text{¡} \arrow[rr, "\approx"] \arrow[dr] & & ! \\
& A \arrow[ur] &
\end{tikzcd}$$

Cf. Definition 145. Thus, $A \in \operatorname{Ob} \operatorname{Ac} \mathcal{C}$; cf. $\mathbf{A}_{\mathrm{Qis}}$.(2.b).

Ad (3) $\Rightarrow$ (2). We have $¡ \in \operatorname{Coac} \mathcal{C} = \operatorname{Ac} \mathcal{C}$; cf. Definition 145.

Thus, we have $¡ \longrightarrow !$; cf. Definition 131. $\square$

Definition 156 (and Stipulation)**.** Let $\mathcal{C}$ be a quasi-model-category.

(1) We call $\mathcal{C}$ *pointed* if we have $! \cong ¡$.

(2) Let $\mathcal{C}$ be pointed. We choose a zero object $*$ in $\mathcal{C}$; cf. Lemma 155.

Remark 157. *Let $\mathcal{C}$ be a pointed quasi-model-category. Let $A \in \operatorname{Ob} \mathcal{C}$.*

The following assertions (1, 2, 3, 4) *are equivalent.*

(1) *We have $A \in \operatorname{Ob} \operatorname{Ac} \mathcal{C}$.*

(2) *We have $A \longrightarrow *$.*

(3) *We have $A \in \operatorname{Ob} \operatorname{Coac} \mathcal{C}$.*

(4) *We have $* \longrightarrow A$.*

Proof. This follows from Remark 93 and Lemma 155. $\square$

3.8.2 The adjunction

For this §3.8.2, let $\mathcal{C}$ be a pointed quasi-model-category.

Recall the construction of the suspension functor in Definition 139 and the loop functor in Definition 151.

Definition 158 (and Lemma)**.** For $X \in \operatorname{Ob} \mathcal{C}$ we choose $A_X \xrightarrow[\approx]{\sigma_X} J_{\mathcal{S}X}$, $B_X \xrightarrow[\approx]{\tau_X} K_{\mathcal{S}X}$ and $X \xrightarrow{\mathfrak{e}_X} \mathcal{O}\mathcal{S}X$ such that the following diagram commutes.

$$\begin{tikzcd}
 & A_X \arrow[rr, "d_X"] \arrow[dd, "\wr\,\sigma_X"] \arrow[dr, phantom, "\mathrm{Q}"] & & \mathcal{S}X \arrow[dd, equal] \\
X \arrow[ur, "a_X"] \arrow[rr, "b_X"'] \arrow[dd, "\mathfrak{e}_X"'] & & B_X \arrow[ur, "c_X"'] \arrow[dd, "\wr\,\tau_X"] & \\
 & J_{\mathcal{S}X} \arrow[rr, "j_{\mathcal{S}X}"'] & & \mathcal{S}X \\
\mathcal{O}\mathcal{S}X \arrow[ur, "m_{\mathcal{S}X}"] \arrow[rr, "l_{\mathcal{S}X}"'] \arrow[r, phantom, "\mathrm{Q}"] & & K_{\mathcal{S}X} \arrow[ur, "k_{\mathcal{S}X}"'] &
\end{tikzcd}$$

Cf. Remarks 149 and 157, Lemma 147 and Lemma 104.(2).

Let $\eta := (\eta_X)_{X \in \operatorname{Ob} \operatorname{Ho} \mathcal{C}}$, where $\eta_X := [\mathfrak{e}_X]$ for $X \in \operatorname{Ob} \operatorname{Ho} \mathcal{C} = \operatorname{Ob} \mathcal{C}$.

We have the transformation $1_{\operatorname{Ho} \mathcal{C}} \xrightarrow{\eta} (\Omega \circ \Sigma)$.

Proof. Suppose given $X \xrightarrow{[f]} Y$ in $\operatorname{Ho}\mathcal{C}$. We need to show that

$$[f] \cdot \eta_Y = \eta_X \cdot (\Omega \circ \Sigma)[f]\,.$$

Thus, it suffices to show that

$$f \cdot \mathfrak{e}_Y \sim \mathfrak{e}_X \cdot \mathcal{O}\mathcal{S}f.$$

We have commutative diagrams

$$\begin{tikzcd}
 & A_X \arrow[rr, "d_X"] \arrow[dd, "\approx\, \alpha_f \cdot \sigma_Y"'] & \arrow[d, phantom, "\mathrm{Q}"] & \mathcal{S}X \arrow[dd, "\mathcal{S}f"] \\
X \arrow[ur, "a_X"] \arrow[rr, "b_X"'] \arrow[dd, "f \cdot \mathfrak{e}_Y"'] & & B_X \arrow[ur, "c_X"'] \arrow[dd, "\approx\, \beta_f \cdot \tau_Y"] & \\
 & J_{\mathcal{S}Y} \arrow[rr, "j_{\mathcal{S}Y}"'] & & \mathcal{S}Y \\
\mathcal{O}\mathcal{S}Y \arrow[ur, "m_{\mathcal{S}Y}"] \arrow[rr, "l_{\mathcal{S}Y}"'] & \arrow[u, phantom, "\mathrm{Q}"] & K_{\mathcal{S}Y} \arrow[ur, "k_{\mathcal{S}Y}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A_X \arrow[rr, "d_X"] \arrow[dd, "\approx\, \sigma_X \cdot \iota_{\mathcal{S}f}"'] & \arrow[d, phantom, "\mathrm{Q}"] & \mathcal{S}X \arrow[dd, "\mathcal{S}f"] \\
X \arrow[ur, "a_X"] \arrow[rr, "b_X"'] \arrow[dd, "\mathfrak{e}_X \cdot \mathcal{O}\mathcal{S}f"'] & & B_X \arrow[ur, "c_X"'] \arrow[dd, "\approx\, \tau_X \cdot \kappa_{\mathcal{S}f}"] & \\
 & J_{\mathcal{S}Y} \arrow[rr, "j_{\mathcal{S}Y}"'] & & \mathcal{S}Y \\
\mathcal{O}\mathcal{S}Y \arrow[ur, "m_{\mathcal{S}Y}"] \arrow[rr, "l_{\mathcal{S}Y}"'] & \arrow[u, phantom, "\mathrm{Q}"] & K_{\mathcal{S}Y} \arrow[ur, "k_{\mathcal{S}Y}"'] &
\end{tikzcd}$$

with A_X, B_X, $J_{\mathcal{S}Y}$ and $K_{\mathcal{S}Y}$ in $\operatorname{Ob}\operatorname{Ac}\mathcal{C} = \operatorname{Ob}\operatorname{Coac}\mathcal{C}$.

Thus, we obtain $f \cdot \mathfrak{e}_Y \sim \mathfrak{e}_X \cdot \mathcal{O}\mathcal{S}f$ by Lemma 150. □

Definition 159 (and Lemma)**.** For $X \in \operatorname{Ob}\mathcal{C}$ we choose $A_{\mathcal{O}X} \xrightarrow[\approx]{\gamma_X} J_X$, $B_{\mathcal{O}X} \xrightarrow[\approx]{\delta_X} K_X$ and $\mathcal{S}\mathcal{O}X \xrightarrow{\mathfrak{g}_X} X$ such that the following diagram commutes.

$$\begin{tikzcd}
 & A_{\mathcal{O}X} \arrow[rr, "d_{\mathcal{O}X}"] \arrow[dd, "\approx\, \gamma_X"'] & \arrow[d, phantom, "\mathrm{Q}"] & \mathcal{S}\mathcal{O}X \arrow[dd, "\mathfrak{g}_X"] \\
\mathcal{O}X \arrow[ur, "a_{\mathcal{O}X}"] \arrow[rr, "b_{\mathcal{O}X}"'] \arrow[dd, equal] & & B_{\mathcal{O}X} \arrow[ur, "c_{\mathcal{O}X}"'] \arrow[dd, "\approx\, \delta_X"] & \\
 & J_X \arrow[rr, "j_X"'] & & X \\
\mathcal{O}X \arrow[ur, "m_X"] \arrow[rr, "l_X"'] & \arrow[u, phantom, "\mathrm{Q}"] & K_X \arrow[ur, "k_X"'] &
\end{tikzcd}$$

Cf. Remarks 135 and 157, Lemma 133 and Lemma 104.(1).

Let $\varepsilon := (\varepsilon_X)_{X \in \operatorname{Ob}\operatorname{Ho}\mathcal{C}}$, where $\varepsilon_X := [\mathfrak{g}_X]$ for $X \in \operatorname{Ob}\operatorname{Ho}\mathcal{C} = \operatorname{Ob}\mathcal{C}$.

We have the transformation $(\Sigma \circ \Omega) \xrightarrow{\varepsilon} 1_{\operatorname{Ho}\mathcal{C}}$.

Proof. This is dual to Definition 158. □

Theorem 160 (Cf. also [**12,** Theorem 2.2])**.** *We have the adjunction* $(\Sigma, \Omega, \eta, \varepsilon)$.

Proof. We need to show that the following diagrams in $[\operatorname{Ho}\mathcal{C}, \operatorname{Ho}\mathcal{C}]$ commute.

$$\begin{tikzcd}
\Sigma \arrow[r, "\Sigma\eta"] \arrow[dr, equal] & \Sigma \circ \Omega \circ \Sigma \arrow[d, "\varepsilon\Sigma"] \\
 & \Sigma
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega \arrow[r, "\eta\Omega"] \arrow[dr, equal] & \Omega \circ \Sigma \circ \Omega \arrow[d, "\Omega\varepsilon"] \\
 & \Omega
\end{tikzcd}$$

By duality, it suffices to show the commutativity of the first diagram.

Suppose given $X \in \operatorname{Ob}\mathcal{C}$. We need to show that $\mathcal{S}\mathfrak{e}_X \cdot \mathfrak{g}_{\mathcal{S}X} \sim 1_{\mathcal{S}X}$.

We have commutative diagrams

$$
\begin{tikzcd}
 & A_X \arrow[rr, "d_X"] \arrow[dd, "\wr\,\sigma_X"] & & \mathcal{S}X \arrow[dd, equal] \\
X \arrow[ur, "a_X"] \arrow[rr, "b_X"'] \arrow[dd, "\mathfrak{e}_X"'] & & B_X \arrow[ur, "c_X"'] \arrow[dd, "\wr\,\tau_X"] \arrow[u, phantom, "\mathrm{Q}"] & \\
 & J_{\mathcal{S}X} \arrow[rr, "j_{\mathcal{S}X}"'] & & \mathcal{S}X \\
\mathcal{O}\mathcal{S}X \arrow[ur, "m_{\mathcal{S}X}"] \arrow[rr, "l_{\mathcal{S}X}"'] \arrow[r, phantom, "\mathrm{Q}"] & & K_{\mathcal{S}X} \arrow[ur, "k_{\mathcal{S}X}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A_X \arrow[rr, "d_X"] \arrow[dd, "\wr\,\alpha_{\mathfrak{e}_X}\cdot\gamma_{\mathcal{S}X}"] & & \mathcal{S}X \arrow[dd, "\mathcal{S}\mathfrak{e}_X\cdot\mathfrak{g}_{\mathcal{S}X}"] \\
X \arrow[ur, "a_X"] \arrow[rr, "b_X"'] \arrow[dd, "\mathfrak{e}_X"'] & & B_X \arrow[ur, "c_X"'] \arrow[dd, "\wr\,\beta_{\mathfrak{e}_X}\cdot\delta_{\mathcal{S}X}"] \arrow[u, phantom, "\mathrm{Q}"] & \\
 & J_{\mathcal{S}X} \arrow[rr, "j_{\mathcal{S}X}"'] & & \mathcal{S}X \\
\mathcal{O}\mathcal{S}X \arrow[ur, "m_{\mathcal{S}X}"] \arrow[rr, "l_{\mathcal{S}X}"'] \arrow[r, phantom, "\mathrm{Q}"] & & K_{\mathcal{S}X} \arrow[ur, "k_{\mathcal{S}X}"'] &
\end{tikzcd}
$$

with A_X, B_X, $J_{\mathcal{S}X}$ and $K_{\mathcal{S}X}$ in $\operatorname{Ob}\operatorname{Ac}\mathcal{C} = \operatorname{Ob}\operatorname{Coac}\mathcal{C}$.

Thus, we obtain $\mathcal{S}\mathfrak{e}_X \cdot \mathfrak{g}_{\mathcal{S}X} \sim 1_{\mathcal{S}X}$ by Lemma 136.(2). □

3.9 A remark on excision

3.9.1 Slim quasi-model-categories

Definition 161. Let $\mathcal{C}$ be an FCQ-category.

(1) A set of commutative quadrangles $\operatorname{SQPO}\mathcal{C}$ in $\mathcal{C}$ is called a *slim set of quasi-pushouts* if it fullfills Definition 96.(1, 2, 4, 5, 6, 7, 8).

(2) A set of commutative quadrangles $\operatorname{SQPB}\mathcal{C}$ in $\mathcal{C}$ is called a *slim set of quasi-pullbacks* if it fullfills Definition 97.(1, 2, 4, 5, 6, 7, 8).

Definition 162.

Let $\mathcal{C}$ be an FCQ-category that has initial and terminal objects; cf. Definition 92. Choose an initial object ¡ and a terminal object ! in $\mathcal{C}$.

Let $\operatorname{SQPO}\mathcal{C}$ be a slim set of quasi-pushouts in $\mathcal{C}$; cf. Definition 161.(1).

Let $\operatorname{SQPB}\mathcal{C}$ be a slim set of quasi-pullbacks in $\mathcal{C}$; cf. Definition 161.(2).

We call

$$(\mathcal{C}, \operatorname{Cof}\mathcal{C}, \operatorname{Fib}\mathcal{C}, \operatorname{Qis}\mathcal{C}, \operatorname{SQPO}\mathcal{C}, \operatorname{SQPB}\mathcal{C})$$

a *slim quasi-model-category* if $\mathbf{Q}_{\mathrm{Cof}}$, $\mathbf{Q}_{\mathrm{Fib}}$ and $\mathbf{Q}_{\mathrm{Braid}}$ hold.

We often refer to just $\mathcal{C}$ as a slim quasi-model-category.

So, a slim quasi-model-category is a category fullfilling all axioms of a quasi-model-category except the excision axioms Definition 96.(3) and Definition 97.(3).

3.9.2 Following Hirschhorn and Reedy

> In this §3.9.2 we follow Hirschhorn [**9,** Prop. 13.1.2] and Reedy [**14,** Th. B] and show that the excision axioms Definition 96.(3) and Definition 97.(3) follow from the other axioms in the saturated case; cf. Corollary 171 below. In the general case, we can only deduce a weaker form of excision; cf. Proposition 170 below.

For this §3.9.2 let $\mathcal{C}$ be a slim quasi-model-category; cf. Definition 162.

Remark 163.

(1) Excision is not used in § 3.4 and § 3.5. In particular, $\mathcal{C}$ is a category with split denominators such that Brown factorisation and Hirschhorn replacement hold; cf. Definition 108, Lemma 105 and Proposition 112.

(2) Excision is used in the context of H-pushouts and H-pullbacks to prove the Brown-Gunnarson gluing lemma Proposition 130 and its dual Proposition 144. In consequence, the construction of the loop and suspension functors and their adjunction, §3.6, §3.7 and §3.8.2, rely on excision.

3.9.2.1 Preparations

Lemma 164. *Suppose given a commutative diagram as follows.*

$$\begin{tikzcd} B \arrow[rr, "1"] \arrow[dr, "i"', "\wr"] & & B \\ & A \arrow[ur, "p"'] & \end{tikzcd}$$

There exists a commutative diagram as follows.

$$\begin{tikzcd} & A \arrow[dr, equal] \arrow[drr, bend left, "p"] & & \\ A \arrow[ur, "1"] \arrow[r, "l", "\approx"'] \arrow[dr, "pi"', "\wr"] & \hat{A} \arrow[u, "p_0"] \arrow[d, "p_1"] & A \arrow[l, "\sigma"'] & B \\ & A \arrow[ur, equal] \arrow[urr, bend right, "p"'] & & \end{tikzcd}$$

In particular, we have $pi \sim 1_A$.

> This lemma is a variant of Lemma 53.(2).

Proof. There exist commutative diagrams as follows.

$$\begin{tikzcd} A \arrow[dr, dotted, "s", "\wr"'] \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "1"'] & & \\ & \bar{A} \arrow[r, "\pi_0"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\mathrm{Q}"] & A \arrow[d, "p"] \\ & A \arrow[r, "p"'] & B \end{tikzcd}
\qquad
\begin{tikzcd} A \arrow[dr, dotted, "k", "\wr"'] \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "pi"', "\wr"] & & \\ & \bar{A} \arrow[r, "\pi_0"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\mathrm{Q}"] & A \arrow[d, "p"] \\ & A \arrow[r, "p"'] & B \end{tikzcd}$$

Cf. $\mathbf{Q}_{\mathrm{Fib}}$.(2), Definition 97.(2,4), $\mathbf{A}_{\mathrm{Qis}}$ and Lemma 104.(2).

There exists a commutative diagram as follows.

$$\begin{tikzcd}
A \arrow[rr, "s", "\approx"'] \arrow[dr, "\sigma"'] & & \bar{A} \\
& \hat{A} \arrow[ur, "\pi"'] &
\end{tikzcd}$$

Cf. $\mathbf{A}_{\mathrm{Fact}}$.(1) and $\mathbf{A}_{\mathrm{Qis}}$.(2.b). Furthermore, there exists $\bar{A} \underset{q}{\overset{\approx}{\dashrightarrow}} \hat{A}$ with $q\pi = 1_{\bar{A}}$; cf. Remark 106.(2) and $\mathbf{A}_{\mathrm{Qis}}$.

Thus, we have the following commutative diagram.

$$\begin{tikzcd}
& A \arrow[dr, equal] \arrow[drr, bend left, "p"] & & \\
A \arrow[ur, "1"] \arrow[r, "kq", "\approx"'] \arrow[dr, "pi"', "\approx"] & \hat{A} \arrow[u, "\pi\pi_0"] \arrow[d, "\pi\pi_1"'] & A \arrow[l, "\sigma"'] \arrow[dl, equal] & B \\
& A \arrow[urr, bend right, "p"'] & &
\end{tikzcd}$$

$\square$

Lemma 165. *Suppose given commutative diagrams as follows.*

$$\begin{tikzcd}
& Y \arrow[dr, equal] \arrow[drr, bend left, "t_Y"] & & \\
X \arrow[ur, "f_0"] \arrow[r, "h"] \arrow[dr, "f_1"'] & \hat{Y} \arrow[u, "p_0"] \arrow[d, "p_1"'] & Y \arrow[l, "s"'] \arrow[dl, equal] & T \\
& Y \arrow[urr, bend right, "t_Y"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[rr, "v"] \arrow[dr, "t_Y"'] & & Z \arrow[dl, "t_Z"] \\
& T &
\end{tikzcd}$$

There exists a commutative diagram as follows.

$$\begin{tikzcd}
& Z \arrow[dr, equal] \arrow[drr, bend left, "t_Z"] & & \\
X \arrow[ur, "f_0 v"] \arrow[r, "k"] \arrow[dr, "f_1 v"'] & \hat{Z} \arrow[u, "q_0"] \arrow[d, "q_1"'] & Z \arrow[l, "z"'] \arrow[dl, equal] & T \\
& Z \arrow[urr, bend right, "t_Z"'] & &
\end{tikzcd}$$

Proof. There exists a commutative diagram as follows.

$$\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d, "s"'] & Z \arrow[d, "t"] \arrow[ddrr, bend left, "1"] \arrow[dddr, bend left, "1"] & & \\
\hat{Y} \arrow[r, "\bar{v}"'] \arrow[ur, phantom, "\mathrm{Q}"] \arrow[drrr, bend right, "p_0 v"'] \arrow[ddrr, bend right, "p_1 v"'] & \overline{Z} \arrow[dr, dotted, "w"] & & \\
& & \tilde{Z} \arrow[r, "z_0"] \arrow[d, "z_1"'] \arrow[dr, phantom, "\mathrm{Q}"] & Z \arrow[d, "t_Z"] \\
& & Z \arrow[r, "t_Z"'] & T
\end{tikzcd}$$

Cf. $\mathbf{Q}_{\text{Cof}}$.(2), $\mathbf{Q}_{\text{Fib}}$.(2), Definition 96.(2), Definition 97.(4) and $\mathbf{Q}_{\text{Braid}}$.

Furthermore, there exists a commutative diagram as follows.

$$\begin{tikzcd}
\overline{Z} \arrow[rr, "w"] \arrow[dr, "i"'] & & \tilde{Z} \\
& \hat{Z} \arrow[ur, "q"'] &
\end{tikzcd}$$

Cf. $\mathbf{A}_{\text{Fact}}$.(1). Thus, we have the following commutative diagram.

$$\begin{tikzcd}
& Z \arrow[drr, bend left, "t_Z"] & & \\
X \arrow[ur, "f_0 v"] \arrow[r, "h\bar{v}i"] \arrow[dr, "f_1 v"'] & \hat{Z} \arrow[u, "qz_0"] \arrow[d, "qz_1"'] & Z \arrow[l, "ti"'] \arrow[ul, equal] \arrow[dl, equal] & T \\
& Z \arrow[urr, bend right, "t_Z"'] & &
\end{tikzcd}$$

Cf. $\mathbf{A}_{\text{Qis}}$ and $\mathbf{A}_{\text{Fib}}$.(2). $\square$

Lemma 166. *Suppose given the following diagram*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[ddr, "t_A"] & B \arrow[dd, "t_B"] \\
A' \arrow[ur, "g"] \arrow[dr, "t_{A'}"'] & \\
& T
\end{tikzcd}$$

in which $at_{A'} = t_A$, $ft_B = t_A$ *and* $gt_B = t_{A'}$.

Suppose given a commutative diagram as follows.

$$\begin{tikzcd}
 & A' \arrow[r, "g"] & B \arrow[dr, equal] \arrow[drr, bend left, "t_B"] & & \\
A \arrow[ur, "a"] \arrow[rr, "h"] \arrow[drr, "f"'] & & \hat{B} \arrow[u, "p_0"] \arrow[d, "p_1"] & B \arrow[l, "s"'] \arrow[dl, equal] & T \\
 & & B \arrow[urr, bend right, "t_B"'] & &
\end{tikzcd}$$

There exist commutative diagrams as follows.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[ddr, "t_A"] & B \arrow[dd, "t_B"] \\
A' \arrow[ur, "\tilde{g}"] \arrow[dr, "t_{A'}"'] & \\
 & T
\end{tikzcd}
\qquad
\begin{tikzcd}
 & B \arrow[dr, equal] & \\
A' \arrow[ur, "g"] \arrow[r, "k"] \arrow[dr, "\tilde{g}"'] & \hat{B} \arrow[u, "p_0"] \arrow[d, "p_1"] & B \arrow[l, "s"'] \arrow[dl, equal] \\
 & B &
\end{tikzcd}$$

This lemma is a variant of Proposition 112.(1).

Proof. There exists a commutative diagram as follows.

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "a"'] & \hat{B} \arrow[d, "p_0"] \\
A' \arrow[r, "g"'] \arrow[ur, dotted, "k"] & B
\end{tikzcd}$$

Cf. $\mathbf{A}_{\mathrm{Lift}}$.(2). The assertion follows by letting $\tilde{g} := kp_1$. □

3.9.2.2 Saturatedness implies excision

Lemma 167. *Suppose given the following commutative diagram.*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] \arrow[ur, phantom, "\mathrm{Q}"] & B'
\end{tikzcd}$$

Suppose given $A' \xrightarrow{v} Z$. *Then there exists* $B' \xrightarrow{u} Z$ *with* $f'u \sim v$.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \arrow[dl, phantom, "\mathrm{Q}"] & \\
A' \arrow[r, "f'"'] \arrow[drr, bend right, "v"'] & B' \arrow[dr, dotted, "u"] & \\
& & Z
\end{tikzcd}
$$

Proof. There exists a commutative diagram

$$
\begin{tikzcd}
B \arrow[rr, "1"] \arrow[dr, "g"', "\sim"] & & B \\
& A \arrow[ur, "f"'] &
\end{tikzcd}
$$

so that $fg \sim 1_A$; cf. Remark 106.(2) and Lemma 53.(2). Thus, we have the diagram

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \arrow[dl, phantom, "\mathrm{Q}"] \arrow[ddr, bend left, "gav"] & \\
A' \arrow[r, "f'"'] \arrow[drr, bend right, "v"'] & B' & \\
& & Z
\end{tikzcd}
$$

in which $av \sim fgav$. Thus, the assertion follows from Corollary 113. $\square$

Lemma 168. *Suppose given a diagram*

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \arrow[dl, phantom, "\mathrm{Q}"] & \\
A' \arrow[r, "f'"'] & B' \arrow[dr, shift left, "u"] \arrow[dr, shift right, "\tilde{u}"'] & \\
& & Z
\end{tikzcd}
$$

where $f'u \sim f'\tilde{u}$. *Then we have* $u \sim \tilde{u}$.

Proof. There exists a commutative diagram as follows.

$$
\begin{tikzcd}
Z_2 \arrow[r, "\pi"] \arrow[d, "\tilde{\pi}"'] \arrow[dr, phantom, "\mathrm{Q}"] & Z \arrow[d] \\
Z \arrow[r] & !
\end{tikzcd}
$$

Cf. $\mathbf{Q}_{\text{Fib}}$ and Definition 97.(4). Since we have $f'u \sim f'\tilde{u}$, there exists $\hat{Z} \xrightarrow{q} Z_2$ and a commutative diagram as follows.

$$\begin{tikzcd}
 & B' \arrow[r, "u"] & Z & \\
A' \arrow[ur, "f'"] \arrow[rr, "h"] \arrow[dr, "f'"'] & & \hat{Z} \arrow[u, "q\pi"] \arrow[d, "q\tilde{\pi}"'] & Z \arrow[l, "s"'] \arrow[ul, equal] \arrow[dl, equal] \\
 & B' \arrow[r, "\tilde{u}"'] & Z &
\end{tikzcd}$$

Cf. Lemma 111.(2). Furthermore, there exists a commutative diagram as follows.

$$\begin{tikzcd}
A' \arrow[r, "f'"] \arrow[d, "h"'] & B' \arrow[d, "h'"] \arrow[ddr, bend left, "\tilde{u}"] \arrow[ddrr, bend left, "u"] & & \\
\hat{Z} \arrow[r, "f''"'] \arrow[ur, phantom, "\mathrm{Q}"] \arrow[drr, bend right, "q\pi"] \arrow[ddrr, bend right, "q\tilde{\pi}"'] & W \arrow[dr, dotted, "w"] & & \\
 & & Z_2 \arrow[r, "\pi"] \arrow[d, "\tilde{\pi}"'] \arrow[dr, phantom, "\mathrm{Q}"] & Z \arrow[d] \\
 & & Z \arrow[r] & !
\end{tikzcd}$$

Cf. $\mathbf{Q}_{\text{Cof}}.(2)$ and $\mathbf{Q}_{\text{Braid}}$.

Moreover, there exists a commutative diagram as follows.

$$\begin{tikzcd}
\hat{Z} \arrow[rr, "f''w"] \arrow[dr, "\zeta"'] & & Z_2 \\
 & \hat{\hat{Z}} \arrow[ur, "\eta"'] &
\end{tikzcd}$$

Cf. $\mathbf{A}_{\text{Fact}}.(1)$. There exists a commutative diagram as follows.

$$\begin{tikzcd}
B \arrow[rr, "1"] \arrow[dr, "g"', "\approx"] & & B \\
 & A \arrow[ur, "f"'] &
\end{tikzcd}$$

Cf. Remark 106.(2) and $\mathbf{A}_{\text{Qis}}$.

By Lemma 164, there exists a commutative diagram as follows.

$$\begin{tikzcd}
 & A \arrow[drrr, bend left, "f"] & & \\
A \arrow[ur, "1"] \arrow[r, "l", "\approx"'] \arrow[dr, "fg"', "\approx"] & \hat{A} \arrow[u, "f_0"] \arrow[d, "f_1"'] & A \arrow[l, "\sigma"'] \arrow[ul, equal] \arrow[dl, equal] & B \\
 & A \arrow[urr, bend right, "f"'] & &
\end{tikzcd}$$

Since we have $ah\zeta\eta = fbh'w$, we may apply Lemma 165 to the commutative diagrams

$$
\begin{tikzcd}
 & A \arrow[dr, equal] \arrow[drr, bend left, "fbh'w"] & & \\
A \arrow[ur, "1"] \arrow[r, "l"] \arrow[dr, "fg"'] & \hat{A} \arrow[u, "f_0"] \arrow[d, "f_1"'] & A \arrow[l, "\sigma"'] & Z \\
 & A \arrow[ur, equal] \arrow[urr, bend right, "fbh'w"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "ah\zeta"] \arrow[dr, "ah\zeta\eta"'] & & \hat{\hat{Z}} \arrow[dl, "\eta"] \\
 & Z &
\end{tikzcd}
$$

to obtain the following commutative diagram.

$$
\begin{tikzcd}
 & \hat{\hat{Z}} \arrow[dr, equal] \arrow[drr, bend left, "\eta"] & & \\
A \arrow[ur, "ah\zeta"] \arrow[r, "\tilde{l}"] \arrow[dr, "fgah\zeta"'] & \check{Z} \arrow[u, "\tilde{f}_0"] \arrow[d, "\tilde{f}_1"'] & \hat{\hat{Z}} \arrow[l, "\check{s}"'] & Z_2 \\
 & \hat{\hat{Z}} \arrow[ur, equal] \arrow[urr, bend right, "\eta"'] & &
\end{tikzcd}
$$

Note that $fgah\zeta\eta = fgfbh'w = fbh'w = ah\zeta\eta$.

By applying Lemma 166 to the preceding diagram and

$$
\begin{tikzcd}
A \arrow[rr, "fgah\zeta"] \arrow[d, "a"'] \arrow[drr, "ah\zeta\eta"] & & \hat{\hat{Z}} \arrow[d, "\eta"] \\
A' \arrow[urr, "h\zeta"] \arrow[rr, "h\zeta\eta"'] & & Z_2
\end{tikzcd}
$$

we obtain the following commutative diagrams.

$$
\begin{tikzcd}
A \arrow[rr, "fgah\zeta"] \arrow[d, "a"'] \arrow[drr, "ah\zeta\eta"] & & \hat{\hat{Z}} \arrow[d, "\eta"] \\
A' \arrow[urr, "H"] \arrow[rr, "h\zeta\eta"'] & & Z_2
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \hat{\hat{Z}} \arrow[dr, equal] & \\
A' \arrow[ur, "h\zeta"] \arrow[r, "\check{l}"] \arrow[dr, "H"'] & \check{Z} \arrow[u, "\tilde{f}_0"] \arrow[d, "\tilde{f}_1"'] & \hat{\hat{Z}} \arrow[l, "\check{s}"'] \\
 & \hat{\hat{Z}} \arrow[ur, equal] &
\end{tikzcd}
$$

There exists a commutative diagram as follows.

$$\begin{tikzcd}
Af \arrow[r, ")"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b"] \arrow[ddr, bend left, "gah\zeta"] & \\
A' \arrow[r, "f'"'] \arrow[drr, bend right, "H"'] & B' \arrow[dr, dotted, "m"] & \\
& & \hat{\hat{Z}}
\end{tikzcd}$$

Cf. Lemma 104.(1). We obtain the following diagram

$$\begin{tikzcd}
& Z \arrow[dr, equal] & \\
B' \arrow[ur, "u"] \arrow[r, "m"] \arrow[dr, "\tilde{u}"'] & \hat{\hat{Z}} \arrow[u, "\eta\pi"] \arrow[d, "\eta\tilde{\pi}"'] & Z \arrow[l, "s\zeta"'] \arrow[dl, equal] \\
& Z &
\end{tikzcd}$$

in which $s\zeta \cdot \eta\pi = 1_Z = s\zeta \cdot \eta\tilde{\pi}$; cf. $\mathbf{A}_{\mathrm{Qis}}$, $\mathbf{A}_{\mathrm{Fib}}$.(2) and $\mathbf{A}_{\mathrm{Cof}}$.(2).

We need to show that $u \sim \tilde{u}$ or, equivalently, $\mathrm{L}\, u = \mathrm{L}\, \tilde{u}$.

Since we have $\mathrm{L}(s\zeta) \in \mathrm{Iso}\,\mathrm{Ho}\,\mathcal{C}$, we obtain $\mathrm{L}(\eta\pi) = (\mathrm{L}(s\zeta))^{-} = \mathrm{L}(\eta\tilde{\pi})$; cf. Corollary 60. Thus, it suffices to show that $m \cdot \eta\pi \sim u$ and $m \cdot \eta\tilde{\pi} \sim \tilde{u}$.

We have the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b"] & \\
A' \arrow[r, "f"'] & B' \arrow[dr, shift left, "m\eta\pi"] \arrow[dr, shift right, "u"'] & \\
& & Z
\end{tikzcd}$$

in which

$$\begin{array}{rcccccccc}
b \cdot u & = & bh'w\pi & = & gfbh'w\pi & = & gah\zeta\eta\pi & = & b \cdot m\eta\pi \\
f' \cdot u & = & hf''w\pi & = & h\zeta\eta\pi & = & H\eta\pi & = & f' \cdot m\eta\pi .
\end{array}$$

Thus, we obtain $u \sim m\eta\pi$; cf. Definition 96.(5).

Similarily, we obtain $\tilde{u} \sim m\eta\tilde{\pi}$. □

Lemma 169. *Suppose given the following commutative diagram.*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

Then we have $\mathrm{L}_{\mathcal{C}}\, f' \in \mathrm{Iso}\,\mathrm{Ho}\,\mathcal{C}$.

Proof. By Lemma 167, there exists $B' \xrightarrow{g'} A'$ with $f'g' \sim 1_{A'}$.

Since we have $f' \cdot g'f' \sim 1_{A'} \cdot f' \sim f' \cdot 1_{B'}$, we obtain $g'f' \sim 1_{B'}$; cf. Lemma 168. □

Proposition 170. Recall that $\mathcal{C}$ is a slim quasi-model-category; cf. Definition 162.
The following assertions (1, 2) *hold.*

(1) *Suppose given the following commutative diagram.*

$$\begin{tikzcd}
A \arrow[r, dashed, "f", "\approx"'] \arrow[d, "a"'] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] \arrow[ur, phantom, "\mathrm{Q}"] & B'
\end{tikzcd}$$

Then we have $\mathrm{L}_{\mathcal{C}}\, f' \in \operatorname{Iso}\operatorname{Ho}\mathcal{C}$.

(2) *Suppose given the following commutative diagram.*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Q}"] & B \arrow[d, "b"] \\
A' \arrow[r, dashed, "\approx", "f'"'] & B'
\end{tikzcd}$$

Then we have $\mathrm{L}_{\mathcal{C}}\, f \in \operatorname{Iso}\operatorname{Ho}\mathcal{C}$.

Proof. *Ad* (1). There exists a commutative diagram as follows.

$$\begin{tikzcd}
A \arrow[r, "s"] \arrow[rrr, bend left, "f", "\approx"'] \arrow[dd, "a"'] & X \arrow[rr, "p"] \arrow[d, "x"] & & B \arrow[dl, "\hat{b}"] \arrow[dd, "b"] \\
& X' \arrow[r, "p'"'] \arrow[ul, phantom, "\mathrm{Q}"] & \hat{B} \arrow[dr, dotted, "v"] \arrow[ul, phantom, "\mathrm{Q}"] & \\
A' \arrow[ur, "s'"'] \arrow[rrr, "f'"'] & & & B'
\end{tikzcd}$$

Cf. $\mathbf{A}_{\mathrm{Fact}}$.(1), $\mathbf{A}_{\mathrm{Qis}}$.(2.b), $\mathbf{Q}_{\mathrm{Cof}}$.(2), Definition 96.(2,4,6) and Lemma 104.(1).

We have $\mathrm{L}\, f' = \mathrm{L}\, s' \cdot \mathrm{L}\, p' \cdot \mathrm{L}\, v \in \operatorname{Iso}\operatorname{Ho}\mathcal{C}$; cf. Corollary 60, Lemma 169 and Lemma 109.(3).

Ad (2). This is dual to (1). □

Corollary 171 (Cf. [**9,** Prop. 13.1.2] and [**14,** Th. B])**.** *Suppose that $\mathcal{C}$ is saturated. Then $\mathcal{C}$ is a quasi-model-category.*

Proof. We have to show that Definition 96.(3) and Definition 97.(3) hold in $\mathcal{C}$.

Since $\mathcal{C}$ is saturated, this follows from Proposition 170. □

Chapter 4

Model categories

4.1 Axioms and elementary properties

4.1.1 Axioms for model categories

Definition 172. Let $\mathcal{M}$ be an FCQ-category that has an initial and a terminal object; cf. Definition 92. Choose an initial object ¡ and a terminal object ! in $\mathcal{M}$.

If the following axioms $\mathbf{M}_{\mathrm{Cof}}$ and $\mathbf{M}_{\mathrm{Fib}}$ hold, we call

$$(\mathcal{M}, \operatorname{Fib}\mathcal{M}, \operatorname{Cof}\mathcal{M}, \operatorname{Qis}\mathcal{M})$$

a *model category.* We often refer to just $\mathcal{M}$ as a model category.

> So altogether, a model category is a category $\mathcal{M}$ with initial object ¡ and terminal object !, with subsets $\operatorname{Fib}\mathcal{M} \subseteq \operatorname{Mor}\mathcal{M}$ of fibrations, $\operatorname{Cof}\mathcal{M} \subseteq \operatorname{Mor}\mathcal{M}$ of cofibrations and $\operatorname{Qis}\mathcal{M} \subseteq \operatorname{Mor}\mathcal{M}$ of quasi-isomorphisms, such that the axioms $\mathbf{A}_{\mathrm{Fib}}$, $\mathbf{A}_{\mathrm{Cof}}$, $\mathbf{A}_{\mathrm{Qis}}$, $\mathbf{A}_{\mathrm{Lift}}$, $\mathbf{A}_{\mathrm{Fact}}$, $\mathbf{M}_{\mathrm{Fib}}$ and $\mathbf{M}_{\mathrm{Cof}}$ hold.

$\mathbf{M}_{\mathrm{Cof}}$ The following assertions (1, 2, 3) hold.

(1) Suppose given $A' \xleftarrow{a} A \xrightarrow{f} B$ in $\mathcal{M}$. There exists a pushout

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] \arrow[ur, phantom, "\lrcorner"] & B'.
\end{tikzcd}$$

(2) Suppose given $B \xleftarrow{a} A \xrightarrow{f} X$ in $\mathcal{M}$. There exists a pushout

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] \arrow[ur, phantom, "\lrcorner"] & B'.
\end{tikzcd}$$

(3) Suppose given $A' \overset{a}{\leftarrow\!\bullet\!-} A \xrightarrow[\approx]{f} B$ in $\mathcal{M}$. There exists a pushout

$$\begin{tikzcd} A \arrow[r, "f", "\approx"'] \arrow[d, "a"', "\bullet" marking] & B \arrow[d, "b"] \arrow[dl, phantom, "\lrcorner" very near start, start anchor={[xshift=-2ex,yshift=2ex]south west}] \\ A' \arrow[r, "\approx", "f'"'] & B'. \end{tikzcd}$$

$\mathbf{M}_{\mathrm{Fib}}$ The following assertions (1, 2, 3) hold.

(1) Suppose given $A' \xrightarrow{f'} B' \overset{b}{\leftarrow\!+\!-} B$ in $\mathcal{M}$. There exists a pullback

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"', "+" marking] \arrow[dr, phantom, "\ulcorner" very near start] & B \arrow[d, "b", "+" marking] \\ A' \arrow[r, "f'"'] & B'. \end{tikzcd}$$

(2) Suppose given $A' \xrightarrow{f'} B' \overset{b}{\leftarrow\!\square\!-} B$ in $\mathcal{M}$. There exists a pullback

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"', "\square" marking] \arrow[dr, phantom, "\ulcorner" very near start] & B \arrow[d, "b", "\square" marking] \\ A' \arrow[r, "f'"'] & B'. \end{tikzcd}$$

(3) Suppose given $A' \xrightarrow[\approx]{f'} B' \overset{b}{\leftarrow\!+\!-} B$ in $\mathcal{M}$. There exists a pullback

$$\begin{tikzcd} A \arrow[r, "f", "\approx"'] \arrow[d, "a"'] \arrow[dr, phantom, "\ulcorner" very near start] & B \arrow[d, "b", "+" marking] \\ A' \arrow[r, "\approx", "f'"'] & B'. \end{tikzcd}$$

Remark 173. Note that the axioms for model categories in this work differ slightly from the axioms originally formulated by Quillen [**12,** Def. I.1.1]. Bousfield and Friedlander have added the axioms $\mathbf{M}_{\mathrm{Cof}}$.(3) and $\mathbf{M}_{\mathrm{Fib}}$.(3); cf. [**1,** Def, 1.2]. They call a model category satisfying these additional axioms a *proper* model category. So, a proper model category in the sense of Bousfield and Friedlander is in particular a model category in the sense of Definition 172. On the other hand, we only suppose certain pushouts and pullbacks to exist, we do not require $\mathcal{M}$ to have all finite limits and colimits.

Definition 174. Let $\mathcal{M}$ be a model category.

(1) We call $\mathcal{M}$ *weakly pointed* if the following assertions (a, b) hold.

 (a) We have $¡ \xrightarrow{\;+\;} !$.

 (b) We have $¡ \xrightarrow{\;\bullet\;} !$.

(2) We call $\mathcal{M}$ *pointed* if we have $¡ \cong !$.

Remark 175. *Let $\mathcal{M}$ be a pointed model category. Then $\mathcal{M}$ is weakly pointed.*

Proof. This follows from $\mathbf{A}_{\mathrm{Cof}}$.(1) and $\mathbf{A}_{\mathrm{Fib}}$.(1). □

4.1.2 Elementary properties

For this §4.1.2, let $\mathcal{M}$ be a model category.

Remark 176. *Suppose given*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] \arrow[ur, phantom, "\lrcorner" near end] & B'. \end{tikzcd}$$

The following assertions (1, 2, 3) *hold.*

(1) *Suppose that* $a \in \operatorname{Cof}\mathcal{M}$. *Then we have* $b \in \operatorname{Cof}\mathcal{M}$.

(2) *Suppose that* $a \in \operatorname{Cof}\mathcal{M} \cap \operatorname{Qis}\mathcal{M}$. *Then we have* $b \in \operatorname{Cof}\mathcal{M} \cap \operatorname{Qis}\mathcal{M}$.

(3) *Suppose that* $a \in \operatorname{Cof}\mathcal{M}$ *and* $f \in \operatorname{Qis}\mathcal{M}$. *Then we have* $f' \in \operatorname{Qis}\mathcal{M}$.

Proof. *Ad* (1). There exists a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "\hat{b}"] \arrow[ddr, bend left, "b"] & \\ A' \arrow[r, "\hat{f}"'] \arrow[ur, phantom, "\lrcorner" near end] \arrow[drr, bend right, "f'"'] & \hat{B} \arrow[dr, "\varphi"] & \\ & & B' \end{tikzcd}$$

in which $\varphi \in \operatorname{Iso}\mathcal{M} \subseteq \operatorname{Cof}\mathcal{M}$; cf. $\mathbf{M}_{\mathrm{Cof}}$.(1), Lemma 35 and $\mathbf{A}_{\mathrm{Cof}}$.(1).

Thus, $b = \hat{b}\varphi \in \operatorname{Cof}\mathcal{M}$; cf. $\mathbf{A}_{\mathrm{Cof}}$.(2).

Ad (2). There exists a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "\hat{b}"] \arrow[ddr, bend left, "b"] & \\ A' \arrow[r, "\hat{f}"'] \arrow[ur, phantom, "\lrcorner" near end] \arrow[drr, bend right, "f'"'] & \hat{B} \arrow[dr, "\varphi"] & \\ & & B' \end{tikzcd}$$

in which $\varphi \in \operatorname{Iso}\mathcal{M} \subseteq \operatorname{Cof}\mathcal{M} \cap \operatorname{Qis}\mathcal{M}$; cf. $\mathbf{M}_{\mathrm{Cof}}$.(2), Lemma 35, $\mathbf{A}_{\mathrm{Cof}}$.(1) and $\mathbf{A}_{\mathrm{Qis}}$.(1).

Thus, $b = \hat{b}\varphi \in \operatorname{Cof}\mathcal{M} \cap \operatorname{Qis}\mathcal{M}$; cf. $\mathbf{A}_{\mathrm{Cof}}$.(2) and $\mathbf{A}_{\mathrm{Qis}}$.(2.a).

Ad (3). There exists a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f", "\approx"'] \arrow[d, "a"'] & B \arrow[d, "\hat{b}"] \arrow[ddr, bend left, "b"] & \\ A' \arrow[r, "\approx", "\hat{f}"'] \arrow[ur, phantom, "\lrcorner" near end] \arrow[drr, bend right, "f'"'] & \hat{B} \arrow[dr, "\varphi", "\approx"'] & \\ & & B' \end{tikzcd}$$

in which $\varphi \in \operatorname{Iso}\mathcal{M} \subseteq \operatorname{Qis}\mathcal{M}$; cf. $\mathbf{M}_{\mathrm{Cof}}$.(3), Lemma 35 and $\mathbf{A}_{\mathrm{Qis}}$.(1).

Thus, $f' = \hat{f}\varphi \in \operatorname{Qis}\mathcal{M}$; cf. $\mathbf{A}_{\mathrm{Qis}}$.(2.a). □

Remark 177. *Suppose given*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B'. \end{tikzcd}$$

The following assertions (1, 2, 3) *hold.*

(1) *Suppose that $b \in \operatorname{Fib}\mathcal{M}$. Then we have $a \in \operatorname{Fib}\mathcal{M}$.*

(2) *Suppose that $b \in \operatorname{Fib}\mathcal{M} \cap \operatorname{Qis}\mathcal{M}$. Then we have $a \in \operatorname{Fib}\mathcal{M} \cap \operatorname{Qis}\mathcal{M}$.*

(3) *Suppose that $b \in \operatorname{Fib}\mathcal{M}$ and $f' \in \operatorname{Qis}\mathcal{M}$. Then we have $f \in \operatorname{Qis}\mathcal{M}$.*

Proof. This is dual to Remark 176. □

4.2 The full subcategory of bifibrant objects

For this §4.2, let $\mathcal{M}$ be a model category.

4.2.1 Cofibrant, fibrant and bifibrant objects

Definition 178 (and Lemma)**.**

(1) Let $\mathcal{M}_{\mathrm{cof}}$ be the full subcategory of $\mathcal{M}$ with $\operatorname{Ob}\mathcal{M}_{\mathrm{cof}} := \{A \in \operatorname{Ob}\mathcal{M} : \text{¡} \rightarrowtail A\}$.
An object $A \in \operatorname{Ob}\mathcal{M}_{\mathrm{cof}}$ is called *cofibrant.*
The category $\mathcal{M}_{\mathrm{cof}}$ is an FCQ-category with initial object ¡; cf. Definition 94.
Furthermore, $\mathbf{Q}_{\mathrm{Cof}}$.(1) *holds in* $\mathcal{M}_{\mathrm{cof}}$.

(2) Let $\mathcal{M}_{\mathrm{fib}}$ be the full subcategory of $\mathcal{M}$ with $\operatorname{Ob}\mathcal{M}_{\mathrm{fib}} := \{A \in \operatorname{Ob}\mathcal{M} : A \twoheadrightarrow \,!\}$.
An object $A \in \operatorname{Ob}\mathcal{M}_{\mathrm{fib}}$ is called *fibrant.*
The category $\mathcal{M}_{\mathrm{fib}}$ is an FCQ-category with terminal object !; cf. Definition 94.
Furthermore, $\mathbf{Q}_{\mathrm{Fib}}$.(1) *holds in* $\mathcal{M}_{\mathrm{fib}}$.

(3) Let $\mathcal{M}_{\mathrm{bif}}$ be the full subcategory of $\mathcal{M}$ with $\operatorname{Ob}\mathcal{M}_{\mathrm{bif}} := \operatorname{Ob}\mathcal{M}_{\mathrm{fib}} \cap \operatorname{Ob}\mathcal{M}_{\mathrm{cof}}$.
An object $A \in \operatorname{Ob}\mathcal{M}_{\mathrm{bif}}$ is called *bifibrant.*
The category $\mathcal{M}_{\mathrm{bif}}$ is an FCQ-category; cf. Definition 94.

Proof. *Ad* (1). Since $\mathcal{M}_{\mathrm{cof}}$ is a full subcategory of the FCQ-category $\mathcal{M}$, the properties $\mathbf{A}_{\mathrm{Cof}}$, $\mathbf{A}_{\mathrm{Fib}}$, $\mathbf{A}_{\mathrm{Qis}}$ and $\mathbf{A}_{\mathrm{Lift}}$ hold in $\mathcal{M}_{\mathrm{cof}}$; cf. Definitions 94 and 172.

In order to show that $\mathcal{M}_{\mathrm{cof}}$ is an FCQ-category, it remains to show that $\mathbf{A}_{\mathrm{Fact}}$ holds in $\mathcal{M}_{\mathrm{cof}}$; cf. Definition 92. By duality, it suffices to show that $\mathbf{A}_{\mathrm{Fact}}.(1)$ holds in $\mathcal{M}_{\mathrm{cof}}$.

Suppose given $A \xrightarrow{f} B$ in $\mathcal{M}_{\mathrm{cof}}$. In $\mathcal{M}$, there exists a commutative diagram as follows.

$$\begin{tikzcd}
¡ \arrow[r, "\bullet" marking] & A \arrow[rr, "f"] \arrow[dr, "i"', "\circ" marking] & & B \\
& & X \arrow[ur, "p"', "/" marking] &
\end{tikzcd}$$

Cf. $\mathbf{A}_{\mathrm{Fact}}.(1)$. By $\mathbf{A}_{\mathrm{Cof}}.(2)$, we have $X \in \mathrm{Ob}\,\mathcal{M}_{\mathrm{cof}}$. Thus, $\mathbf{A}_{\mathrm{Fact}}.(1)$ holds in $\mathcal{M}_{\mathrm{cof}}$.

By $\mathbf{A}_{\mathrm{Cof}}.(1)$, we have $¡ \xrightarrow{1} ¡$. Thus, $¡ \in \mathrm{Ob}\,\mathcal{M}_{\mathrm{cof}}$.

By definition of $\mathrm{Ob}\,\mathcal{M}_{\mathrm{cof}}$, the property $\mathbf{Q}_{\mathrm{Cof}}.(1)$ holds in $\mathcal{M}_{\mathrm{cof}}$.

Ad (2). This is dual to (1).

Ad (3). This follows as in (1) and (2). □

Remark 179. *Suppose that $\mathcal{M}$ is weakly pointed;* cf. Definition 174.(1).

Then we have $! \in \mathrm{Ob}\,\mathcal{M}_{\mathrm{bif}}$ *and* $¡ \in \mathrm{Ob}\,\mathcal{M}_{\mathrm{bif}}$.

Furthermore, $\mathbf{Q}_{\mathrm{Cof}}.(1)$ *and* $\mathbf{Q}_{\mathrm{Fib}}.(1)$ *hold in* $\mathcal{M}_{\mathrm{bif}}$.

Proof. This follows from Definition 178.(1, 2); cf. Definition 174.(1). □

4.2.2 Quasi-pushouts

In this §4.2.2, we define R-pushouts in $\mathcal{M}_{\mathrm{bif}}$ – R as in "replaced" – which will play the role of quasi-pushouts in $\mathcal{M}_{\mathrm{bif}}$ in the sense of Definition 96; cf. Definition 180 below. As soon as we have shown that the set of R-pushouts $\mathrm{RPO}\,\mathcal{M}_{\mathrm{bif}}$ is a set of quasi- pushouts in $\mathcal{M}_{\mathrm{bif}}$, we will mainly refer to R-pushouts as quasi-pushouts; cf. Proposition 189 below.

Definition 180. A commutative quadrangle

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"', "\bullet" marking] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

in $\mathcal{M}_{\mathrm{bif}}$ is called an *R-pushout*, if the following assertion $(*)$ holds.

(∗) Suppose given a commutative diagram in $\mathcal{M}$ as follows.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dd, "a"', "\bullet" marking] & & B \arrow[dl, "\hat{b}", "\bullet" marking] \arrow[dd, "b"] \\
& \hat{B} \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, "u"] & \\
A' \arrow[ur, "\hat{f}"'] \arrow[rr, "f'"'] & & B'
\end{tikzcd}$$

Then we have $u \in \operatorname{Qis} \mathcal{M} \cap \operatorname{Cof} \mathcal{M}$.

To indicate that (A, B, A', B') is an R-pushout, we often write

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"', "\bullet" marking] \arrow[dr, phantom, "\mathrm{R}"] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] & B'.
\end{tikzcd}$$

We denote the set of all R-pushouts in $\mathcal{M}_{\mathrm{bif}}$ by $\operatorname{RPO} \mathcal{M}_{\mathrm{bif}}$.

Remark 181. *Suppose given a commutative quadrangle*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"', "\bullet" marking] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] & B'.
\end{tikzcd}$$

The following assertions (1, 2) *are equivalent.*

(1) *The commutative quadrangle* (A, B, A', B') *is an R-pushout;* cf. Definition 180.

(2) *There exists a commutative diagram as follows.*

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dd, "a"', "\bullet" marking] & & B \arrow[dl, "\hat{b}", "\bullet" marking] \arrow[dd, "b"] \\
& \hat{B} \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, "u", "\circ" marking] & \\
A' \arrow[ur, "\hat{f}"'] \arrow[rr, "f'"'] & & B'
\end{tikzcd}$$

Proof. *Ad* (1) ⇒ (2). By $\mathbf{M}_{\mathrm{Cof}}$.(1), we have a commutative diagram as follows.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dd, "a"', "\bullet" marking] & & B \arrow[dl, "\hat{b}", "\bullet" marking] \arrow[dd, "b"] \\
& \hat{B} \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, dotted, "u"] & \\
A' \arrow[ur, "\hat{f}"'] \arrow[rr, "f'"'] & & B'
\end{tikzcd}$$

Since (1) holds, we have $u \in \operatorname{Qis} \mathcal{M} \cap \operatorname{Cof} \mathcal{M}$.

Ad $(2) \Rightarrow (1)$. Suppose given the following commutative diagram.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dd, "a"'] & & B \arrow[dl, "\tilde b"] \arrow[dd, "b"] \\
& \tilde B \arrow[dr, "\tilde u"] & \\
A' \arrow[ur, "\tilde f"] \arrow[rr, "f'"'] & & B'
\end{tikzcd}$$

We have to show that $\tilde u \in \operatorname{Qis} \mathcal{M} \cap \operatorname{Cof} \mathcal{M}$; cf. Definition 180.

By Lemma 35, we have the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "\tilde b"] \arrow[ddr, bend left, "\hat b"] & \\
A' \arrow[r, "\tilde f"'] \arrow[drr, bend right, "\hat f"'] \arrow[ur, phantom, "\lrcorner"] & \tilde B \arrow[dr, "v"] & \\
& & \hat B.
\end{tikzcd}$$

in which $v \in \operatorname{Iso} \mathcal{M} \subseteq \operatorname{Qis} \mathcal{M} \cap \operatorname{Cof} \mathcal{M}$; cf. $\mathbf{A}_{\text{Cof}}$.(1) and $\mathbf{A}_{\text{Qis}}$.(1).

Since we have $\tilde f \tilde u = f' = \tilde f v u$ and $\tilde b \tilde u = b = \tilde b v u$, we obtain $\tilde u = vu$; cf. Definition 29.

In consequence, $\tilde u = vu \in \operatorname{Qis} \mathcal{M} \cap \operatorname{Cof} \mathcal{M}$; cf. $\mathbf{A}_{\text{Cof}}$.(2) and $\mathbf{A}_{\text{Qis}}$.(2.a). □

Remark 182. Suppose given

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] \arrow[ur, phantom, "\lrcorner"] & B'.
\end{tikzcd}$$

in $\mathcal{M}_{\text{bif}}$. Then (A, B, A', B') is an R-pushout.

Lemma 183. *Suppose given*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{R}"] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] & B'.
\end{tikzcd}$$

in $\mathcal{M}_{\text{bif}}$ *. The following assertions* (1, 2, 3) *hold.*

(1) *We have* $b \in \operatorname{Cof} \mathcal{M}$.

(2) *Suppose that* $a \in \operatorname{Cof} \mathcal{M} \cap \operatorname{Qis} \mathcal{M}$. *Then we have* $b \in \operatorname{Cof} \mathcal{M} \cap \operatorname{Qis} \mathcal{M}$.

(3) *Suppose that* $f \in \operatorname{Qis} \mathcal{M}$. *Then we have* $f' \in \operatorname{Qis} \mathcal{M}$.

Proof. By Remark 181, there exists a commutative diagram in $\mathcal{M}$ as follows.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dd, "a"'] & & B \arrow[dl, "\hat{b}"] \arrow[dd, "b"] \\
& \hat{B} \arrow[dr, "u"] & \\
A' \arrow[ur, "\hat{f}"'] \arrow[rr, "f'"'] & & B'
\end{tikzcd}$$

Ad (1). We have $b = \hat{b}u \in \operatorname{Cof}\mathcal{M}$; cf. $\mathbf{A}_{\mathrm{Cof}}$.(2).

Ad (2). We have $b = \hat{b}u \in \operatorname{Cof}\mathcal{M} \cap \operatorname{Qis}\mathcal{M}$; cf. Remark 176.(2), $\mathbf{A}_{\mathrm{Cof}}$.(2) and $\mathbf{A}_{\mathrm{Qis}}$.(2.a).

Ad (3). We have $f' = \hat{f}u \in \operatorname{Qis}\mathcal{M}$; cf. Remark 176.(3) and $\mathbf{A}_{\mathrm{Qis}}$.(2.a). □

Lemma 184. *Suppose given the following commutative diagram in* $\mathcal{M}_{\mathrm{bif}}$ *.*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

Then the following assertions (1, 2) *are equivalent.*

(1) *We have* (A, B, A', B') *in* $\operatorname{RPO}\mathcal{M}_{\mathrm{bif}}$*.*

(2) *We have* (A, A', B, B') *in* $\operatorname{RPO}\mathcal{M}_{\mathrm{bif}}$*.*

Proof. This follows from Remark 31 and Remark 176.(1). □

Lemma 185. *Suppose given a diagram*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] & \\
A' \arrow[r, "f'"'] & B' \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & T
\end{tikzcd}$$

in $\mathcal{M}_{\mathrm{bif}}$ *in which* $f'u = f'v$ *and* $bu = bv$.

Then we have $u \sim v$ *in* $\mathcal{M}_{\mathrm{bif}}$ *;* cf. Definition 95.

Proof. By Remark 181, there exists a commutative diagram in $\mathcal{M}$ as follows.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dd, "a"'] & & B \arrow[dl, "\hat{b}"] \arrow[dd, "b"] \\
& \hat{B} \arrow[dr, "i"] & \\
A' \arrow[ur, "\hat{f}"'] \arrow[rr, "f'"'] & & B'
\end{tikzcd}$$

Since we have $\hat{f} \cdot iu = \hat{f} \cdot iv$ and $\hat{b} \cdot iu = \hat{b} \cdot iv$, we obtain $iu = iv$; cf. Definition 29.

Thus, we have commutative diagrams

$$
\begin{tikzcd}
\hat{B} \arrow[r, "i"] \arrow[d, "i"'] & B' \arrow[d, "i_0"] \arrow[ddr, bend left, "1", "\approx"'] & \\
B' \arrow[r, "i_1"'] \arrow[drr, bend right, "1"', "\approx"] & \tilde{B} \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, dotted, "\tilde{t}", "\approx"'] & \\
& & B'
\end{tikzcd}
\qquad
\begin{tikzcd}
\hat{B} \arrow[r, "i"] \arrow[d, "i"'] & B' \arrow[d, "i_0"] \arrow[ddr, bend left, "u"] & \\
B' \arrow[r, "i_1"'] \arrow[drr, bend right, "v"'] & \tilde{B} \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, dotted, "\tilde{h}"] & \\
& & T
\end{tikzcd}
$$

in $\mathcal{M}$; cf. $\mathbf{M}_{\mathrm{Cof}}$.(2), Remark 176.(2) and $\mathbf{A}_{\mathrm{Qis}}$.

Additionally, we have a commutative diagram in $\mathcal{M}$ as follows.

$$
\begin{tikzcd}
¡ \arrow[r] & B' \arrow[r, "i_0"] & \tilde{B} \arrow[rr] \arrow[dr, "j"'] & & ! \\
& & & \bar{B} \arrow[ur, "p"'] &
\end{tikzcd}
$$

In particular, we have $\bar{B} \in \operatorname{Ob} \mathcal{M}_{\mathrm{bif}}$. Furthermore, we have commutative diagrams

$$
\begin{tikzcd}
\tilde{B} \arrow[r, "\tilde{t}"] \arrow[d, "j"'] & B' \arrow[d] \\
\bar{B} \arrow[r] \arrow[ur, dotted, "t", "\approx"'] & !
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
\tilde{B} \arrow[r, "\tilde{h}"] \arrow[d, "j"'] & T \arrow[d] \\
\bar{B} \arrow[r] \arrow[ur, dotted, "h"] & !
\end{tikzcd}
$$

in $\mathcal{M}$; cf. $\mathbf{A}_{\mathrm{Lift}}$.(1) and $\mathbf{A}_{\mathrm{Qis}}$.(2.b). Since we have the commutative diagram

$$
\begin{tikzcd}
B' \arrow[r, equal] \arrow[d, equal] & B' \arrow[r, "u"] \arrow[d, "i_0 j"'] & T \arrow[r, equal] \arrow[d, equal] & T \arrow[d, equal] \\
B' \arrow[d, equal] & \bar{B} \arrow[l, "t"', "\approx"] \arrow[r, "h"] & T \arrow[r, equal] \arrow[d, equal] & T \arrow[d, equal] \\
B' \arrow[r, equal] & B' \arrow[r, "v"'] \arrow[u, "i_1 j"] & T \arrow[r, equal] & T
\end{tikzcd}
$$

in $\mathcal{M}_{\mathrm{bif}}$, we have $u \sim v$ in $\mathcal{M}_{\mathrm{bif}}$. $\square$

Lemma 186. *Suppose given*

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[d, "c"] \\
A' \arrow[r, "f'"'] \arrow[ur, phantom, "\mathrm{R}"] & B' \arrow[r, "g'"'] \arrow[ur, phantom, "\mathrm{R}"] & C'
\end{tikzcd}
$$

in $\mathcal{M}_{\mathrm{bif}}$. *Then* (A, C, A', C') *is in* $\operatorname{RPO} \mathcal{M}_{\mathrm{bif}}$.

Proof. In $\mathcal{M}$, we have a commutative diagram as follows.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[ddd, "a"'] & & B \arrow[rrr, "g"] \arrow[ddd, "b"] \arrow[ddl, "\hat{b}"] & & & C \arrow[ddd, "c"] \arrow[dll, "\tilde{c}"] \arrow[ddl, "\hat{c}"] \\
& & & \tilde{C} \arrow[dr, dotted, "w"] & & \\
& \hat{B} \arrow[uurr, "\tilde{g}"] \arrow[dr, "u"] & & & \hat{C} \arrow[dr, "v"] & \\
A' \arrow[ur, "\hat{f}"] \arrow[rr, "f'"'] & & B' \arrow[rrr, "g'"'] \arrow[urr, "\hat{g}"] & & & C'
\end{tikzcd}$$

Cf. Remark 181 and $\mathbf{M}_{\mathrm{Cof}}$.(1).

It suffices to show that $wv \in \operatorname{Qis} \mathcal{M} \cap \operatorname{Cof} \mathcal{M}$; cf. Remark 181 and Lemma 32.(1). Thus, it suffices to show that $w \in \operatorname{Qis} \mathcal{M} \cap \operatorname{Cof} \mathcal{M}$; cf. $\mathbf{A}_{\mathrm{Cof}}$.(2) and $\mathbf{A}_{\mathrm{Qis}}$.(2.a).

Since $(B, C, B', \hat{C})$ and $(B, C, \hat{B}, \tilde{C})$ are pushouts, so is $(\hat{B}, \tilde{C}, B', \hat{C})$; cf. Lemma 34.(2).

Thus, we have $w \in \operatorname{Qis} \mathcal{M} \cap \operatorname{Cof} \mathcal{M}$; cf. Remark 176.(2). $\square$

Lemma 187. *Suppose given*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a_0"'] & B \arrow[d, "b_0"] \\
A' \arrow[r, "f'"'] \arrow[d, "a_1"'] \arrow[r, phantom, "\mathrm{R}", shift left=2ex] & B' \arrow[d, "b_1"] \\
A'' \arrow[r, "f''"'] \arrow[r, phantom, "\mathrm{R}", shift left=2ex] & B''
\end{tikzcd}$$

in $\mathcal{M}_{\mathrm{bif}}$ *. Then* (A, B, A'', B'') *is in* $\operatorname{RPO} \mathcal{M}_{\mathrm{bif}}$ *.*

Proof. In $\mathcal{M}$, we have a commutative diagram as follows.

$$\begin{tikzcd}
A \arrow[rrr, "f"] \arrow[dd, "a_0"'] & & & B \arrow[dll, "\hat{b}"] \arrow[dd, "b_0"] \\
& \hat{B} \arrow[drr, "u"] \arrow[dd, "\tilde{b}"] & & \\
A' \arrow[ur, "\hat{f}"] \arrow[rrr, "f'"'] \arrow[ddd, "a_1"'] & & & B' \arrow[ddl, "\bar{b}"] \arrow[ddd, "b_1"] \\
& \tilde{B} \arrow[dr, dotted, "v"'] & & \\
& & \bar{B} \arrow[dr, "w"] & \\
A'' \arrow[uur, "\tilde{f}"] \arrow[urr, "\bar{f}"'] \arrow[rrr, "f''"'] & & & B''
\end{tikzcd}$$

Cf. Remark 181 and $\mathbf{M}_{\mathrm{Cof}}.(1)$.

By Remark 181, it suffices to show that $vw \in \operatorname{Qis}\mathcal{M} \cap \operatorname{Cof}\mathcal{M}$. Thus, it suffices to show that $v \in \operatorname{Qis}\mathcal{M} \cap \operatorname{Cof}\mathcal{M}$; cf. $\mathbf{A}_{\mathrm{Cof}}.(2)$ and $\mathbf{A}_{\mathrm{Qis}}.(2.a)$.

Since $(A', B', A'', \bar{B})$ and $(A', \hat{B}, A'', \tilde{B})$ are pushouts, so is $(\hat{B}, B', \tilde{B}, \bar{B})$; cf. Lemma 34.(1).

Thus, we have $v \in \operatorname{Qis}\mathcal{M} \cap \operatorname{Cof}\mathcal{M}$; cf. Remark 176.(2). □

Lemma 188. *Suppose given a commutative diagram in $\mathcal{M}_{\mathrm{bif}}$ as follows.*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{R}"] & B \arrow[d, "b"] \arrow[ddr, bend left, "y"] & & \\
A' \arrow[r, "f'"'] \arrow[ddr, bend right, "x"'] & B' \arrow[ddrr, bend left, "u"] & & \\
& & & \\
& & T \arrow[r, "s"'] & S
\end{tikzcd}$$

Then there exists $B' \xrightarrow{v} T$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \arrow[ddr, bend left, "y"] & & \\
A' \arrow[r, "f'"', "\mathrm{R}"] \arrow[drr, bend right, "x"'] & B' \arrow[dr, dotted, "v"] \arrow[drr, bend left, "u"] & & \\
& & T \arrow[r, "s"'] & S
\end{tikzcd}$$

Proof. In $\mathcal{M}$, we have a commutative diagram as follows.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dd, "a"'] & & B \arrow[dl, "\hat{b}"] \arrow[dd, "b"] \\
& \hat{B} \arrow[dr, "i"] & \\
A' \arrow[ur, "\hat{f}"] \arrow[rr, "f'"'] & & B'
\end{tikzcd}$$

Cf. Remark 181. Furthermore, we have a commutative diagram in $\mathcal{M}$ as follows.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "\hat{b}"] \arrow[ddr, bend left, "y"] & \\
A' \arrow[r, "\hat{f}"'] \arrow[drr, bend right, "x"'] & \hat{B} \arrow[dr, dotted, "\hat{v}"] & \\
& & T
\end{tikzcd}$$

Since $\hat{f} \cdot iu = xs = \hat{f} \cdot \hat{v}s$ and $\hat{b} \cdot iu = ys = \hat{b} \cdot \hat{v}s$, we obtain $iu = \hat{v}s$; cf. Definition 29.

Moreover, we have a commutative diagram as follows.

$$\begin{tikzcd}
\hat{B} \arrow[r, "\hat{v}"] \arrow[d, "i"'] & T \arrow[d, "s"] \\
B' \arrow[r, "u"'] \arrow[ur, dotted, "v"] & S
\end{tikzcd}$$

Cf. $\mathbf{A}_{\mathrm{Lift}}$.(1).

Since $bv = \hat{b}iv = \hat{b}\hat{v} = y$ and $f'v = \hat{f}iv = \hat{f}\hat{v} = x$, we have the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \arrow[ddr, bend left, "y"] & & \\
A' \arrow[r, "f'"', "\mathrm{R}"] \arrow[drr, bend right, "x"'] & B' \arrow[dr, "v"] \arrow[drr, bend left, "u"] & & \\
& & T \arrow[r, "s"'] & S
\end{tikzcd}$$

in $\mathcal{M}_{\mathrm{bif}}$. □

Proposition 189. *The set* $\operatorname{RPO} \mathcal{M}_{\text{bif}}$ *is a set of quasi-pushouts in* $\mathcal{M}_{\text{bif}}$ *;* cf. Definition 96.

Proof. We have to show that $\operatorname{RPO} \mathcal{M}_{\text{bif}}$ fullfills the assertions (1–8) from Definition 96.

Ad (1). This follows from Definition 180 and Lemma 183.(1).

Ad (2). This follows from Lemma 183.(2).

Ad (3). This follows from Lemma 183.(3).

Ad (4). This follows from Lemma 184.

Ad (5). This follows from Lemma 185.

Ad (6). This follows from Lemma 186.

Ad (7). This follows from Lemma 187.

Ad (8). This follows from Lemma 188. □

4.2.3 Quasi-pullbacks

Definition 190. A commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b", "\shortmid" marking] \\ A' \arrow[r, "f'"'] & B'. \end{tikzcd}$$

in $\mathcal{M}_{\text{bif}}$ is called an *R-pullback*, if the following assertion $(*)$ holds.

$(*)$ Suppose given a commutative diagram in $\mathcal{M}$ as follows.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "u"] \arrow[dd, "a"'] & & B \arrow[dd, "b", "\shortmid" marking] \\ & \hat{A} \arrow[ur, "\hat{f}"'] \arrow[dl, "\hat{a}", "\shortmid" marking] \arrow[dr, phantom, "\ulcorner", very near start] & \\ A' \arrow[rr, "f'"'] & & B' \end{tikzcd}$$

Then we have $u \in \operatorname{Qis} \mathcal{M} \cap \operatorname{Fib} \mathcal{M}$.

To indicate that (A, B, A', B') is an R-pullback, we often write

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{R}"] & B \arrow[d, "b", "\shortmid" marking] \\ A' \arrow[r, "f'"'] & B'. \end{tikzcd}$$

We denote the set of all R-pullbacks in $\mathcal{M}_{\text{bif}}$ by $\operatorname{RPB} \mathcal{M}_{\text{bif}}$.

Remark 191.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] & B'.
\end{tikzcd}
$$

in $\mathcal{M}_{\mathrm{bif}}$. *The following assertions* (1, 2) *are equivalent.*

(1) *The commutative quadrangle* (A, B, A', B') *is an R-pullback;* cf. Definition 190.

(2) *There exists a commutative diagram as follows.*

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "u"] \arrow[dd, "a"'] & & B \arrow[dd, "b"] \\
& \hat{A} \arrow[ur, "\hat{f}"'] \arrow[dl, "\hat{a}"] \arrow[dr, phantom, "\ulcorner", very near start] & \\
A' \arrow[rr, "f'"'] & & B'
\end{tikzcd}
$$

Proof. This is dual to Remark 181. □

Proposition 192. *The set* $\mathrm{RPB}\,\mathcal{M}_{\mathrm{bif}}$ *is a set of quasi-pullbacks in* $\mathcal{M}_{\mathrm{bif}}$ *;* cf. Definition 97.

Proof. This is dual to Proposition 189. □

4.2.4 Quasi-model-structure

Recall that $\mathcal{M}$ is a model category; cf. Definition 172.

Recall that $\mathcal{M}_{\mathrm{bif}}$ is an FCQ-category; cf. Definition 178.(3).

Recall that $\mathrm{RPO}\,\mathcal{M}_{\mathrm{bif}}$ is a set of quasi-pushouts in $\mathcal{M}_{\mathrm{bif}}$; cf. Definition 180 and Proposition 189.

Recall that $\mathrm{RPB}\,\mathcal{M}_{\mathrm{bif}}$ is a set of quasi-pullbacks in $\mathcal{M}_{\mathrm{bif}}$; cf. Definition 190 and Proposition 192.

Theorem 193. *The following assertions* (1, 2) *hold.*

(1) *Suppose that* $\mathcal{M}$ *is weakly pointed;* cf. Definition 174.(1).

Then $(\mathcal{M}_{\mathrm{bif}}\,, \mathrm{Cof}\,\mathcal{M}_{\mathrm{bif}}\,, \mathrm{Fib}\,\mathcal{M}_{\mathrm{bif}}\,, \mathrm{Qis}\,\mathcal{M}_{\mathrm{bif}}\,, \mathrm{RPO}\,\mathcal{M}_{\mathrm{bif}}\,, \mathrm{RPB}\,\mathcal{M}_{\mathrm{bif}})$ *is a quasi-model-category;* cf. Definition 100.

(2) *Suppose that* $\mathcal{M}$ *is pointed;* cf. Definition 174.(2).

Then $(\mathcal{M}_{\mathrm{bif}}\,, \mathrm{Cof}\,\mathcal{M}_{\mathrm{bif}}\,, \mathrm{Fib}\,\mathcal{M}_{\mathrm{bif}}\,, \mathrm{Qis}\,\mathcal{M}_{\mathrm{bif}}\,, \mathrm{RPO}\,\mathcal{M}_{\mathrm{bif}}\,, \mathrm{RPB}\,\mathcal{M}_{\mathrm{bif}})$ *is a pointed quasi-model-category;* cf. Definition 156.(1).

Proof. *Ad* (1). By Remark 179, the FCQ-category $\mathcal{M}_{\text{bif}}$ has initial and terminal objects and fulfills $\mathbf{Q}_{\text{Cof}}.(1)$ and $\mathbf{Q}_{\text{Fib}}.(1)$.

It remains to show that $\mathcal{M}_{\text{bif}}$ fulfills $\mathbf{Q}_{\text{Braid}}$, $\mathbf{Q}_{\text{Cof}}.(2)$ and $\mathbf{Q}_{\text{Fib}}.(2)$.

Ad $\mathbf{Q}_{\text{Braid}}$. Suppose given a commutative diagram in $\mathcal{M}_{\text{bif}}$ as follows.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \arrow[dd, phantom, ""] \arrow[rrdd, bend left, "u"] \arrow[rrddd, bend left, "v" near start] & & \\
A' \arrow[r, "f'"'] \arrow[urr, phantom, ""] \arrow[rrrd, bend right, "u'"] \arrow[rrdd, bend right, "v'"'] & B' \arrow[ul, phantom, "\mathrm{R}"] & & \\
& & X \arrow[r, "g"] \arrow[d, "x"'] \arrow[dr, phantom, "\mathrm{R}"] & Y \arrow[d, "y"] \\
& & X' \arrow[r, "g'"'] & Y'
\end{tikzcd}$$

By Remarks 181 and 191, there exist commutative diagrams in $\mathcal{M}$ as follows.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dd, "a"'] & & B \arrow[dl, "\hat{b}"] \arrow[dd, "b"] \\
& \hat{B} \arrow[dr, "s"] & \\
A' \arrow[ur, "\hat{f}"] \arrow[rr, "f'"'] & & B'
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "t"] \arrow[dd, "x"'] & & Y \arrow[dd, "y"] \\
& \hat{X} \arrow[ur, "\hat{g}"'] \arrow[dl, "\hat{x}"] & \\
X' \arrow[rr, "g'"'] & & Y'
\end{tikzcd}$$

Furthermore, there exist commutative diagrams in $\mathcal{M}$ as follows.

$$\begin{tikzcd}
B \arrow[dr, dotted, "h"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\
& \hat{X} \arrow[r, "\hat{g}"] \arrow[d, "\hat{x}"'] & Y \arrow[d, "y"] \\
& X' \arrow[r, "g'"'] & Y'
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[dr, dotted, "h'"] \arrow[drr, bend left, "u'"] \arrow[ddr, bend right, "v'"'] & & \\
& \hat{X} \arrow[r, "\hat{g}"] \arrow[d, "\hat{x}"'] & Y \arrow[d, "y"] \\
& X' \arrow[r, "g'"'] & Y'
\end{tikzcd}$$

Since we have $fh \cdot \hat{g} = ah' \cdot \hat{g}$ and $fh \cdot \hat{x} = ah' \cdot \hat{x}$, we obtain $fh = ah'$; cf. Definition 30.

There exists a commutative diagram in $\mathcal{M}$ as follows.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "\hat{b}"] \arrow[ddr, bend left, "h"] & \\
A' \arrow[r, "\hat{f}"'] \arrow[drr, bend right, "h'"'] & \hat{B} \arrow[dr, dotted, "\hat{h}"] & \\
& & \hat{X}
\end{tikzcd}$$

By $\mathbf{A}_{\mathrm{Lift}}.(1)$ and $\mathbf{A}_{\mathrm{Fib}}.(2)$, there exists a commutative diagram in $\mathcal{M}$ as follows.

$$\begin{tikzcd}
\hat{B} \arrow[r, "\hat{h}"] \arrow[d, "s"'] & \hat{X} \arrow[d] \\
B' \arrow[r] \arrow[ur, dotted, "\tilde{w}"] & !
\end{tikzcd}$$

Moreover, we have a commutative diagram in $\mathcal{M}$ as follows.

$$\begin{tikzcd}
¡ \arrow[r] \arrow[d] & X \arrow[d, "t"] \\
B' \arrow[r, "\tilde{w}"'] \arrow[ur, dotted, "w"] & \hat{X}
\end{tikzcd}$$

Cf. $\mathbf{A}_{\mathrm{Lift}}.(2)$. So we have $swt = \hat{h}$.

Thus, we have the following commutative diagram in $\mathcal{M}_{\mathrm{bif}}$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \arrow[ddr, bend left, "v"] \arrow[ddrr, bend left, "u"] & & \\
A' \arrow[r, "f'"'] \arrow[ur, phantom, "\mathrm{R}"] \arrow[drrr, bend right, "u'"'] \arrow[ddrr, bend right, "v'"'] & B' \arrow[dr, "w"] & & \\
& & X \arrow[r, "g"] \arrow[d, "x"'] & Y \arrow[d, "y"] \\
& & X' \arrow[r, "g'"'] \arrow[ur, phantom, "\mathrm{R}"] & Y'
\end{tikzcd}$$

Ad $\mathbf{Q}_{\mathrm{Cof}}.(2)$. Suppose given $A' \xleftarrow{a} A \xrightarrow{f} B$ in $\mathcal{M}_{\mathrm{bif}}$.

In $\mathcal{M}$, there exists a commutative diagram as follows.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] \arrow[ur, phantom, "\lrcorner"] & B'
\end{tikzcd}$$

Cf. $\mathbf{M}_{\mathrm{Cof}}.(1)$. Furthermore, we have a commutative diagram

$$\begin{tikzcd}
¡ \arrow[r] & B \arrow[r, "b"] & B' \arrow[rr] \arrow[dr, "i"'] & & ! \\
& & & \hat{B} \arrow[ur] &
\end{tikzcd}$$

in $\mathcal{M}$. In particular, $\hat{B} \in \operatorname{Ob} \mathcal{M}_{\text{bif}}$. Therefore, we have the commutative diagram

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dd, "a"'] \arrow[dr, phantom, "\lrcorner" near end] & & B \arrow[dl, "b"] \arrow[dd, "bi"] \\
& B' \arrow[dr, "i"] & \\
A' \arrow[ur, "f'"'] \arrow[rr, "f'i"'] & & \hat{B}
\end{tikzcd}$$

in $\mathcal{M}$. Thus, $(A, B, A', \hat{B})$ is an R-pushout in $\mathcal{M}_{\text{bif}}$, i.e., a quasi-pushout; cf. Remark 181.

Ad $\mathbf{Q}_{\text{Fib}}$.(2). This is dual to $\mathbf{Q}_{\text{Cof}}$.(2).

Ad (2). Since (1) holds, this follows from Remark 175; cf. Definition 156.(1). □

4.3 Further properties of $\mathcal{M}_{\text{bif}}$

In this §4.3 we give properties of $\mathcal{M}_{\text{bif}}$ which might turn out to be useful in the future.

Remark 194. *The following assertions* (1, 2) *hold.*

(1) *Suppose given* $A \xrightarrow{a} A'$ *in* $\mathcal{M}_{\text{bif}}$ *. Then the commutative diagram*

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "a"'] & A \arrow[d, "a"] \\
A' \arrow[r, equal] & A'
\end{tikzcd}$$

is an R-pushout; cf. Definition 180.

(2) *Suppose given* $B \xrightarrow{b} B'$ *in* $\mathcal{M}_{\text{bif}}$ *. Then the commutative diagram*

$$\begin{tikzcd}
B \arrow[r, equal] \arrow[d, "b"'] & B \arrow[d, "b"] \\
B' \arrow[r, equal] & B'
\end{tikzcd}$$

is an R-pullback; cf. Definition 190.

Proof. *Ad* (1). Since (A, A, A', A') is a pushout, this follows from Remark 182.

Ad (2). This is dual to (1). □

Lemma 195. *Suppose given a commutative diagram in $\mathcal{M}_{\mathrm{bif}}$ as follows.*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{R}"] & B \arrow[d, "b"] \arrow[ddr, bend left, "u"] & & \\
A' \arrow[r, "f'"'] \arrow[ddrr, bend right, "u'"'] & B' \arrow[dddr, bend right, "v'"'] \arrow[drr, bend left, "v"] & & \\
 & & X \arrow[r, "g"] \arrow[d, "x"'] \arrow[dr, phantom, "\mathrm{R}"] & Y \arrow[d, "y"] \\
 & & X' \arrow[r, "g'"'] & Y'
\end{tikzcd}$$

There exists $B' \xrightarrow{w} X$ *such that the following diagram commutes.*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{R}"] & B \arrow[d, "b"] \arrow[ddr, bend left, "u"] & & \\
A' \arrow[r, "f'"'] \arrow[ddrr, bend right, "u'"'] & B' \arrow[dddr, bend right, "v'"'] \arrow[drr, bend left, "v"] \arrow[dr, dotted, "w"] & & \\
 & & X \arrow[r, "g"] \arrow[d, "x"'] \arrow[dr, phantom, "\mathrm{R}"] & Y \arrow[d, "y"] \\
 & & X' \arrow[r, "g'"'] & Y'
\end{tikzcd}$$

Proof. By Remarks 181 and 191, there exist commutative diagrams in $\mathcal{M}$ as follows.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dd, "a"'] & & B \arrow[dl, "\hat{b}"] \arrow[dd, "b"] \\
 & \hat{B} \arrow[dr, "s"] & \\
A' \arrow[ur, "\hat{f}"] \arrow[rr, "f'"'] & & B'
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "t"] \arrow[dd, "x"'] & & Y \arrow[dd, "y"] \\
 & \hat{X} \arrow[ur, "\hat{g}"'] \arrow[dl, "\hat{x}"] & \\
X' \arrow[rr, "g'"'] & & Y'
\end{tikzcd}$$

Furthermore, there exist commutative diagrams in $\mathcal{M}$ as follows.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "\hat{b}"] \arrow[ddr, bend left, "u"] & \\
A' \arrow[r, "\hat{f}"'] \arrow[drr, bend right, "u'"'] & \hat{B} \arrow[dr, dotted, "\hat{u}"] & \\
 & & X
\end{tikzcd}
\qquad
\begin{tikzcd}
B' \arrow[dr, dotted, "\hat{v}"] \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "v'"'] & & \\
 & \hat{X} \arrow[r, "\hat{g}"] \arrow[d, "\hat{x}"'] & Y \arrow[d, "y"] \\
 & X' \arrow[r, "g'"'] & Y'
\end{tikzcd}$$

Since we have

$$\begin{array}{ccccccc}
b\hat{v} \cdot \hat{g} & = & bv & = & ug & = & ut \cdot \hat{g} \\
b\hat{v} \cdot \hat{x} & = & bv' & = & ux & = & ut \cdot \hat{x},
\end{array}$$

we obtain $b\hat{v} = ut$; cf. Definition 30. Since we have

$$\begin{array}{ccccccc} f'\hat{v} \cdot \hat{g} & = & f'v & = & u'g & = & u't \cdot \hat{g} \\ f'\hat{v} \cdot \hat{x} & = & f'v' & = & u'x & = & u't \cdot \hat{x}, \end{array}$$

we obtain $f'\hat{v} = u't$; cf. Definition 30. Since we have

$$\begin{array}{ccccccc} \hat{b} \cdot \hat{u}t & = & ut & = & b\hat{v} & = & \hat{b} \cdot s\hat{v} \\ \hat{f} \cdot \hat{u}t & = & u't & = & f'\hat{v} & = & \hat{f} \cdot s\hat{v}, \end{array}$$

we obtain $\hat{u}t = s\hat{v}$; cf. Definition 29.

Thus, we have the following commutative diagram in $\mathcal{M}$.

$$\begin{tikzcd} \hat{B} \arrow[r, "\hat{u}"] \arrow[d, "s"'] & X \arrow[d, "t"] \\ B' \arrow[r, "\hat{v}"'] \arrow[ur, dotted, "w"] & \hat{X} \end{tikzcd}$$

Cf. $\mathbf{A}_{\text{Lift}}$. In consequence, we have the following commutative diagram in $\mathcal{M}_{\text{bif}}$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \arrow[drr, bend left, "u"] \arrow[dl, phantom, "\mathrm{R}"] & & \\ A' \arrow[r, "f'"'] \arrow[drr, bend right, "u'"'] & B' \arrow[dr, "w"] \arrow[drrr, bend left, "v"] \arrow[ddr, bend right, "v'"'] & & \\ & & X \arrow[r, "g"] \arrow[d, "x"'] \arrow[dr, phantom, "\mathrm{R}"] & Y \arrow[d, "y"] \\ & & X' \arrow[r, "g'"'] & Y' \end{tikzcd}$$

□

Bibliography

[1] Aldridge K. Bousfield and Eric M. Friedlander. Homotopy theory of Γ-spaces, spectra, and bisimplicial sets. In M. G. Barratt and M. E. Mahowald, editors, *Geometric Applications of Homotopy Theory II*, page 80–130, Berlin, Heidelberg, 1978. Springer Berlin Heidelberg.

[2] Kenneth S. Brown. Abstract homotopy theory and generalized sheaf cohomology. *Trans. Amer. Math. Soc.*, 186:419–458, 1973. `doi:10.2307/1996573`.

[3] Ronald Brown. *Elements of modern topology.* McGraw-Hill, New York-Toronto-London, 1968.

[4] Ronald Brown and Philip R. Heath. Coglueing homotopy equivalences. *Math. Z.*, 113:313–325, 1970. `doi:10.1007/BF01110331`.

[5] William G. Dwyer, Philip S. Hirschhorn, Daniel M. Kan, and Jeffrey H. Smith. *Homotopy limit functors on model categories and homotopical categories*, volume 113 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2004.

[6] Peter Gabriel and Michel Zisman. *Calculus of fractions and homotopy theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer-Verlag New York, Inc., New York, 1967.

[7] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory.* Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, 2009. Reprint of the 1999 edition.

[8] Thomas E.W. Gunnarsson. *Abstract Homotopy Theory and Related Topics.* PhD thesis, Chalmers University of Technology Göteborg, 1978.

[9] Philip S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2003.

[10] Klaus Heiner Kamps. Kan-Bedingungen und abstrakte Homotopiethoerie. *Math. Z.*, 124:215–236, 1972. `doi:10.1007/BF01110801`.

[11] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1998.

[12] Daniel G. Quillen. *Homotopical algebra.* Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York, 1967.

[13] Andrei Radulescu-Banu. Cofibrations in homotopy theory. 2006. `arXiv:math/0610009v4`.

[14] Christopher L. Reedy. Homotopy theory of model categories. 1974. URL: `http://www-math.mit.edu/~psh/reedy.pdf`.

[15] Horst Schubert. *Categories.* Springer-Verlag, New York-Heidelberg, 1972. Translated from the German by Eva Gray.

[16] Sebastian Thomas. On the 3-arrow calculus for homotopy categories. *Homology Homotopy Appl.*, 13(1):89–119, 2011. `doi:10.4310/HHA.2011.v13.n1.a5`.

[17] Sebastian Thomas. *A calculus of fractions for the homotopy category of a Brown cofibration category.* PhD thesis, RWTH Aachen, 2012. URL: `http://publications.rwth-aachen.de/record/210492/files/4478.pdf`.

[18] Friedhelm Waldhausen. Algebraic K-theory of spaces. In *Algebraic and geometric topology*, volume 1126 of *Lecture Notes in Mathematics*, pages 318–419. Springer, Berlin, 1985.