Gendo-symmetric algebras, canonical comultiplication, bar cocomplex and dominant dimension

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Abstract

To each endomorphism algebra A of a generator over a symmetric algebra, first a canonical comultiplication (possibly without a counit) is constructed and then a bar cocomplex. The algebras A are characterised by the existence of these data. The dominant dimension of A is shown to be determined by the exactness of the cocomplex at its beginning terms.

Keywords Endomorphism algebra, Bar cocomplex, Dominant dimension
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1 Introduction

The aim of this article is to exhibit a so far hidden structure on a class of finite dimensional associative algebras that are of interest both from an abstract and an applied point of view. We are going to derive the existence of a *comultiplication* from homological and ring theoretical properties of finite dimensional associative algebras, and we are going to use this comultiplication to define a *bar cocomplex* from which we can read off a crucial homological property, the *dominant dimension*, of the algebra we started with. The class of algebras covered by our results includes all symmetric algebras as well as many quasi-hereditary algebras occurring in algebraic Lie theory. We call these algebras gendo-symmetric and we show that they are characterised by admitting a comultiplication with the properties used for defining the bar cocomplex.

The term 'gendo-symmetric' is meant to indicate that one characterisation of these algebras is as endomorphism rings of generators (modules containing each indecomposable projective module at least once as a direct summand) over a symmetric algebra.

The main results of this article are briefly summarised as follows:

Main results. *(a) (Theorem 2.4 and Proposition 2.8) Let A be a gendo-symmetric algebra. Then A has a coassociative comultiplication that is an A-bimodule morphism. In addition, there is a compatible counit if and only if A is symmetric.*

(b) (Theorem 3.6) Let A be a gendo-symmetric algebra. Then there exists a bar cocomplex

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for A, using the comultiplication in (a). Conversely, let B be an algebra admitting a comultiplication such that a bar cocomplex can be defined as for gendo-symmetric algebras. Then B is gendo-symmetric.

(c) (Theorem 3.6 and Theorem 3.8) Let A be gendo-symmetric. Then exactness of the bar cocomplex in (b) and dominant dimension of A determine each other; the bar cocomplex is exact at places $0, \ldots, n-1$ *if and only if* $\operatorname{dom.dim}(A) \geq n$*. Let M be an A-module of dominant dimension at least* 2*. Then changing coefficients in the bar cocomplex produces a complex whose exactness corresponds in the same way with* $\operatorname{dom.dim}(M)$*.*

Gendo-symmetric algebras are defined by a special case of the *Morita-Tachikawa correspondence*, which shows that algebras of dominant dimension at least two are exactly the *endomorphism rings of generator-cogenerators* over an algebra (which in our case is assumed to be symmetric). A generator-cogenerator is a module that up to isomorphism contains each indecomposable projective or injective module at least once as a direct summand. An algebra A of dominant dimension at least two has a faithful projective-injective module, say Ae, and there is a *double centraliser property* on this bimodule Ae, that is, $A \cong \operatorname{End}_{eAe}(Ae)$. Classical Schur-Weyl duality between Schur algebras and group algebras of symmetric groups as well as Soergel's structure theorem for the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ are examples of this situation [12]. Hence the class of gendo-symmetric algebras contains these and many other examples from algebraic Lie theory as well as symmetric algebras and Auslander algebras. Our construction of a comultiplication on a gendo-symmetric algebra A uses the homological properties of A. Even for the examples from algebraic Lie theory there is no visible connection to the Hopf algebras on group algebras and universal enveloping algebras arising in this context. These Hopf algebra structures are, in fact, not known to induce comultiplications on Schur algebras or blocks of $\mathcal{O}$. We will, however, show that the new comultiplication constructed here can be used to reprove, for instance, classical Schur-Weyl duality without reference to dominant dimension.

The characterisation of gendo-symmetric algebras in Theorem 3.6 should be compared with Abrams' result that *Frobenius algebras* are exactly the finite-dimensional algebras with a comultiplication and a compatible counit. This result in turn is the non-commutative extension of the 'equivalence' between two-dimensional topological quantum field theories and commutative Frobenius algebras. In contrast to the class of Frobenius algebras, the class of gendo-symmetric algebras includes many interesting non-semisimple algebras of finite global dimension.

The dominant dimension of a Schur algebra $S(n, r)$ in the 'stable range' $n \geq r$ has been determined explicitly in [6]. There, we have shown that it controls exactly the *quality of the Schur functor*. That is, dominant dimension determines the degrees in which the Schur functor identifies *Yoneda extension groups* between Weyl filtered modules and their images under the Schur functor, which are dual Specht filtered. In more general contexts, dominant dimension also can be characterised by vanishing of extension groups. The bar cocomplex introduced here provides a new tool for determining the dominant dimension of an algebra or a module, which at the same time relates this dimension to other concepts of homological algebra, such as the bar complex. In [5], the comultiplication on classical Schur algebras will be constructed explicitly and it will be related to a coefficient space called 'Doty coalgebra'. These results go beyond the 'stable range' discussed at the end of the present paper, and they also extend our knowledge on dominant dimension of Schur algebras.

This article is organised as follows: In the second section we first collect results on dominant dimension and recall results of [7] in order to define the class of gendo-symmetric algebras by four equivalent conditions. Then we construct a comultiplication on gendo-symmetric algebras, directly from the defining double centraliser property, and we check the properties of this comultiplication. Moreover, we show that in addition there is a counit for this comultiplication if and only if the algebra is symmetric.

In the third section we define an analogue of the bar complex, using the new comultiplication instead of multiplication. Using this complex (called the bar cocomplex), we characterise gendo-symmetric algebras by the existence of a comultiplication with natural properties. At the same time we show that the bar cocomplex of a gendo-symmetric algebra determines precisely the dominant dimension of the algebra. At the end of this section, we discuss the example of a classical Schur algebra in the 'stable range' and explain how this instance of Schur-Weyl duality can be reproven using our new tools. A more general discussion of this situation, covering also the more difficult case of the classical Schur algebra beyond the 'stable range', can be found in the parallel article [5].

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2 Gendo-symmetric algebras and a new comultiplication

After first collecting some information on dominant dimension, we will define the main objects of this article, gendo-symmetric algebras. Then we construct a comultiplication on these algebras and check its properties.

Throughout, all algebras and modules are finite dimensional over a field k unless stated otherwise. By A-mod and A-bimod, we denote the categories of left modules and bimodules respectively over an algebra A, and by D the usual k-duality functor $\mathrm{Hom}_k(-, k)$.

2.1 Dominant dimension

Let A be an algebra. The *dominant dimension* of a left A-module M, which we denote by $\mathrm{dom.dim}\, M$, is the maximal number t (or ∞) having the following property: let $0 \to M \to I^0 \to I^1 \to \cdots \to I^t \to \cdots$ be a minimal injective resolution of M, then I^j is projective for all $j < t$ (or ∞). The dominant dimension of a right A-module is defined similarly. It is clear that $\mathrm{dom.dim}\, {}_AA = \mathrm{dom.dim}\, A_A$ for which we write $\mathrm{dom.dim}\, A$.

If $\mathrm{dom.dim}\, A \geq 1$, then there exists a unique (up to isomorphism) minimal faithful right A-module. It must be projective and injective, therefore of the form eA for some idempotent e in A. Note eA is a generator-cogenerator as a left eAe-module. If further $\mathrm{dom.dim}\, A \geq 2$, then eA is a faithfully balanced bimodule. That is, there is a double centraliser property, namely $A \cong \mathrm{End}_{eAe}(eA)$ canonically. The dominant dimension of A as an A-bimodule is called the Nakayama dimension of A/k by Müller in [14] where he proved that it is invariant under Morita equivalences and under arbitrary field extensions, and that it equals the dominant dimension of A.

Lemma 2.1 *Let A be an algebra. If there is an injective morphism $A \to \mathrm{Hom}_A(\mathrm{D}(A), {}_AA)$ of right A-modules, then* $\mathrm{dom.dim}\, A \geq 1$.

Proof. See [7, Lemma 3.1]. □

Lemma 2.2 *Let A be an algebra and e an idempotent in A such that Ae is faithful. Then an element $z \in A$ belongs to $Z(A)$, the center of A if and only if $zae = aeze$ for any $a \in A$. In particular the morphism $\phi : Z(A) \to Z(eAe)$ sending z to eze is injective. Moreover, if $\mathrm{End}_{eAe}(Ae) \cong A$ then ϕ is an isomorphism.*

Proof. Assume $zae = aeze$ for all $a \in A$. Since Ae is faithful, there exists an embedding of left A-modules $\varphi : A \to (Ae)^{\oplus m}$ for some m. Let $\varphi(1) = (a_1e, \ldots, a_me)$ where $a_i \in A$. Then for any $x \in A$,

$$\varphi(xz) = (xza_1e, \ldots, xza_me) = (xa_1eze, \ldots, xa_meze) = (zxa_1e, \ldots, zxa_me) = \varphi(zx).$$

Consequently $xz = zx$ for any $x \in A$, i.e., $z \in Z(A)$. The other direction is trivial.

To see ϕ is injective, observe that $ez = 0$ for $z \in Z(A)$ implies $\varphi(z) = 0$ hence $z = 0$. If further $\mathrm{End}_{eAe}(Ae) \cong A$, then any $u \in Z(eAe)$ regarded as an eAe-endomorphism of Ae by right multiplication defines a unique element w in A such that $wae = aeu$ for any $a \in A$. Clearly $u = ewe = \phi(w)$ and $\varphi(xw) = (xwa_1e, \ldots, xwa_ne) = (xa_1eu, \ldots, xa_neu) = (wxa_1e, \ldots, wxa_ne) = \varphi(wx)$ for any $x \in A$. As φ is injective, this implies $w \in Z(A)$. □

2.2 A comultiplication for gendo-symmetric algebras

Now we are going to introduce the class of algebras studied in this article. The following definition is based on [7, Theorem 3.2], which provides the equivalence of the four conditions.

Definition 2.3 *A finite dimensional k-algebra A is called* **gendo-symmetric** *if it satisfies one of the following equivalent conditions:*

(a) $\mathrm{dom.dim}\, A \geq 2$ *and* $\mathrm{D}(Ae) \cong eA$ *as (eAe, A)-bimodules, where Ae is a basic faithful projective injective A-module,*

(b) $\mathrm{Hom}_A({}_A\mathrm{D}(A), {}_AA) \cong A$ *as A-bimodules,*

(c) $\mathrm{D}(A) \otimes_A \mathrm{D}(A) \cong \mathrm{D}(A)$ *as A-bimodules,*

(d) *A is the endomorphism algebra of a generator over a symmetric algebra.*

Construction of the comultiplication.

Let A be gendo-symmetric. The definition contains several isomorphisms, which will be used as ingredients for a multiplication on $\mathrm{D}(A)$. Dualising then yields a comultiplication on A. We recall now the isomorphisms from the proof of [7, Theorem 3.2].

First we observe that the bimodule isomorphisms in (b), (c) are unique up to multiples of invertible central elements in A, and by Lemma 2.2 so is the isomorphism in (a). Fix an (eAe, A)-bimodule isomorphism $\iota : eA \cong \mathrm{D}(Ae)$. By the double centraliser property $\mathrm{End}_{eAe}(eA) \cong A$ (Section 2.1), there is an A-bimodule isomorphism $\gamma : Ae \otimes_{eAe} eA \cong \mathrm{D}(A)$ such that $\gamma(ae \otimes eb)(x) = \iota(ebx)(ae)$ for $a, b, x \in A$. Hence there is an isomorphism in (c)

$$\mathrm{D}(A) \otimes_A \mathrm{D}(A) \cong (Ae \otimes_{eAe} eA) \otimes_A (Ae \otimes_{eAe} eA) \cong Ae \otimes_{eAe} eA \xrightarrow{\gamma} \mathrm{D}(A)$$

where the first isomorphism is $\gamma^{-1} \otimes_A \gamma^{-1}$.

Let m be the composition of the canonical A-bimodule morphism $\mathrm{D}(A) \otimes_k \mathrm{D}(A) \to \mathrm{D}(A) \otimes_A \mathrm{D}(A)$ with the isomorphism in (c) above:

$$m : \mathrm{D}(A) \otimes_k \mathrm{D}(A) \to \mathrm{D}(A) \otimes_A \mathrm{D}(A) \cong D(A).$$

Below, we will check that m defines an associative multiplication on $\mathrm{D}(A)$, possibly without a unit. Dualising m yields

$$\Delta : A \longrightarrow {}_AA \otimes_k A_A$$

such that $(f \otimes g)\Delta(a) = m(g \otimes f)(a)$ for any $f, g \in \mathrm{D}(A)$ and $a \in A$.

We also will use the isomorphism in (b)

$$\mathrm{Hom}_A(\mathrm{D}(A), {}_AA) \cong \mathrm{Hom}_A(Ae \otimes_{eAe} eA, A) \cong \mathrm{Hom}_{eAe}(eA, eA) \cong A$$

where the first isomorphism is $\mathrm{Hom}_A(\gamma, A)$. Let

$$\Theta : \mathrm{D}(A) \to {}_AA$$

be the inverse image of $1 \in A$ under this isomorphism in (b). Then $(\Theta \circ \gamma)(ae \otimes eb) = aeb$ for $a, b \in A$, in particular Θ is an A-bimodule morphism with $e\Theta = \iota^{-1}$. Moreover, we will check below that $\Theta : \mathrm{D}(A) \to A$ preserves multiplications.

Theorem 2.4 . *Let A be gendo-symmetric. Then*

$$\Delta : A \longrightarrow {}_AA \otimes_k A_A$$

is a coassociative comultiplication on A.

The proof of the Theorem will consist of the two Lemmas below. Prior to that we remark that the above construction of m and of Δ can be carried out in a more general situation, which requires more technical assumptions; we refrain from providing details and stick to the well-defined class of gendo-symmetric algebras.

We keep the notations introduced above.

Lemma 2.5 *The map m satisfies*

$$m(1 \otimes m) = m(m \otimes 1)$$

as k-morphisms from $\mathrm{D}(A) \otimes_k \mathrm{D}(A) \otimes_k \mathrm{D}(A)$ to $\mathrm{D}(A)$, and

$$\Theta(m(f \otimes g)) = \Theta(f)\Theta(g)$$

for any $f, g \in \mathrm{D}(A)$.

Proof. For any $a, b, c, d, x, y \in A$, the definitions of m and Θ imply equalities

$$m(\gamma(ae \otimes eb) \otimes \gamma(ce \otimes ed)) = \gamma(ae \otimes eb \otimes ce \otimes ed) = \gamma(aebce \otimes ed),$$

$$\begin{aligned} m(1 \otimes m)(\gamma(ae \otimes eb) \otimes \gamma(ce \otimes ed) \otimes \gamma(xe \otimes ey)) &= m(\gamma(ae \otimes eb) \otimes \gamma(cedxe \otimes ey)) \\ &= \gamma(aebcedxe \otimes ey) \\ m(m \otimes 1)(\gamma(ae \otimes eb) \otimes \gamma(ce \otimes ed) \otimes \gamma(xe \otimes ey)) &= m(\gamma(aebce \otimes ed) \otimes \gamma(xe \otimes ey)) \\ &= \gamma(aebcedxe \otimes ey) \end{aligned}$$

and $(\Theta \circ m)(\gamma(ae \otimes eb) \otimes \gamma(ce \otimes ed)) = \Theta(\gamma(aebce \otimes ed)) = aebced = (aeb)(ced)$. As γ is an isomorphism, the results follow. □

Lemma 2.6 *Let* $\Delta : A \to {}_AA \otimes_k A_A$ *be as above. Then*

(1) Δ *is an* A*-bimodule morphism.*

(2) $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$.

(3) $\mathrm{Im}(\Delta) = \{\sum u_i \otimes v_i \mid \sum u_i a \otimes v_i = \sum u_i \otimes a v_i,\ \forall a \in A\}$.

Proof. (1) By definition of Δ, there are equalities for $a, b, c, d, u, v \in A$

$$\begin{aligned} (\gamma(ae \otimes eb) \otimes \gamma(ce \otimes ed))\Delta(uv) &= m(\gamma(ce \otimes ed) \otimes \gamma(ae \otimes eb))(uv) \\ &= \gamma(cedae \otimes eb)(u1v) = \gamma(vcedae \otimes ebu)(1) \\ (\gamma(ae \otimes eb) \otimes \gamma(ce \otimes ed))(u\Delta(v)) &= (\gamma(ae \otimes ebu) \otimes \gamma(ce \otimes ed))\Delta(v) \\ &= \gamma(cedae \otimes ebu)(v) = \gamma(vcedae \otimes ebu)(1) \\ (\gamma(ae \otimes eb) \otimes \gamma(ce \otimes ed))(\Delta(u)v) &= (\gamma(ae \otimes eb) \otimes \gamma(vce \otimes ed))\Delta(u) \\ &= \gamma(vcedae \otimes eb)(u) = \gamma(vcedae \otimes ebu)(1) \end{aligned}$$

So $\Delta(uv) = u\Delta(v) = \Delta(u)v$, i.e., Δ is an A-bimodule morphism.

(2) For any $f, g, h \in \mathrm{D}(A)$ and $u \in A$, let $\Delta(u) = \sum u_i \otimes v_i$, then

$$\begin{aligned} (f \otimes g \otimes h)(1 \otimes \Delta)\Delta(u) &= \sum f(u_i)(g \otimes h)\Delta(v_i) = \sum f(u_i)m(h \otimes g)(v_i) \\ &= (f \otimes m(h \otimes g))\Delta(u) = m(m(h \otimes g) \otimes f)(u) \\ (f \otimes g \otimes h)(\Delta \otimes 1)\Delta(u) &= \sum (f \otimes g)\Delta(u_i)h(v_i) = \sum m(g \otimes f)(u_i)h(v_i) \\ &= (m(g \otimes f) \otimes h)\Delta(u) = m(h \otimes m(g \otimes f))(u) \end{aligned}$$

By Lemma 2.5, it follows that $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$.

(3) Let $\Sigma = \{\sum u_i \otimes v_i \mid \sum u_i a \otimes v_i = \sum u_i \otimes a v_i,\ \forall a \in A\}$. Then $\mathrm{Im}(\Delta) \subseteq \Sigma$. Indeed, for any $u \in A$, let $\Delta(u) = \sum u_i \otimes v_i$. Then for any $f, g \in \mathrm{D}(A)$ and $a \in A$

$$(f \otimes g)(\sum u_i a \otimes v_i) = (af \otimes g)\Delta(u) = m(g \otimes af)(u)$$

$$(f \otimes g)(\sum u_i \otimes a v_i) = (f \otimes ga)\Delta(u) = m(ga \otimes f)(u)$$

By definition of m, there is an equality $m(ga\otimes f) = m(g\otimes af)$. Thus $\Delta(u) \in \Sigma$. Conversely for each $\theta = \sum u_i \otimes v_i \in \Sigma$, there is a k-linear map $\mathrm{D}(A) \to A$, denoted by $\hat{\theta}$, such that $\hat{\theta}(f) = \sum f(u_i)v_i$ for any $f \in \mathrm{D}(A)$. Since for any $a \in A$, $\sum u_i a\otimes v_i = \sum u_i \otimes a v_i$, it follows

$$\hat{\theta}(af) = \sum(af)(u_i)v_i = \sum f(u_i a)v_i = \sum f(u_i)av_i = a\hat{\theta}(f)$$

$\hat{\theta}$ is a left A-module morphism, i.e., $\hat{\theta} \in \mathrm{Hom}_A(\mathrm{D}(A), A) \cong A$. Sending θ to $\hat{\theta}$ defines an injective map $\Sigma \to \mathrm{Hom}_A(\mathrm{D}(A), {}_AA)$. On the other hand, m being surjective implies that Δ is injective. It now follows from $\mathrm{Im}(\Delta) \subseteq \Sigma$ that $\mathrm{Im}(\Delta) = \Sigma$. □

Any k-linear map $\tilde{\Delta} : A \to A \otimes_k A$ induces a map $\mu : \mathrm{D}(A) \otimes_k \mathrm{D}(A) \to \mathrm{D}(A)$ such that $\mu(f \otimes g)(a) = (g \otimes f)\tilde{\Delta}(a)$ for any $f, g \in \mathrm{D}(A)$ and $a \in A$. If $\tilde{\Delta}$ satisfies (1), (2) and (3) above, then μ induces an A-bimodule isomorphism $\mathrm{D}(A) \otimes_A \mathrm{D}(A) \cong \mathrm{D}(A)$. Indeed, by the proof of Lemma 2.6, (1) implies μ is an A-bimodule morphism and (3) implies that μ factors through $\mathrm{D}(A) \otimes_A \mathrm{D}(A)$ and $\mathrm{Im}(\tilde{\Delta}) = \mathrm{Hom}_A(\mathrm{D}(A), {}_AA) \cong A$ as A-bimodules. Therefore, $\tilde{\Delta}$ and Δ differ only by an invertible central element z in A, so that $\tilde{\Delta}(a) = \Delta(az)$ for all $a \in A$.

Corollary 2.7 *The comultiplication Δ is unique up to precomposing it with multiplication by an invertible central element.*

Thus we call Δ the *canonical comultiplication* attached to the gendo-symmetric algebra A.

For any left A-module M, the restriction map $M \cong \mathrm{Hom}_A(A, M) \to \mathrm{Hom}_{eAe}(eA, eM)$ induces

$$\eta_M : M \to \mathrm{Hom}_{eAe}(eA, eM) \cong \mathrm{Hom}_A(Ae \otimes_{eAe} eA, M) \cong \mathrm{Hom}_A(\mathrm{D}(A), M) \hookrightarrow {}_AA \otimes_k M$$

where the second isomorphism is $\mathrm{Hom}_A(\gamma, M)$. Clearly η_M is a left A-module morphism.

Proposition 2.8 *Let A be a gendo-symmetric k-algebra with the canonical comultiplication $\Delta : A \to {}_AA \otimes_k A_A$. Then*

(1) *(A, Δ) has a counit if and only if A is symmetric.*

(2) *Let $\Delta(1) = \sum x_i \otimes y_i$. Then $\Delta(1) = \sum y_i \otimes x_i$ and for any $a, b \in A$,*

$$\sum \gamma(ae \otimes eb)(y_i)x_i = aeb = \sum \gamma(ae \otimes eb)(x_i)y_i.$$

(3) *$\Delta = \eta_A$ and for any left A-module M, there are identities*

  (i) *$\eta_M = \Delta \otimes_A \mathrm{Id}_M$, i.e., $\eta_M(m) = \sum x_i \otimes y_i m$ for any $m \in M$.*

  (ii) *$(\Delta \otimes 1)\eta_M = (1 \otimes \eta_M)\eta_M$.*

Proof. (1) If $\delta \in \mathrm{D}(A)$ is a counit of (A, Δ), then $m(\delta \otimes f)(a) = (f \otimes \delta)\Delta(a)$ which equals $f(1 \otimes \delta)\Delta(a) = f(a)$, similarly $m(f \otimes \delta)(a) = (\delta \otimes f)\Delta(a) = f(a)$ for any $a \in A$. So δ is a unit of $(\mathrm{D}(A), m)$. Let u be the image of δ under $\Theta : \mathrm{D}(A) \to A$. Then, $\Theta m(\delta \otimes \gamma(ae \otimes eb)) = \Theta(\gamma(ae \otimes eb))$ implies $uaeb = aeb$ for any $a, b \in A$ by Lemma 2.5. Hence $u = 1$ since AeA is a faithful left A-module. As a result, Θ is surjective as an A-bimodule morphism and thus an isomorphism by comparing dimensions. So A is symmetric.

Conversely, if A is symmetric, then there is an A-bimodule isomorphism $\beta : \mathrm{D}(A) \cong A$, hence an invertible central element z in A such that $\beta^{-1}(\beta(f)\beta(g)) = zm(f \otimes g)$ for all $f, g \in \mathrm{D}(A)$. Let $\delta = z\beta^{-1}(1)$. Then $\beta(m(f \otimes \delta)) = \beta(f)$, i.e., $m(f \otimes \delta) = f$ and similarly $m(\delta \otimes f) = f$. So δ is an unit of $(\mathrm{D}(A), m)$, hence a counit of (A, Δ).

(2) Under the identification $\gamma : Ae \otimes_{eAe} eA \cong \mathrm{D}(A)$ so that $\gamma(ae \otimes eb)(x) = \iota(ebx)(ae)$ for $a, b, x \in A$, we have by definition of $\Delta : A \to A \otimes_k A$, for any $a, b, c, d \in A$,

$$\gamma(aebce \otimes ed)(1) = (\gamma(ce \otimes ed) \otimes \gamma(ae \otimes eb))\Delta(1) = \sum \iota(edx_i)(ce) \cdot \iota(eby_i)(ae).$$

That is $\iota(ed)(aebce) = \iota(edx_i)(\sum \iota(eby_i)(ae) \cdot ce) = \iota(ed)(\sum \iota(eby_i)(ae) \cdot x_i ce)$. So

$$aebce = \sum \iota(eby_i)(ae) \cdot x_i ce = \sum \gamma(ae \otimes eb)(y_i) \cdot x_i ce.$$

Since Ae is faithful as a left A-module, it follows that $aeb = \sum \gamma(ae \otimes eb)(y_i) \cdot x_i$. On the other hand, $\gamma(ce \otimes ed)(aeb) = \iota(edaeb)(ce) = \iota(eb)(cedae)$ and

$$\begin{aligned} \gamma(ce \otimes ed)(\sum \gamma(ae \otimes eb)(x_i)y_i) &= (\gamma(ae \otimes eb) \otimes \gamma(ce \otimes ed)) \sum x_i \otimes y_i \\ &= (\gamma(ae \otimes eb) \otimes \gamma(ce \otimes ed))\Delta(1) \\ &= m(\gamma(ce \otimes ed) \otimes \gamma(ae \otimes eb))(1) \\ &= \gamma(cedae \otimes eb)(1) = \iota(eb)(cedae) \\ &= \gamma(ce \otimes ed)(aeb) \end{aligned}$$

Therefore $\sum \gamma(ae \otimes eb)(y_i)x_i = aeb = \sum \gamma(ae \otimes eb)(x_i)y_i$. In particular $\sum x_i \otimes y_i = \sum y_i \otimes x_i$.

(3) By definition of η_M, the map $\eta_M(m) : \mathrm{D}(A) \to M$ satisfies $\eta_M(m)(\gamma(ae \otimes eb)) = aebm$ which equals $\sum \gamma(ae \otimes eb)(x_i)y_i m$ by (2) for any $m \in M$. It follows that $\eta_M(m) = \sum x_i \otimes y_i m$ and in particular $\Delta = \eta_A$. The identity in (ii) follows from

$$\begin{aligned} (\Delta \otimes 1)\eta_M(m) &= \sum \Delta(x_j) \otimes y_j m = \sum_{i,j} x_i \otimes y_i x_j \otimes y_j m \\ (1 \otimes \eta_M)\eta_M(m) &= \sum x_i \otimes \eta_M(y_i m) = \sum_{i,j} x_i \otimes x_j \otimes y_j y_i m \end{aligned}$$

and $\sum_j y_i x_j \otimes y_j = \Delta(y_i) = \sum_j x_j \otimes y_j y_i$ in $A \otimes_k A$. □

Remark 2.9 *Gendo-symmetric algebras extend the subclass $\mathscr{A}$ of quasi-hereditary algebras introduced in [6]. These include the algebras on both sides of classical Schur-Weyl duality and of Soergel's structure theorem for the BGG-category O.*

In [9], Kerner and Yamagata investigated two variations of gendo-symmetric algebras. One, called Morita algebras, is defined by replacing symmetric algebras with self-injective algebras in Definition 2.3(d). The other one is defined by relaxing the condition on the bimodule isomorphisms in Definition 2.3(c) to be just one-sided module isomorphisms. It has been pointed out in [9] that the latter variation is not closed under Morita equivalences.

With respect to Morita equivalences, gendo-symmetric algebras are well-behaved. Unlike Hopf algebras, but like symmetric algebras their defining properties, and thus also the existence of a comultiplication are preserved under Morita equivalences.

Proposition 2.10 *Let A be a gendo-symmetric k-algebra.*

(1) *If F/k is any field extension, then $A_F = A \otimes_k F$ is gendo-symmetric.*

(2) *If B is Morita equivalent to A, then B is gendo-symmetric.*

Proof. (1) Note that $\mathrm{D}_F(A) := \mathrm{Hom}_F(A_F, F) \cong \mathrm{D}(A) \otimes_k F$ and $(\mathrm{D}(A) \otimes_A \mathrm{D}(A)) \otimes_k F$ is canonically isomorphic to $\mathrm{D}_F(A) \otimes_{A_F} \mathrm{D}_F(A)$ as A_F-bimodules. By Definition 2.3(c), A being gendo-symmetric implies so for A_F.

(2) Let B be Morita equivalent to A, i.e., there is a projective generator P in A-mod such that $B \cong \mathrm{End}_A(P)^{op}$. Then $M \mapsto \mathrm{Hom}_A(P, A) \otimes_A M \otimes_A P$ defines an equivalence $\mathcal{F}$ from A-bimod to B-bimod satisfying $\mathcal{F}(X \otimes_A Y) \cong \mathcal{F}(X) \otimes_B \mathcal{F}(Y)$ for any X, Y in A-bimod and $\mathcal{F}(A) \cong B$, $\mathcal{F}(\mathrm{D}(A)) \cong \mathrm{D}(B)$. Now A being gendo-symmetric yields

$$\mathrm{D}(B) \otimes_B \mathrm{D}(B) \cong \mathcal{F}(\mathrm{D}(A) \otimes_A \mathrm{D}(A)) \cong \mathcal{F}(\mathrm{D}(A)) \cong \mathrm{D}(B)$$

in B-bimod. By Definition 2.3(c) B is gendo-symmetric. $\square$

In addition to the general characterisation of dominant dimension due to Müller [14], see also [4, 6], there is the following one for gendo-symmetric algebras [7].

Proposition 2.11 *Let A be a gendo-symmetric algebra and $n \geq 2$ a natural number. Then for any left A-module M, $\mathrm{dom.dim}\, M \geq n$ if and only if $\mathrm{Hom}_A(\mathrm{D}(A), M) \cong M$ and $\mathrm{Ext}^i_A(\mathrm{D}(A), M) = 0$ for $i = 1, 2, \ldots, n-2$.*

As $\mathrm{Hom}_A(\mathrm{D}(A), M) \cong M$ implies $\mathrm{dom.dim}\, M \geq 2$, the following canonical map

$$M \cong \mathrm{Hom}_A(A, M) \to \mathrm{Hom}_{eAe}(eA, eM) \to \mathrm{Hom}_A(Ae \otimes_{eAe} eA, M) \cong \mathrm{Hom}_A(\mathrm{D}(A), M)$$

is an isomorphism. Thus the isomorphism $M \cong \mathrm{Hom}_A(\mathrm{D}(A), M)$ in the Proposition above may be required to be canonical.

3 Bar cocomplex, characterising gendo-symmetric algebras and determining their dominant dimension

In the first two subsections, an analogue of the bar complex is defined that uses the comultiplication of gendo-symmetric algebras. This complex then is used to formulate and prove the main results of this article, characterising the class of gendo-symmetric algebras and determining their dominant dimension.

3.1 Bar complex

Let Λ be an arbitrary associative k-algebra (not necessarily unital or finite dimensional) and $m : \Lambda \otimes_k \Lambda \to \Lambda$ be the multiplication map. Consider the bar complex:

$$\mathscr{B}^{\bullet}_{\Lambda} : \qquad \cdots \xrightarrow{\partial_n} \Lambda^{\otimes n+1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} \Lambda \otimes_k \Lambda \otimes_k \Lambda \xrightarrow{\partial_1} \Lambda \otimes_k \Lambda \xrightarrow{m} \Lambda \to 0$$

where the boundary map $\partial_n : \Lambda^{\otimes n+2} \to \Lambda^{\otimes n+1}$ is defined by

$$\partial_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes m(a_i \otimes a_{i+1}) \otimes a_{i+2} \otimes \cdots \otimes a_{n+1}$$

If Λ is unital, the bar complex $\mathscr{B}^\bullet_\Lambda$ is known as the bar resolution which is exact with the homotopy map $s_n : \Lambda^{\otimes n} \to \Lambda^{\otimes n+1}$ given by $s_n(a_0 \otimes \cdots \otimes a_{n-1}) = 1 \otimes a_0 \otimes \cdots \otimes a_{n-1}$. However, if Λ is not unital, the complex $\mathscr{B}^\bullet_\Lambda$ will be far from being exact. In such case, let $R = k \oplus \Lambda$ be the unital algebra with multiplication given by $(x + a) \cdot (y + b) = xy + ya + xb + m(a, b)$ for $x, y \in k$ and $a, b \in \Lambda$. Since R is an augmented k-algebra with Λ being the augmented ideal, we can consider the normalised bar resolution of R:

$$\overline{\mathscr{B}}^\bullet_R : \qquad \cdots \to R \otimes_k \Lambda^{\otimes n} \otimes_k R \to \cdots \to R \otimes_k \Lambda \otimes_k R \to R \otimes_k R \to R \to 0$$

where the boundary maps are given by the same recipe as above. Using these notations we get:

Lemma 3.1 *There is a canonical isomorphism* $\Lambda \otimes_R \overline{\mathscr{B}}^\bullet_R \otimes_R \Lambda \cong \mathscr{B}^\bullet_\Lambda$ *if and only if* m *induces an isomorphism* $\Lambda \otimes_R \Lambda \cong \Lambda$. *In this case* $\mathrm{H}^{\geq -1}(\mathscr{B}^\bullet_\Lambda) = 0$ *and* $\mathrm{H}^{-i-1}(\mathscr{B}^\bullet_\Lambda) = \mathrm{Tor}^R_i(\Lambda, \Lambda)$ *for* $i \geq 1$.

Proof. $\overline{\mathscr{B}}^\bullet_R$ is a free resolution of R in R-bimod and Λ is an ideal of R. So $\Lambda \otimes_R \overline{\mathscr{B}}^\bullet_R \otimes_R \Lambda$ has cohomologies $\mathrm{Tor}^R_i(\Lambda, \Lambda)$ at degree $-i-1$ for $i \geq 1$ and zero at higher degrees. On the other hand, there is a natural chain map $\{\alpha_i\}$ from $\Lambda \otimes_R \overline{\mathscr{B}}^\bullet_R \otimes_R \Lambda$ to $\mathscr{B}^\bullet_\Lambda$ where the maps α_i are canonical isomorphisms for $i \leq -1$ and $\alpha_0 : \Lambda \otimes_R \Lambda \to \Lambda$ is given by $\alpha_0(a, b) = m(a, b)$ for $a, b \in A$. Therefore, $\{\alpha_i\}$ is an isomorphism if and only if α_0 is an isomorphism. $\square$

Now let A be a finite dimensional k-algebra with a unit. Assume that there is a k-linear map $m : {}_A\mathrm{D}(A) \otimes_k \mathrm{D}(A)_A \to \mathrm{D}(A)$ satisfying the following properties

(m1) m is an A-bimodule morphism.

(m2) m is surjective.

(m3) $m(m \otimes 1) = m(1 \otimes m)$.

(m4) $m(f \cdot a \otimes g) = m(f \otimes a \cdot g)$ for any $f, g \in \mathrm{D}(A)$ and $a \in A$.

Then $(\mathrm{D}(A), m)$ is an associative k-algebra, not necessarily unital. Let $R = k \oplus \mathrm{D}(A)$ be the augmented k-algebra associated with $(\mathrm{D}(A), m)$. Then $\mathrm{D}(A)$ carries both the structure of an R-bimodule and of an A-bimodule and by (m1) it is also an (A, R)-module and an (R, A)-module. Observe that (m3) implies m factors through $\overline{m} : {}_A\mathrm{D}(A) \otimes_R \mathrm{D}(A)_A \to \mathrm{D}(A)$ and (m4) implies m factors through ${}_A\mathrm{D}(A) \otimes_A \mathrm{D}(A)_A \to \mathrm{D}(A)$ which we denote by $\widetilde{m}$. We keep these notations to state:

Proposition 3.2 *If* $\overline{m} : {}_A\mathrm{D}(A) \otimes_R \mathrm{D}(A)_A \cong \mathrm{D}(A)$ *as* A*-bimodules, then* A *is gendo-symmetric.*

Proof. By Definition 2.3(c), it suffices to prove $\widetilde{m} : {}_A\mathrm{D}(A) \otimes_A \mathrm{D}(A)_A \to \mathrm{D}(A)$ is an isomorphism between A-bimodules. (m2) implies $\widetilde{m}$ is surjective, thus $\widetilde{m}^* : A \to \mathrm{Hom}_A(\mathrm{D}(A), {}_AA)$ is injective between A-bimodules. By Lemma 2.1 we have $\mathrm{dom.\,dim}\, A \geq 1$, furthermore by [9, Corollary 2.6], there exists an idempotent $e \in A$ such that Ae is faithful and $\mathrm{D}(Ae) \cong eA$. In particular, Ae is projective, injective and faithful. Multiplying by e on both sides of $\widetilde{m}$ yields $e\,\mathrm{D}(A) \otimes_A \mathrm{D}(A) \cong eA \otimes_A \mathrm{D}(A) \cong e\,\mathrm{D}(A)$ as right A-modules, i.e., $e \cdot \ker(\widetilde{m}) = 0$.

Consider the following diagram in A-mod which is obviously commutative

$$\begin{array}{ccccccc} {}_A\mathrm{D}(A) \otimes_k \mathrm{D}(A) \otimes_k \mathrm{D}(A) & \xrightarrow{m\otimes 1 - 1\otimes m} & {}_A\mathrm{D}(A) \otimes_k \mathrm{D}(A) & \xrightarrow{m} & \mathrm{D}(A) & \longrightarrow & 0 \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{Id}} & & \\ \ker(\widetilde{m}) & \hookrightarrow & \mathrm{D}(A) \otimes_A \mathrm{D}(A) & \xrightarrow{\widetilde{m}} & \mathrm{D}(A) & \longrightarrow & 0 \end{array}$$

where the morphism π exists by exactness of the first row which follows from Lemma 3.1 and $\overline{m}$ being an isomorphism. Note that π is clearly surjective and ${}_A\mathrm{D}(A) \otimes_k \mathrm{D}(A) \otimes_k \mathrm{D}(A)$, as an injective left A-module, is a quotient of a direct sum of copies of the projective injective module Ae since $\mathrm{dom.\,dim}\, A \geq 1$. It follows that $\ker(\widetilde{m})$ is a quotient of $(Ae)^{\oplus n}$ for some n, hence $e \cdot \ker(\widetilde{m}) = 0$ implies $\ker(\widetilde{m}) = 0$, i.e., $\widetilde{m}$ is isomorphic. □

3.2 The bar cocomplex

Let A be a gendo-symmetric algebra. Using the notations of Section 2.2 and Lemma 2.5, there is a canonical multiplication $m : \mathrm{D}(A) \otimes_k \mathrm{D}(A) \to \mathrm{D}(A)$ and there is an A-bimodule morphism $\Theta : \mathrm{D}(A) \to A$ preserving multiplication. By Proposition 2.8(1), $(\mathrm{D}(A), m)$ does, in general, not have a unit. Therefore, we introduce the augmented k-algebra $R = k \oplus \mathrm{D}(A)$ and a natural k-algebra morphism $\theta : R \to A$ given by $\theta(x + f) = x + \Theta(f)$ for $x \in k, f \in \mathrm{D}(A)$. The corresponding restriction functor induced by θ is denoted by

$$\mathrm{Res} : A\text{-mod} \longrightarrow R\text{-mod}$$

For any left A-module M, the R-module $\mathrm{Res}(M)$ equals M as a k-vector space and $\gamma(ae \otimes eb) \cdot m = aebm$ for any $a, b \in A, m \in M$. In particular $\mathrm{D}(A)$ as a left ideal of R coincides with $\mathrm{Res}(\mathrm{D}(A))$ since $m(\gamma(ae \otimes eb) \otimes \gamma(xe \otimes ey)) = \gamma(aebxe \otimes ey) = aeb\gamma(xe \otimes ey)$ for any $a, b, x, y \in A$. Similarly ${}_RA$, the k-dual of the right ideal $\mathrm{D}(A)$ of R, coincides with $\mathrm{Res}(A)$. Therefore, we can simplify notation by identifying $\mathrm{Res}(M)$ with M for any left A-module whenever it is clear from the context which module structure is taken.

Res has a right adjoint functor $\mathrm{Hom}_R(A, -) : R\text{-mod} \to A\text{-mod}$, i.e., there is a natural isomorphism $\mathrm{Hom}_A(X, \mathrm{Hom}_R(A, V)) \cong \mathrm{Hom}_R(\mathrm{Res}(X), V)$ for any left A-module X and left R-module V. In other words, there is the following commutative diagram

$$\begin{array}{ccc} R\text{-mod} & \xrightarrow{\mathrm{Hom}_R(A,-)} & A\text{-mod} \\ & {\scriptstyle \mathrm{Hom}_R(\mathrm{Res}(X),-)} \searrow \quad \swarrow {\scriptstyle \mathrm{Hom}_A(X,-)} & \\ & \mathrm{Vect}_k & \end{array}$$

Note that $\mathrm{Hom}_R(A, \mathrm{D}(R)) \cong \mathrm{D}(A)$, i.e., $\mathrm{Hom}_R(A, -)$ sends injective R-modules to injective A-modules. We obtain a first quadrant Grothendieck spectral sequence

$$E_2^{p,q} = \mathrm{Ext}_A^p(X, \mathrm{Ext}_R^q(A, V)) \Rightarrow \mathrm{Ext}_R^{p+q}(\mathrm{Res}(X), V) \qquad (*)$$

for any $X \in A$-mod and $V \in R$-mod. Standard results on spectral sequences imply:

Proposition 3.3 *Let Y be a left A-module with $\mathrm{Ext}^i_R(A,Y) = 0$ for $1 \le i \le n$. Then $\mathrm{Ext}^i_A(X, \mathrm{Hom}_R(A,Y)) \cong \mathrm{Ext}^i_R(X,Y)$ for any $X \in A$-mod and $0 \le i \le n$. Moreover, there is a five term exact sequence*

$$0 \to \mathrm{Ext}^{n+1}_A(X, \mathrm{Hom}_R(A,Y)) \to \mathrm{Ext}^{n+1}_R(X,Y) \to \mathrm{Hom}_A(X, \mathrm{Ext}^{n+1}_R(A,Y)) \\ \to \mathrm{Ext}^{n+2}_A(X, \mathrm{Hom}_R(A,Y)) \to \mathrm{Ext}^{n+2}_R(X,Y)$$

Proposition 3.4

(1) *The Jacobson radical of R is spanned by $\{\gamma(a_i e \otimes e b_i) \mid a_i$ or $b_i \in \mathrm{rad}(A)\}$.*

(2) $\mathrm{Res}(I)$ *is projective injective for any projective injective left A-module I.*

(3) *If A is symmetric, then $R \cong k \times A$ as k-algebras.*

(4) *If A is not symmetric, then for any simple left A-module L,* $\mathrm{Res}(L)$ *is simple if $eL \neq 0$ or a direct sum of copies of trivial modules otherwise. Moreover every simple left R-module is either trivial or of the form* $\mathrm{Res}(L)$ *for some simple left A-module L.*

Proof. It is clear that J, the span of $\{\gamma(a_i e \otimes e b_i) \mid a_i$ or $b_i \in \mathrm{rad}(A)\}$ is a nilpotent ideal of R and $\overline{R} = R/J \cong k \oplus \overline{A}e \otimes_{e\overline{A}e} e\overline{A}$ where $\overline{A} = A/\mathrm{rad}(A)$. To proceed, let $\{e_1, \ldots, e_n\}$ be a complete set of pairwise orthogonal primitive idempotents in A so that $e = e_1 + \cdots + e_r$. Let D_i be the division algebra $e_i\overline{A}e_i$ for each i. Then by Wedderburn-Artin Theorem [3], $e_i\overline{A}e_j \neq 0$ implies $D_i \cong D_j$ and $\dim_{D_i} e_i\overline{A}e_j = \dim_{D_j} e_j\overline{A}e_i = 1$. Note that Ae is basic as a left A-module by Section 2.2, i.e., $e_1, \ldots, e_r$ are pairwise non-conjugate. If $e_i\overline{A}e \neq 0$, then $\dim_{D_i} e_i\overline{A}e = 1$ and $\dim_{D_i} e\overline{A}e_i = 1$.

Let $X = \{1 \le i \le n \mid e_i\overline{A}e \neq 0\}$ and choose for each $i \in X$ the elements $E_i \in e_i\overline{A}e$ and $F_i \in e\overline{A}e_i$ such that $F_iE_i = \overline{e_j}$ in $e\overline{A}e$ where e_j is uniquely determined by $ee_j \neq 0$ and $e_i\overline{A}e_j \neq 0$. Then $E_iF_i = \overline{e_i}$ in $e_i\overline{A}e_i$ and $\epsilon_i = E_i \otimes F_i$ satisfies $\epsilon_i^2 = \epsilon_i$, $\epsilon_i\epsilon_l = 0$ for $i \neq l$ and $(\sum \epsilon_i)w = w = w(\sum \epsilon_i)$ for all $w \in \overline{A}e \otimes_{e\overline{A}e} e\overline{A}$. Moreover, the left $\overline{R}$-module $\widehat{L}_i := (\overline{A}e \otimes_{e\overline{A}e} e\overline{A})\epsilon_i$ is isomorphic to $\mathrm{Res}(\overline{A}e_i)$ and $\mathrm{End}_{\overline{R}}(\widehat{L}_i) \cong D_i$. As a result, $\widehat{L}_i$ is a simple left $\overline{R}$-module and

$$\overline{R} \cong k(1 - \sum \epsilon_i) \oplus \bigoplus_{i \in X} \widehat{L}_i$$

i.e., $\overline{R}$ is semisimple. Equivalently J equals the Jacobson radical of R and all non-isomorphic simple left R-modules are k (trivial module when $\sum \epsilon_i \neq 1$) and $\widehat{L}_i = \mathrm{Res}(\overline{A}e_i)$ for $1 \le i \le r$. When A is not symmetric, $\mathrm{Res}(L)$ is a direct sum of copies of trivial R-modules for any simple left A-module L with $eL = 0$. This proves (1), (3) and (4).

(2) It is enough to show that $\mathrm{Res}(Ae)$ is projective and injective. Since $e\,\mathrm{D}(A) \cong \mathrm{D}(Ae)$ as both right A-modules and right R-modules, we have $\mathrm{D}(\mathrm{Res}(Ae)) = \mathrm{D}(Ae)_R \cong e\,\mathrm{D}(A)_R \cong \gamma(e \otimes e)R$ as right R-modules. So $\mathrm{Res}(Ae) \cong R\gamma(e \otimes e)$ is projective and injective. □

For any left A-module, let $\xi_M : M \cong \mathrm{Hom}_A(A, M) \to \mathrm{Hom}_R(A, M)$ be the canonical embedding of left A-modules induced by the functor Res.

Lemma 3.5 *Let $n \ge 2$ be an integer.*

(1) $\mathrm{Hom}_R(\mathrm{D}(A), A) \cong \mathrm{Hom}_R(A, A) \cong A$ *as A-bimodules.*

(2) $\mathrm{Ext}^i_R(A, A) = 0$ *for* $1 \leq i \leq \mathrm{dom.dim}\, A - 1$.

(3) $\{M \in A\text{-mod} \mid \xi_M$ *is an isomorphism*$\}$ *is closed under taking submodules.*

(4) *If* $\mathrm{dom.dim}\, M \geq n$*, then* ξ_M *is an isomorphism and* $\mathrm{Ext}^i_R(A, M) = 0$ *for* $1 \leq i \leq n-1$.

Proof. (1) Observe that $\mathrm{Hom}_{eAe}(eA_R, eA) \cong {}_RA$ as (R, A)-bimodules. By Proposition 3.4 $eA \otimes_R \mathrm{D}(A) \cong \gamma(e \otimes e)\, \mathrm{D}(A) \cong eA$ as (eAe, A)-bimodules. Therefore,

$$\begin{aligned}\mathrm{Hom}_R(\mathrm{D}(A), A) &\cong \mathrm{Hom}_R(\mathrm{D}(A), \mathrm{Hom}_{eAe}(eA_R, eA)) \\ &\cong \mathrm{Hom}_{eAe}(eA \otimes_R \mathrm{D}(A), eA) \cong \mathrm{Hom}_{eAe}(eA, eA) \cong A\end{aligned}$$

Similarly, $\mathrm{Hom}_R(A, A) \cong A$ as A-bimodules.

To prove (3), let $0 \to X \to M \to Y \to 0$ be any short exact sequence in A-mod with ξ_M being an isomorphism. Applying the left exact functor $\mathrm{Hom}_R(A, \mathrm{Res}(-))$ yields the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & M & \longrightarrow & Y & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \xi_X} & & \downarrow{\scriptstyle \xi_M} & & \downarrow{\scriptstyle \xi_Y} & & \\ 0 & \longrightarrow & \mathrm{Hom}_R(A, X) & \longrightarrow & \mathrm{Hom}_R(A, M) & \longrightarrow & \mathrm{Hom}_R(A, Y) & \longrightarrow & \end{array}$$

Since ξ_X, ξ_Y are injective and ξ_M is an isomorphism, it follows that ξ_X must be an isomorphism.

(2) is a special case of (4), which we are going to prove now. Let $0 \to M \to I^0 \to I^1 \to \cdots \xrightarrow{\delta} I^{n-1} \to I^n \to \cdots$ be a minimal injective resolution of M in A-mod. Then $I^0, \ldots, I^{n-1}$ are projective injective left A-modules, hence projective injective as left R-modules by Proposition 3.4(2). So $\mathrm{Res}(M)$ has an injective resolution in R-mod of the form

$$0 \to \mathrm{Res}(M) \to \mathrm{Res}(I^0) \to \cdots \xrightarrow{\mathrm{Res}(\delta)} \mathrm{Res}(I^{n-1}) \to E^n \to \cdots$$

Since $\mathrm{Im}(\mathrm{Res}(\delta)) = \mathrm{Res}(\mathrm{Im}(\delta))$, it follows by (1) and (3) above that $\mathrm{Hom}_R(A, I^i) \cong I^i$ and $\mathrm{Hom}_R(A, \mathrm{Im}(\mathrm{Res}(\delta))) \cong \mathrm{Im}\, \mathrm{Res}(\delta)$ canonically for $0 \leq i \leq n-1$. Applying $\mathrm{Hom}_R(A, -)$ yields the following long exact sequence

$$0 \to \mathrm{Hom}_R(A, M) \to \mathrm{Hom}_R(A, I^0) \to \cdots \to \mathrm{Hom}_R(A, I^{n-1}) \to \mathrm{Hom}_R(A, E^n)$$

In particular ξ_M is an isomorphism since $n \geq 2$ and $\mathrm{Ext}^i_R(A, M) = 0$ for $1 \leq i \leq n-1$. $\square$

Example. Let A be the k-algebra given by quiver and relations: $1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \,/\langle \beta\alpha \rangle$. It is well-known that A is Morita equivalent to the Schur algebra $S_k(2, 2)$ if k is an infinite field of characteristic 2, see [8]. A has a k-basis $\{e_1, e_2, \alpha, \beta, \alpha\beta\}$ so $\mathrm{D}(A)$ has the dual basis $\{e_1^*, e_2^*, \alpha^*, \beta^*, (\alpha\beta)^*\}$. Let $C = k[x]/(x^2)$. Then $A \cong \mathrm{End}_C(k \oplus C)$, i.e., A is a gendo-symmetric algebra. The multiplication rule on $\mathrm{D}(A)$ described in Section 2.2 is given by

	e_1^*	e_2^*	α^*	β^*	$(\alpha\beta)^*$
e_1^*	0	0	0	0	0
e_2^*	0	0	0	0	e_2^*
α^*	0	0	0	e_1^*	α^*
β^*	0	0	e_2^*	0	0
$(\alpha\beta)^*$	0	e_2^*	0	β^*	$(\alpha\beta)^*$

The corresponding augmented algebra R is given by the same quiver as A but with relations $\langle \alpha\beta\alpha, \beta\alpha\beta \rangle$. In particular $\mathrm{dom.dim}\, A = 2$, R is symmetric, $\mathrm{Ext}_R^1(A, A) = 0$ and $\mathrm{Ext}_R^2(A, A)$ is 1-dimensional.

To each gendo-symmetric algebra A, we attach a *bar cocomplex* as follows.

$$\mathscr{C}_A^\bullet : \qquad 0 \to A \xrightarrow{\Delta} A \otimes_k A \xrightarrow{\delta^1} A \otimes_k A \otimes_k A \to \cdots \xrightarrow{\delta^{n-1}} A^{\otimes n+1} \to \cdots$$

where Δ is a canonical comultiplication of A and the differential $\delta^r : A^{\otimes r+1} \to A^{\otimes r+2}$ is given by the following rule: for any $a_0, \ldots, a_r \in A$

$$\delta^r(a_0 \otimes \cdots \otimes a_r) = \sum_{i=0}^{r} (-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes \Delta(a_i) \otimes a_{i+1} \otimes \cdots \otimes a_r.$$

More generally, for each left A-module M, there is a bar cocomplex

$$\mathscr{C}_M^\bullet : \qquad 0 \to M \xrightarrow{\eta_M} A \otimes_k M \xrightarrow{\delta_M^1} A \otimes_k A \otimes_k M \to \cdots \xrightarrow{\delta_M^n} A^{\otimes n+1} \otimes_k M \to \cdots$$

where $\eta_M : M \to A \otimes_k M$ is the canonical morphism (Section 2.2) and the differential map $\delta_M^n : A^{\otimes n} \otimes_k M \to A^{\otimes n+1} \otimes_k M$ is given by:

$$\begin{aligned} \delta_M^n(a_0 \otimes \cdots \otimes a_{n-1} \otimes m) = & \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes \Delta(a_i) \otimes a_{i+1} \otimes \cdots \otimes a_{n-1} \otimes m \\ & + (-1)^n a_0 \otimes \cdots \otimes a_{n-1} \otimes \eta_M(m) \end{aligned}$$

for $a_0, \ldots, a_{n-1} \in A$ and $m \in M$.

3.3 The characterisations of gendo-symmetric algebras and their dominant dimension

The main result of this article is the following characterisation of the dominant dimension of a gendo-symmetric algebra in terms of exactness of the bar cocomplex. At the same time, the class of gendo-symmetric algebras is given another homological characterisation, in terms of the bar cocomplex.

Theorem 3.6 *Let A be a finite dimensional k-algebra and $n \geq 2$ an integer. Then A is a gendo-symmetric algebra with* $\mathrm{dom.dim}\, A \geq n$ *if and only if there is an A-bimodule morphism $\Delta : A \to {}_A A \otimes_k A_A$ satisfying*
(1) *Δ is injective;*

(2) $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$ *and*
(3) $\mathrm{Im}(\Delta) \subseteq \{\sum u_i \otimes v_i \in A \otimes_k A \mid \sum u_i a \otimes v_i = \sum u_i \otimes a v_i,\ \forall a \in A\}$ *such that the complex*

$$\mathscr{C}_A^\bullet: \qquad 0 \to A \xrightarrow{\Delta} A \otimes_k A \xrightarrow{\delta^1} A \otimes_k A \otimes_k A \to \cdots \xrightarrow{\delta^{n-1}} A^{\otimes n+1} \to \cdots$$

has cohomologies $\mathrm{H}^i(\mathscr{C}_A^\bullet) = 0$ *for* $0 \le i \le n-1$, *where the differential* $\delta^r : A^{\otimes r+1} \to A^{\otimes r+2}$ *is given by: for any* $a_0, \dots, a_r \in A$

$$\delta^r(a_0 \otimes \cdots \otimes a_r) = \sum_{i=0}^{r} (-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes \Delta(a_i) \otimes a_{i+1} \otimes \cdots \otimes a_r.$$

Proof. If A is gendo-symmetric, then the canonical comultiplication Δ from Section 2.2 satisfies (1), (2) and (3) by Lemma 2.6. To calculate the cohomologies of $\mathscr{C}_A^\bullet$, set $X = \mathrm{D}(A)$, $Y = A$ and $V = \mathrm{Res}(A)$. By Lemma 3.5, $A \cong \mathrm{Hom}_R(A, Y)$ and for $1 \le i \le n-1$ there is $\mathrm{Ext}_R^i(A, A) = 0$. Thus by rewriting Tor as Ext and then applying Proposition 3.3,

$$\mathrm{Tor}_i^R(\mathrm{D}(A), \mathrm{D}(A)) \cong \mathrm{D}\,\mathrm{Ext}_R^i(\mathrm{D}(A), A) \cong \mathrm{D}\,\mathrm{Ext}_A^i(\mathrm{D}(A), A)$$

for $0 \le i \le n-1$. As a result, the canonical multiplication m on $\mathrm{D}(A)$ (see Section 2.2) induces $\mathrm{D}(A) \otimes_R \mathrm{D}(A) \cong \mathrm{D}(A)$ and $\mathrm{Tor}_i^R(\mathrm{D}(A), \mathrm{D}(A)) = 0$ for $1 \le i \le n-2$ by Proposition 2.11. Applying Lemma 3.1, the bar complex

$$\mathscr{B}_{\mathrm{D}(A)}^\bullet: \qquad \cdots \xrightarrow{\partial_{r-1}} \mathrm{D}(A)^{\otimes r} \to \cdots \to \mathrm{D}(A) \otimes_k \mathrm{D}(A) \to \mathrm{D}(A) \to 0$$

has cohomologies $\mathrm{H}^{-i}(\mathscr{B}_{\mathrm{D}(A)}^\bullet) = 0$ for $0 \le i \le n-1$. For $r \ge 1$, let $\pi_r : A^{\otimes r} \to \mathrm{D}(\mathrm{D}(A)^{\otimes r})$ be the k-linear map defined by

$$\pi_r(a_1 \otimes \cdots \otimes a_r)(f_1 \otimes \cdots \otimes f_r) = (-1)^{\frac{r(r+1)}{2}} \prod_{i=1}^{r} f_i(a_{r+1-i})$$

Then the following diagram commutes

$$\begin{array}{ccc} A^{\otimes r} & \xrightarrow{\delta^{r-1}} & A^{\otimes r+1} \\ \downarrow{\scriptstyle \pi_r} & & \downarrow{\scriptstyle \pi_{r+1}} \\ \mathrm{D}(\mathrm{D}(A)^{\otimes r}) & \xrightarrow{\partial_{r-1}^*} & \mathrm{D}(\mathrm{D}(A)^{\otimes r+1}) \end{array}$$

Indeed, for any $a_1, \dots, a_r \in A$ and $f_0, \dots, f_r \in \mathrm{D}(A)$, there are equalities

$$\begin{aligned} &\partial_{r-1}^* \pi_r(a_1 \otimes \cdots \otimes a_r)(f_0 \otimes \cdots \otimes f_r) \\ &= \sum_{i=0}^{r-1} (-1)^i \pi_r(a_1 \otimes \cdots \otimes a_r)(f_0 \otimes \cdots \otimes f_{i-1} \otimes m(f_i, f_{i+1}) \otimes f_{i+2} \otimes \cdots \otimes f_r) \\ &= \sum_{i=0}^{r-1} (-1)^{i + \frac{r(r+1)}{2}} \prod_{j=1}^{r-i-1} f_{r+1-j}(a_j) m(f_i, f_{i+1})(a_{r-i}) \prod_{j=0}^{i-1} f_j(a_{r-j}) \end{aligned}$$

By the identity $(-1)^{i+\frac{r(r+1)}{2}} = (-1)^{r-i-1+\frac{(r+1)(r+2)}{2}}$, the last term equals

$$\begin{aligned}
&\sum_{i=0}^{r-1}(-1)^{r-i-1+\frac{(r+1)(r+2)}{2}}\prod_{j=1}^{r-i-1} f_{r+1-j}(a_j)(f_{i+1},f_i)\Delta(a_{r-i})\prod_{j=0}^{i-1} f_j(a_{r-j}) \\
&= \sum_{i=0}^{r-1}(-1)^{r-i-1}\pi_{r+1}(a_1\otimes\cdots\otimes\Delta(a_{r-i})\otimes\cdots\otimes a_r)(f_0\otimes\cdots\otimes f_r) \\
&= \pi_{r+1}\delta^{r-1}(a_1\otimes\cdots\otimes a_r)(f_0\otimes\cdots\otimes f_r)
\end{aligned}$$

Therefore $\{\pi_i\}$ defines a chain isomorphism from $\mathscr{C}_A^\bullet$ to the dual of $\mathscr{B}_{\mathrm{D}(A)}^\bullet$. In particular, $\mathrm{H}^i(\mathscr{C}_A^\bullet) = 0$ for $0 \le i \le n-1$.

Conversely, assume that there is an A-bimodule morphism $\Delta : A \to {}_AA \otimes_k A_A$ satisfying (1),(2), (3) and $\mathrm{H}^i(\mathscr{C}_A^\bullet) = 0$ for $0 \le i \le n-1$. Define $m : \mathrm{D}(A) \otimes_k \mathrm{D}(A) \to \mathrm{D}(A)$ by $m(f \otimes g)(a) = (g \otimes f)\Delta(a)$ for $a \in A, f, g \in \mathrm{D}(A)$. Then m satisfies (m1)-(m4) in Section 3.1. In particular $(\mathrm{D}(A), m)$ is an associative k-algebra, and the chain map $\{\pi_i\}$ above induces an isomorphism from the bar complex of $(\mathrm{D}(A), m)$ to the dual of $\mathscr{C}_A^\bullet$. Note that $n \ge 2$. As $\mathrm{H}^{-1}(\mathscr{B}_{\mathrm{D}(A)}^\bullet) = \mathrm{H}^0(\mathscr{B}_{\mathrm{D}(A)}^\bullet) = 0$, the map m induces $\mathrm{D}(A) \otimes_R \mathrm{D}(A) \cong \mathrm{D}(A)$ where R is the augmented k-algebra associated to $(\mathrm{D}(A), m)$. Applying Proposition 3.2 shows that A is gendo-symmetric with Δ a canonical comultiplication. Let $\kappa = \operatorname{dom.dim} A$. Then by Lemma 3.5 and Proposition 3.3,

$$\mathrm{Ext}_A^i(\mathrm{D}(A), A) \cong \mathrm{Ext}_R^i(\mathrm{D}(A), A) \cong \mathrm{D}\,\mathrm{H}^{-i-1}(\mathscr{B}_{\mathrm{D}(A)}^\bullet) \cong \mathrm{H}^{i+1}(\mathscr{C}_A^\bullet)$$

for $0 \le i \le \kappa - 1$. Applying Proposition 2.11, we have $\mathrm{Ext}_A^{\kappa-1}(\mathrm{D}(A), A) \ne 0$ and therefore $\mathrm{H}^\kappa(\mathscr{C}_A^\bullet) \ne 0$. So $\operatorname{dom.dim} A = \kappa \ge n$, since $\mathrm{H}^i(\mathscr{C}_A^\bullet) = 0$ for $0 \le i \le n-1$. □

Remark 3.7 *This characterisation of gendo-symmetric algebras may be compared with a result by Abrams, who proved that a finite dimensional algebra has a coassociative comultiplication and a counit if and only if it is a Frobenius algebra. This result includes the equivalence between two-dimensional toplogical quantum field theories and commutative Frobenius algebras, rigorously proved also by Abrams. See [1, 2, 11] for precise formulations and details.*

The bar cocomplex $\mathscr{C}_A^\bullet$ attached to the gendo-symmetric algebra A has the following universal property with respect to change of coefficients. In particular, there is a combinatorial characterisation of dominant dimensions of left A-modules.

Theorem 3.8 *Let A be a gendo-symmetric algebra and $n \ge 2$ an integer. Let M be a left A-module with* $\operatorname{dom.dim} M \ge 2$. *Then $\mathscr{C}_M^\bullet \cong \mathscr{C}_A^\bullet \otimes_A M$, and* $\operatorname{dom.dim} M \ge n$ *if and only if the bar cocomplex $\mathscr{C}_M^\bullet$ has cohomologies $\mathrm{H}^i(\mathscr{C}_M^\bullet) = 0$ for $0 \le i \le n-1$.*

Proof. By Proposition 2.8(3) $\mathscr{C}_M^\bullet \cong \mathscr{C}^\bullet \otimes_A M$ canonically. Moreover, $\operatorname{dom.dim} M \ge 2$ implies $\xi_M : M \cong \mathrm{Hom}_R(A, M)$ by Lemma 3.5(4). Therefore by Proposition 2.11

$$M \cong \mathrm{Hom}_A(\mathrm{D}(A), M) \cong \mathrm{Hom}_A(\mathrm{D}(A), \mathrm{Hom}_R(A, M)) \cong \mathrm{Hom}_R(\mathrm{D}(A), M)$$

canonically. Equivalently $\mathrm{D}(M) \otimes_R \mathrm{D}(A) \cong \mathrm{D}(M) \otimes_A \mathrm{D}(A) \cong \mathrm{D}(M)$ canonically so that for any $n^* \in \mathrm{D}(M)$ and $f \in \mathrm{D}(A)$, $n^* \otimes f$ is sent to $n^* \otimes f$ and then to $n^*\Theta(f)$, where Θ is the morphism from $\mathrm{D}(A)$ to A defined in Section 2.2. Let $\pi_1 : \mathrm{D}(M) \otimes_R \mathrm{D}(A) \to \mathrm{D}(M)$ be the isomorphism which sends $n^* \otimes f$ to $-n^*\Theta(f)$ for $n^* \in \mathrm{D}(M)$ and $f \in \mathrm{D}(A)$. For $r \geq 2$, let $\pi_r : \mathrm{D}(M) \otimes_k \mathrm{D}(A)^{\otimes r-1} \to \mathrm{D}(A^{\otimes r-1} \otimes_k M)$ be the k-linear isomorphism

$$\pi_r(n^* \otimes f_1 \otimes \ldots \otimes f_{r-1})(a_1 \otimes \ldots \otimes a_{r-1} \otimes m) = (-1)^{\frac{r(r+1)}{2}} n^*(m) \prod_{i=1}^{r-1} f_{r-i}(a_i)$$

Then $\{\pi_r\}_{r\geq 1}$ is a chain isomorphism from $\mathrm{D}(M) \otimes_R \overline{\mathscr{B}}^{\bullet}_R \otimes_R \mathrm{D}(A)$ to the dual of $\mathscr{C}^{\bullet}_M$, i.e., the following diagrams commute ($r \geq 3$)

$$\begin{array}{ccc} \mathrm{D}(M) \otimes_k \mathrm{D}(A) & \xrightarrow{d^0_M} & \mathrm{D}(M) \otimes_R \mathrm{D}(A) \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} \\ \mathrm{D}(A \otimes_k M) & \xrightarrow{\eta^*_M} & \mathrm{D}(M) \end{array} \qquad \begin{array}{ccc} \mathrm{D}(M) \otimes_k \mathrm{D}(A)^{\otimes r-1} & \xrightarrow{d^{r-2}_M} & \mathrm{D}(M) \otimes_k \mathrm{D}(A)^{\otimes r-2} \\ \downarrow{\scriptstyle \pi_r} & & \downarrow{\scriptstyle \pi_{r-1}} \\ \mathrm{D}(A^{\otimes r-1} \otimes_k M) & \xrightarrow{(\delta^{r-2}_M)^*} & \mathrm{D}(A^{\otimes r-2} \otimes_k M) \end{array}$$

Indeed, let $\Delta(1) = \sum x_i \otimes y_i \in A \otimes_k A$, then for $m \in M, n^* \in \mathrm{D}(M)$, $f, f_1, \ldots, f_{r-1} \in \mathrm{D}(A)$ and $a_1, \ldots, a_{r-2} \in A$ we have $\pi_1 d^0_M(n^* \otimes f)(m) = -n^*(\Theta(f)m)$ and

$$\eta^*_M \pi_2(n^* \otimes f)(m) = \pi_2(n^* \otimes f) \sum x_i \otimes y_i m = -\sum f(x_i) n^*(y_i m) = -n^*(\sum f(x_i) y_i m)$$

which equals $-n^*(\Theta(f)m)$ since by Proposition 2.8(2) $\Theta(f) = \sum f(x_i)y_i$. In general as in the proof of Theorem 3.6 a routine check together with $(f \otimes n^*)\eta_M(m) = \sum f(x_i)n^*(y_i m) = n^*(\Theta(f)m)$ yields $(\delta^{r-2}_M)^* \pi_r = \pi_{r-1} d^{r-2}_M$. As a result,

$$\mathrm{H}^{i+1}(\mathscr{C}^{\bullet}_M) = \mathrm{H}^{-i-1}(\mathrm{D}(M) \otimes_R \overline{\mathscr{B}}_R \otimes_R \mathrm{D}(A)) \cong \mathrm{D}\,\mathrm{Ext}^i_R(\mathrm{D}(A), M)$$

for all $i \geq 1$. Let $d = \mathrm{dom.dim}\, M$. Then by Proposition 2.11 and Proposition 3.3 we have $\mathrm{Ext}^i_R(\mathrm{D}(A), M) \cong \mathrm{Ext}^i_A(\mathrm{D}(A), M) = 0$ for $1 \leq i \leq d-2$ and

$$\mathrm{Ext}^{d-1}_R(\mathrm{D}(A), M) \cong \mathrm{Ext}^{d-1}_A(\mathrm{D}(A), M) \neq 0$$

In particular $\mathrm{H}^i(\mathscr{C}^{\bullet}_M) = 0$ for $0 \leq i \leq d-1$ and $\mathrm{H}^d(\mathscr{C}^{\bullet}_M) \neq 0$. □

3.4 Example: Classical Schur-Weyl duality

As an illustration of Theorem 3.6, we give an alternative proof of the double centraliser property for Schur algebras, and an approach to calculate their dominant dimensions. Let k be an infinite field, n and r be two natural numbers. Let $I(n, r)$ be the set of multi-indices $(i_1, \ldots, i_r)$ with $i_\rho \in \{1, \ldots, n\}$. Clearly the k-space $A_k(n, r)$ of r-th homogenous polynomials in the n^2 indeterminants $\{c_{i,j} \mid 1 \leq i, j \leq n\}$ is spanned by the monomials $c_{\underline{i},\underline{j}} = c_{i_1,j_1} \cdots c_{i_r j_r}$ where $\underline{i}, \underline{j} \in I(n, r)$. Note that $A_k(n, r)$ has a natural coalgebra structure with the comultiplication $\Delta : A_k(n, r) \to A_k(n, r) \otimes A_k(n, r)$ and the counit $\varepsilon : A_k(n, r) \to k$ given by

$$\Delta(c_{\underline{i},\underline{j}}) = \sum_{\underline{k} \in I(n,r)} c_{\underline{i},\underline{k}} \otimes c_{\underline{k},\underline{j}}, \qquad \varepsilon(c_{\underline{i},\underline{j}}) = \delta_{\underline{i},\underline{j}}$$

The Schur algebra $S_k(n,r)$ is then defined to be the k-dual of $A_k(n,r)$. Let $\mathbf{M}_n(k)$ be the monoid of $n \times n$ matrices over k. Recall the left and right actions of $\mathbf{M}_n(k)$ on $A_k(n,r)$

$$g \cdot c_{\underline{i},\underline{j}} = \sum_{\underline{k} \in I(n,r)} c_{\underline{i},\underline{k}} \cdot g_{k_1,j_1} \cdots g_{k_r,j_r}$$
$$c_{\underline{i},\underline{j}} \cdot g = \sum_{\underline{k} \in I(n,r)} g_{i_1,k_1} \cdots g_{i_r,k_r} \cdot c_{\underline{k},\underline{j}}$$

for $g = (g_{i,j}) \in \mathbf{M}_n(k)$ and $\underline{i}, \underline{j} \in I(n,r)$. Both actions factor through the natural actions of $S_k(n,r)$ on $A_k(n,r)$, see for example [8] for a detailed account. Consider the following k-linear map introduced in [5]

$$\mu : A_k(n,r) \otimes_k A_k(n,r) \to A_k(n,r)$$
$$\mu(c_{\underline{i},\underline{j}} \otimes c_{\underline{k},\underline{l}}) = \begin{cases} \sum_{\sigma \in \mathrm{Stab}(\underline{i})} c_{\underline{k},\underline{j}\,\sigma\tau} & \underline{l} = \underline{i} \cdot \tau \text{ for some } \tau \in \Sigma_r; \\ 0, & \text{else.} \end{cases}$$

where Σ_r is the symmetric group on r-letters, $\underline{i} \cdot \sigma$ is defined to be $(i_{\sigma(1)}, \ldots, i_{\sigma(r)})$ for any $\underline{i} \in I(n,r)$ and $\sigma \in \Sigma_r$, and $\mathrm{Stab}(\underline{i}) = \{\sigma \in \Sigma_r \mid \underline{i}\,\sigma = \underline{i}\}$. It is a routine check that μ is well-defined and satisfies (m1), (m3) and (m4) in Section 3.1, see also [5]. If $n \geq r$, then $\underline{w} = (1, \ldots, r)$ belongs to $I(n,r)$ and $\mu(c_{\underline{w},\underline{j}} \otimes c_{\underline{i},\underline{w}}) = c_{\underline{i},\underline{j}}$, so μ also satisfies (m2). We claim that in $W = A_k(n,r) \otimes_{S_k(n,r)} A_k(n,r)$

(1) $c_{\underline{i},\underline{j}} \otimes c_{\underline{k},\underline{l}} = 0$ unless $\underline{l} = \underline{i}\,\sigma$ for some $\sigma \in \Sigma_r$.

(2) If $n \geq r$, then W is spanned by $\{c_{\underline{w},\underline{j}} \otimes c_{\underline{i},\underline{w}} \mid \underline{i}, \underline{j} \in I(n,r)\}$.

Condition (1) is easy to check, see also [5]. To prove (2), it suffices to write $c_{\underline{k},\underline{j}} \otimes c_{\underline{i},\underline{k}} \in W$ for any $\underline{i}, \underline{j}, \underline{k} \in I(n,r)$ into a linear combination of the given elements. Choose $g = (g_{i,j})$ in $\mathbf{M}_n(k)$ such that $g_{i,k_i} = x_i$ in k for $i = 1, \ldots, r$ and all other entries are zero. Then in W

$$\begin{aligned} 0 &= c_{\underline{w},\underline{j}} \cdot g \otimes c_{\underline{i},\underline{k}} - c_{\underline{w},\underline{j}} \otimes g \cdot c_{\underline{i},\underline{k}} \\ &= x_1 x_2 \cdots x_r (c_{\underline{k},\underline{j}} \otimes c_{\underline{i},\underline{k}} - \sum_{\sigma \in \mathrm{Stab}(\underline{k})} c_{\underline{w},\underline{j}} \otimes c_{\underline{i},\underline{w}\sigma}) + (x_1^2 H_1 + \cdots + x_r^2 H_r) \end{aligned}$$

As k is an infinite field, we have $c_{\underline{k},\underline{j}} \otimes c_{\underline{i},\underline{k}} = \sum_{\sigma \in \mathrm{Stab}(\underline{k})} c_{\underline{w},\underline{j}} \otimes c_{\underline{i}\,\sigma,\underline{w}}$ in W. For each $\sigma \in \Sigma_r$, we choose g to be a permutation matrix so that $c_{\underline{w}\sigma,\underline{w}}(g) = 1$. Then by similar arguments as above, $c_{\underline{w},\underline{j}\,\sigma} \otimes c_{\underline{i}\,\sigma,\underline{w}} = c_{\underline{w},\underline{j}} \otimes c_{\underline{i},\underline{w}}$ in W for any $\underline{i}, \underline{j} \in I(n,r)$. As a result, when $n \geq r$

$$\dim_k A_k(n,r) \otimes_{S_k(n,r)} A_k(n,r) \leq \binom{n^2 + r - 1}{r} = \dim_k A_k(n,r).$$

In particular μ induces $A_k(n,r) \otimes_{S_k(n,r)} A_k(n,r) \cong A_k(n,r)$ as $S_k(n,r)$-bimodules, which by Definition 2.3(3) implies $S_k(n,r)$ is gendo-symmetric. Hence any faithful projective and injective left $S_k(n,r)$-module satisfies the double centraliser property and the dominant dimension of $S_k(n,r)$ equals $\max\{d \mid \mathrm{H}^i(\mathscr{C}^\bullet_{S_k(n,r)}) = 0 \mid 0 \leq i \leq d\} + 1$. We remark that in the case $n \geq r \geq p = \mathrm{char}(k) > 0$, we have found a closed formula $\mathrm{dom.dim}\, S_k(n,r) = 2(p-1)$ [6]. The case $n < r$ is much more subtle since the Schur algebra may have dominant dimension zero. See [5] for some further investigation and more results under the mild condition $r \leq n(p-1)$.

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