ADJOINT FUNCTORS AND EQUIVALENCES OF RELATED CATEGORIES - TILTING THEORY FOR CATEGORIES

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Abstract.

For rings R, E, a bimodule ${}_RP_E$ creates an adjoint functor pair

$$\mathrm{Hom}_R(P,-) : R\text{-Mod} \to \text{E-Mod}, \quad \mathrm{P} \otimes_{\mathrm{E}} - : \text{E-Mod} \to \text{R-Mod},$$

with functorial morphisms - unit and counit - denoted by

$$\varepsilon_N : P \otimes_E \mathrm{Hom}_R(P,N) \to N, \quad p \otimes f \mapsto (p)f,$$
$$\eta_X : X \to \mathrm{Hom}_R(P, P \otimes_E X), \quad x \mapsto [p \mapsto p \otimes x],$$

where $N \in R$-Mod and $X \in E$-Mod. If $E = \mathrm{End}_R(P)$, P is $*$-*module* if and ony if η_X is an epimorphism, for all $X \in E$-Mod, and ε_N is a monomorphism, for all $N \in R$-Mod. The proof is based on special properties of module categories and the question arises as to what can be shown if one only imposes conditions on unit and counit and no other properties of the base category are known. For this we consider the following setting.

Given an adjunction $F \dashv G : \mathbb{B} \to \mathbb{A}$ between any categories $\mathbb{A}$, $\mathbb{B}$, with unit $\eta : \mathrm{Id}_{\mathbb{A}} \to GF$ and counit $\varepsilon : FG \to \mathrm{Id}_{\mathbb{B}}$, there are

(i) a monad $\mathbf{T} = (GF, G\varepsilon F, \eta)$ on $\mathbb{A}$, with category $\mathbb{A}_{\mathbf{T}}$ of $\mathbf{T}$-modules,
(ii) a comonad $\mathbf{S} = (FG, F\eta G, \varepsilon)$ on $\mathbb{B}$, with category $\mathbb{B}^{\mathbf{S}}$ of $\mathbf{S}$-comodules,
(iii) comparison functors $G_{[1]} : \mathbb{B} \to \mathbb{A}_{\mathbf{T}}$ and $F^{\underline{1}} : \mathbb{A} \to \mathbb{B}^{\mathbf{S}}$.

The indices on the comparison functors indicate that they are the beginning of (co)monadic decompositions. Note that $\mathbf{T}$ need not be a ring and $\mathbf{S}$ not a coring even if $\mathbb{A}, \mathbb{B}$ are Grothendieck categories. Clearly, if ε and η are isomorphisms, one gets an equivalence between the categories $\mathbb{A}$ and $\mathbb{B}$. The case that ε is (pointwise) an extremal monomorphism and η is (pointwise) an (extremal) epimorphism is considered in [1]. It is not difficult to see that one gets an equivalence between $\mathbb{A}_{\mathbf{T}}$ and $\mathbb{B}^{\mathbf{S}}$ if and only if $\mathbf{T}$ is an idempotent monad (equivalently, $\mathbf{S}$ is an idempotent comonad).

In the talk we report on further progress in related research and present equivalences of other categories obtained by a given adjunction.

References

[1] Clark, J. and Wisbauer, R., *Idempotent monads and pairs of $\star$-functors*, J. Pure Applied Algebra **215**(2), 145-153 (2011).