SCHUR ALGEBRAS OF BRAUER ALGEBRAS, II

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Abstract. A classical problem of invariant theory and of Lie theory is to determine endomorphism rings of representations of classical groups, for instance of tensor powers of the natural module (Schur-Weyl duality) or of full direct sums of tensor products of exterior powers (Ringel duality). In this article, the endomorphism rings of full direct sums of tensor products of symmetric powers over symplectic and orthogonal groups are determined. These are shown to be isomorphic to Schur algebras of Brauer algebras as defined in [24]. This implies structural properties of the endomorphism rings, such as double centraliser properties, quasi-hereditary, and a universal property, as well as a classification of simple modules.

1. Introduction

Let G be a classical group defined over an algebraically closed field k, E its natural module and $E^{\otimes r}$ the r-fold tensor product. Classical Schur-Weyl duality determines the centraliser algebra $\mathrm{End}_G(E^{\otimes r})$. When G equals the general linear group GL_n, the centraliser is a quotient of the group algebra $k\Sigma_r$ of the symmetric group. When G is orthogonal or symplectic, the centraliser algebra is a quotient of a Brauer algebra. For $n \geq r$, the symmetric group acts faithfully on the tensor space; the Brauer algebra acts faithfully on the tensor space for $n \geq 2r$. In such a situation, classical invariant theory and its characteristic-free versions, in particular, work by Schur, Brauer, Weyl, De Concini and Procesi, and others, provides much information. Additional work is needed to determine the structure of the centraliser algebras and their representation theory, which are far from being known.

Keep G, but replace the tensor space $E^{\otimes r}$ by a (full) direct sum of tensor products of either exterior or symmetric powers of the natural module. When choosing a full direct sum of tensor products of exterior powers in type A, Donkin [12] has shown that the endomorphism algebra is a type A Schur algebra; in fact, for $n \geq r$ this assertion is the Ringel self-duality of the classical Schur algebra. Adamovich and Rybnikov [1] have extended this result about the endomorphism ring of a direct sum of tensor products of exterior powers to cover also certain orthogonal and symplectic situations. The case of symmetric powers has remained open.

The main result of this article determines the endomorphism rings of a full direct sum of tensor products of symmetric powers, for all classical groups over an algebraically closed field of any characteristic. While in type A the centraliser algebra of a direct sum of tensor products of symmetric powers is again the classical type A Schur algebra, unexpectedly a different algebra is coming up in the orthogonal and symplectic case:

Theorem 1.1. *Let $G \subset GL_n$ be an orthogonal or symplectic group, over an algebraically closed field k. Assume $n \geq 2r$ when G is a symplectic group, and $n > 2r$ when G is*

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an orthogonal group. Denote by $B_r = B_r(\delta)$ the Brauer algebra with non-zero parameter $\delta \in k$. Fix the parameter $\delta = -n \in k$ when G is a symplectic group, and $\delta = n \in k$ when G is an orthogonal group.

Then the centraliser algebra

$$C := \mathrm{End}_G(\bigoplus_{\lambda \vdash r-2l, 0 \leq l \leq \frac{r}{2}} \mathrm{Sym}^\lambda E)$$

is isomorphic to the Schur algebra $S_B(n,r)$ of the Brauer algebra B_r.

The Schur algebra $S_B(n,r)$ has been defined in [24] as the endomorphism algebra

$$S_B(n,r) = \mathrm{End}_{B_r}(\bigoplus_{\lambda \vdash r-2l, 0 \leq l \leq \frac{r}{2}} M(l,\lambda))$$

of the permutation modules (introduced by Hartmann and Paget [22]) of the corresponding Brauer algebra B_r. Both algebras, B_r and $S_B(n,r)$, are defined combinatorially, and they are related by a Schur-Weyl duality. The inverse Schur functor (see Lemma 3.4) sends permutation modules $M(l,\lambda)$ to symmetric powers $\mathrm{Sym}^\lambda E$. Using this, Theorem 1.1 establishes a direct connection between $S_B(n,r)$ and the representation theory of classical groups. Here and throughout, when G is the orthogonal or symplectic group inside GL_n, the parameter δ of the Brauer algebra is taken to be non-zero in k and fixed as $\pm n$. Moreover, when dealing with an orthogonal group, we assume the ground field k to have characteristic different from two.

When the group G is even orthogonal or symplectic, its action on tensor space and on the symmetric powers is via a generalised Schur algebra that is associated with a saturated set of highest weights. In general, the action factors through the enveloping algebra of G in $\mathrm{End}_k(E^{\otimes r})$. This algebra will be denoted by $S_{env}(G)$, see Section 2.1; in the case of even orthogonal or symplectic groups, $S_{env}(G)$ equals the generalised Schur algebra just mentioned.

Corollary 1.2. *Let G, n and δ be as in 1.1. Then there is a Schur-Weyl duality between the algebra $S_{env}(G)$ and the algebra $C \simeq S_B(n,r)$, on the bimodule*

$$M := \bigoplus_{\lambda \vdash r-2l, 0 \leq l \leq \frac{r}{2}} \mathrm{Sym}^\lambda E,$$

that is, the following two equations hold true:

$$C = \mathrm{End}_{S_{env}(G)}(M) \qquad \textit{and} \qquad S_{env}(G) = \mathrm{End}_C(M).$$

With the tensor space $E^{\otimes r}$ being a direct summand of M, this Schur-Weyl duality on the bimodule M extends the classical Schur-Weyl duality (due to Brauer [2] in characteristic zero and [8, 9, 16, 35] in general) on tensor space.

Apart from relating two different situations, the isomorphism in Theorem 1.1 moreover transports much structure and information (developed in [24] and also in [22, 21]) from the Schur algebra $S_B(n,r)$ to the centraliser algebra C — see Section 3.10 for a more detailed formulation:

Corollary 1.3. *Let C be defined as in Theorem 1.1.*

(a) *The algebra C has an integral form with an explicit basis, which is independent of the ground field k and of its characteristic.*
(b) *The algebra C carries a quasi-hereditary structure, that is, mod-C is a highest weight category.*
(c) *The global (cohomological) dimension of C is finite.*
(d) *There is a Schur-Weyl duality between C and the Brauer algebra B_r.*
(e) *When the characteristic is different from two or three, the algebra C satisfies a universal property that makes it unique up to Morita equivalence: It is the quasi-hereditary 1-cover of the Brauer algebra B_r in the sense of Rouquier* [33].
(f) *The simple C-modules are parametrised by the disjoint union of all partitions of the non-negative integers of the form $r, r-2, r-4, \ldots$.*

These properties are shared by the members of a much larger family of algebras $S_B(n, r, \delta)_{\delta \in k}$, specialising to the symplectic and the orthogonal case for $\delta = -n$ and $\delta = n$, respectively.

From the point of view of invariant theory and of Lie theory, the results of this article describe the previously unknown endomorphism ring of a classical object, as well as its ring structure, its representation theory and its homological properties. From the point of view of the more recent — and now quickly expanding — theory of Brauer algebras and their Schur algebras, Theorem 1.1 gives a Lie theoretical meaning to these Schur algebras, which turn out to be the third players in a triangle of six Schur functors mutually relating Brauer algebras, their Schur algebras, and the enveloping algebras of orthogonal or symplectic groups, on the full direct sum of tensor powers of symmetric powers. This triangle replaces the familiar type A situation of just two algebras being in Schur-Weyl duality, which provides a classical connection between Lie theory and combinatorics.

This article is organised as follows: Section 2 collects definitions and notation as well as some results on Schur-Weyl duality for classical groups, Brauer algebras and various Schur algebras. Section 3 is devoted to the proof of Theorem 1.1. Sections 3.9 and 3.10 explain and prove Corollaries 1.2 and 1.3, respectively. Finally, Subsection 3.11 puts the various Schur functors, and three different algebras, together into one commuting triangle.

2. Schur-Weyl duality, Brauer algebras and Schur algebras

A main theme of this article is Schur-Weyl duality and its various manifestations. This is motivated by classical Schur-Weyl duality. Here, $G = \mathrm{GL}_n(k)$ acts on tensor space $E^{\otimes r}$ by diagonal extension of its action on the natural module E. The symmetric group Σ_r acts by place permutation on tensor space. The two actions commute and do, in fact, centralise each other. When $n \geq r$, this means

$$\mathrm{End}_G(E^{\otimes r}) = k\Sigma_r \text{ and } \mathrm{End}_{\Sigma_r}(E^{\otimes r}) = S_{env}(G),$$

where the enveloping algebra $S_{env}(G)$ of G in $\mathrm{End}_k(E^{\otimes r})$ is isomorphic to the classical type A Schur algebra $S(n, r)$. When $n < r$, the group algebra $k\Sigma_r$ has to be replaced by

a certain known quotient algebra. In characteristic zero, classical Schur-Weyl duality is due to Schur [34], in general it follows from results of Carter and Lusztig [5], De Concini and Procesi [8], and Green [19, Theorem 2.6c].

The injective modules over $S(n,r)$ are direct summands of direct sums of tensor products of symmetric powers. Indeed, the coalgebra $A(n,r) = S(n,r)^*$ dual to the Schur algebra is for $n \geq r$ a full sum of tensor products of symmetric powers, see [19]. Therefore, the endomorphism ring of a full direct sum of tensor powers of symmetric powers (the analogue of the algebra C in type A) is Morita equivalent, for a suitable choice of multiplicities even isomorphic, to the Schur algebra $S(n,r)$ itself. Moreover, tensor space $E^{\otimes r}$ is a full direct sum of permutation modules $M^\lambda = k\uparrow_{k\Sigma_\lambda}^{k\Sigma_r}$ (with λ running through all compositions of r) over the symmetric group Σ_r. Therefore, the type A analogue of the algebra $S_B(n,r)$ is the (type A) Schur algebra $S(n,r)$ itself. In types different from A, there is no such coincidence any more.

2.1. **Brauer algebras and Schur-Weyl dualities.** Let k be a commutative domain, and choose a parameter $\delta \in k$. Let r be a natural number. The Brauer algebra $B_r(\delta)$ of degree r for parameter δ is defined to be the vector space with k-basis given by the set of all Brauer diagrams on $2r$ vertices. A Brauer diagram is a diagram whose vertices are arranged in two rows of r vertices each, and there are r edges between the vertices such that each vertex is incident to precisely one edge. Brauer diagrams are considered up to homotopy, thus the dimension of $B_r(\delta)$ is $(2r-1)!! = (2r-1)\cdot(2r-3)\cdots 3\cdot 1$. To multiply two Brauer diagrams, say b_1 and b_2, the diagrams are concatenated, with b_1 drawn on top of b_2, and any closed loops appearing are removed, to give a Brauer diagram d. The result of the multiplication then is, by definition, $b_1 \cdot b_2 = \delta^c d$, where c is the number of closed loops removed. Typically the parameter δ is understood from the context, and we will denote the Brauer algebra by B_r or just B. Brauer algebras were introduced in [2] in the context of generalising Schur-Weyl duality from general linear groups to orthogonal and symplectic subgroups. For more details and examples see for instance [2, 3, 21, 22, 25, 27]. The restriction of the parameter $\delta = \pm n$ is necessary to obtain an action of the Brauer algebra $B_r(\delta)$ on the generalised symmetric powers $\mathrm{Sym}^\lambda E$. In characteristic zero, Brauer algebras are semisimple for non-integral parameter.

Let k be an algebraically closed field of characteristic $p \geq 0$ and let n, r be positive integers. Let E be an n-dimensional k-vector space and let ω be a non-degenerate symmetric bilinear form on E. The orthogonal group relative to ω is

$$\mathrm{O}_n = \{g \in \mathrm{GL}_n \mid \omega(gx, gy) = \omega(x,y) \text{ for all } x, y \in E\ \}.$$

Similarly for $n = 2m$ even positive integer, let ω be a non-degenerate skew-symmetric bilinear form on E. The symplectic group relative to ω is

$$\mathrm{Sp}_n = \{g \in \mathrm{GL}_n \mid \omega(gx, gy) = \omega(x,y) \text{ for all } x, y \in E\ \}.$$

In the following, we let $G \in \{\mathrm{Sp}_n, \mathrm{O}_n\}$. The classical groups GL_n, Sp_n and O_n operate on E by matrix multiplication. This action extends diagonally to an action on the tensor space $E^{\otimes r}$.

Brauer diagrams can be interpreted as G-homomorphisms in the following way: Assume E has basis $\{v_1, \ldots, v_n\}$, and let $\{v^1, \ldots, v^n\}$ be the dual basis of E with respect to the

invariant form ω. Define

$$\vartheta = \sum_{i=1}^{n} v_i \otimes v^i.$$

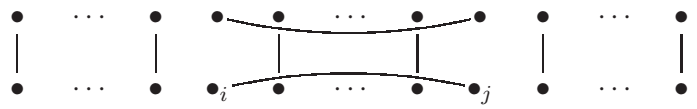
Then ϑ is G-invariant (see [18, 4.3.2.]). For $1 \leq i, j \leq n$, define the (i,j)th contraction operator $C_{i,j} : E^{\otimes r} \to E^{\otimes r-2}$ by

$$C_{i,j}(x_1 \otimes \cdots \otimes x_r) = \omega(x_i, x_j)\, x_1 \otimes \cdots \otimes \hat{x_i} \otimes \cdots \otimes \hat{x_j} \otimes \cdots \otimes x_r$$

where we omit the ith vector x_i and the jth vector x_j in the tensor product. Moreover, the (i,j)th expansion operator $D_{i,j} : E^{\otimes r-2} \to E^{\otimes r}$ is defined by

$$D_{i,j}(x_1 \otimes \cdots \otimes x_{r-2}) = \sum_{t=1}^{n} x_1 \otimes \cdots \otimes v_t \otimes \cdots \otimes v^t \otimes \cdots \otimes x_{r-2}.$$

Here v_t is in the ith position and v^t is in the jth position. Setting $b_{ij} = D_{i,j} \circ C_{i,j}$, it is easily checked that $b_{i,j} = b_{j,i}$. By (1) below, all elements in $\mathrm{End}_G(E^{\otimes r})$ coincide with elements in the Brauer algebra B_r. In particular, the element $b_{i,j}$ coincides with the Brauer diagram

$b_{i,j} =$ 

with the horizontal edges between vertices i and j. Here the top row horizontal arc corresponds to the contraction operator, and the arc in the bottom row corresponds to the expansion operator.

Diagrams consisting of $r - 2l$ through strings connecting top and bottom vertices (and l arcs at corresponding top and bottom places), naturally correspond to elements of a symmetric group Σ_{r-2l}. Such elements are G-endomorphisms of tensor space factoring through the smaller tensor space $E^{\otimes r-2l}$. Every Brauer diagram can be factorised as a product of contraction operators, an element of a symmetric group Σ_{r-2l} and then a product of expansion operators. This factorisation is the basic ingredient of the cellular structure of the Brauer algebra, for details see [27].

From now on, we assume $n \geq 2r$ in case G is a symplectic group and $n > 2r$ in the orthogonal case. Then the Brauer algebra with parameter $\pm n$ acts faithfully on tensor space $E^{\otimes r}$. Results by Brauer [2] in characteristic zero, and in general by De Concini–Procesi [8], Oehms [31], Dipper–Doty–Hu [9, 16] and Tange [35] extend classical Schur-Weyl duality to orthogonal and symplectic subgroups, implying in particular the following two isomorphisms:

$$B_r(n) = \mathrm{End}_{O_n}(E^{\otimes r}), \qquad B_r(-n) = \mathrm{End}_{Sp_n}(E^{\otimes r}). \tag{1}$$

$$S_{env}(\mathrm{O}(n)) = \mathrm{End}_{B_r(n)}(E^{\otimes r}), \qquad S_{env}(\mathrm{Sp}(n)) = \mathrm{End}_{B_r(-n)}(E^{\otimes r}). \tag{2}$$

Recall that here $S_{env}(G)$ denotes the enveloping algebra in $\mathrm{End}_k(E^{\otimes r})$ of the respective group. A version of Schur-Weyl duality involving $\mathrm{Hom}_G(E^{\otimes s}, E^{\otimes t})$ with s and t not necessarily equal can be found in [35]: In this version, tensor space $E^{\otimes r}$ is replaced by a direct sum $\oplus_{s=0}^{r} E^{\otimes s}$. In Theorem 3 of [35], Schur-Weyl duality is shown for this situation; the statement and the conditions coincide with those of usual Schur-Weyl

duality. (This also works in the orthogonal case, see Remark 3 in [35].) Here, basis elements of G-homomorphisms between tensor spaces of different degrees are represented by generalised Brauer diagrams (called (u, v)-diagrams in [35]) with u vertices in the top row and t vertices in the bottom row. See [35, Section 3] for explanations and details. Generalised Brauer diagrams with not necessarily equal numbers of vertices in top and bottom row are the morphisms in the category of Brauer diagrams, as described in detail in [28], where classical results of invariant theory are also discussed in detail, and extended.

When G is a symplectic or an even orthogonal group, the enveloping algebra $S_{env}(G)$ is a generalised Schur algebra in the sense of Donkin, which gives it additional relevance, as follows: The classical type A Schur algebra defined by Green [19] provides a framework to study the polynomial representation theory of the general linear group GL_n. In fact, the algebra $S_{env}(G)$ in this case coincides with Green's algebra $S(n, r)$, and the modules over $S(n, r)$ are the polynomial representations of G that are homogeneous of degree r. Donkin [11] generalised this concept to rational representations of reductive groups associated with finite saturated sets of weights. Generalised Schur algebras are quasi-hereditary, so their module categories are highest weight categories in the sense of Cline–Parshall–Scott [6]. The union of these module categories exhausts the category of rational representations of the given group. When G is a symplectic group, the set of weights occuring in $E^{\otimes r}$ is saturated, and $S_{env}(\mathrm{Sp}_n)$ coincides with the generalised Schur algebra associated with this set of weights. A similar result holds true for even orthogonal groups. In the case of odd orthogonal groups, the set of weights in $E^{\otimes r}$ is not saturated. Hence for n odd, $S_{env}(\mathrm{O}_n)$ is in general not a generalised Schur algebra. It is, however, a direct summand of a generalised Schur algebra. Our assumption $n > 2r$ in the orthogonal case ensures that the enveloping algebras $S_{env}(\mathrm{O}_n)$ and $S_{env}(\mathrm{SO}_n)$ of the orthogonal and the special orthogonal group, both acting on tensor space, do coincide. The same is true for the corresponding generalised Schur algebras. See [15, Section 4] and [29, 30] for details. This will allow us in Subsection 3.3 to use Brundan's results [4] on restriction from general linear to special orthogonal groups in order to get information on restriction to orthogonal groups.

2.2. **Schur algebras of Brauer algebras.** Schur algebras $S_B(n, r)$ of Brauer algebras have been studied in the preceding article [24]. These algebras are endomorphism algebras of direct sums of permutation modules of Brauer algebras, which have been defined by Hartmann and Paget [22]. For $l \leq \frac{r}{2}$ and $\lambda \vdash r - 2l$, the permutation module $M(l, \lambda)$ is defined as

$$M(l, \lambda) = M^\lambda \otimes_{k\Sigma_{r-2l}} e_l B_r$$

where

(3) $e_l = \frac{1}{\delta^l} \cdot$ 

with l arcs in top and bottom row, respectively, and M^λ is the permutation module (indexed by λ) associated with the symmetric group Σ_{r-2l}.

By definition, for any fixed parameter $\delta \neq 0$, the Schur algebra $S_B(n, r) := S_B(n, r, \delta)$ is the endomorphism ring of the direct sum $\oplus_{l,\lambda} M(l, \lambda)$ of all permutation modules of the

Brauer algebra:

$$S_B(n,r) = \mathrm{End}_{B_r}(\bigoplus_{\lambda \vdash r-2l, 0 \leq l \leq \frac{r}{2}} M(l,\lambda)).$$

We drop the parameter δ in notation; later on, it will be assumed to be n or $-n$ when we work with the orthogonal or the symplectic group, respectively.

In [24, Theorem 7.1] it has been shown, in particular, that $S_B(n,r)$ is a quasi-hereditary algebra. So its module category is a highest weight category. Moreover, $S_B(n,r)$ is related to the Brauer algebra B_r by a Schur-Weyl duality on the direct sum $\oplus M(l,\lambda)$ of the permutation modules. This Schur-Weyl duality is different from that stated in Corollary 1.2, but related to it; see Section 3.11 below for more information.

In [24, Theorem 5.3] an explicit basis of the Schur algebra $S_B(n,r)$ has been constructed, consisting of B-homomorphisms

$$\phi_{u,\pi,\sigma} : M(l,\lambda) \to M(m,\mu) \tag{4}$$

with

$$\begin{array}{rll} \pi \in \Sigma_{r-2m} & \text{a representative of} & \Sigma_\mu \backslash \Sigma_{r-2m} / (\Sigma_{r-2u} \times H_{u-m}), \\ \sigma \in \Sigma_{r-2l} & \text{a representative of} & (\Sigma_\nu \times H_{u-l}) \backslash \Sigma_{r-2l} / \Sigma_\lambda, \end{array}$$

where $\Sigma_\nu = \Sigma_{r-2u} \cap \pi^{-1} \Sigma_\mu \pi$.

Then the B-homomorphism

$$\phi_{u,\pi,\sigma} : M(l,\lambda) \longrightarrow M(m,\mu)$$

is explicitly given on a generator of $M(l,\lambda)$ by

$$\phi_{u,\pi,\sigma}(\Sigma_\lambda \cdot id \otimes e_l) = \sum_{\alpha \in \Sigma_\lambda \cap \sigma^{-1}(\Sigma_\nu \times H_{u-l})\sigma \backslash \Sigma_\lambda} (\Sigma_\mu \cdot id \otimes e_{\pi,u}) \sigma \cdot \alpha$$

(see [24, Subsection 5.3]).

By [24, Section 10], such a basis element $\phi_{u,\pi,\sigma}$ corresponds to a triple, say $(v, w, \xi(\hat{\sigma}))$, defined as follows:

$$\begin{array}{r} v \in V_{u-m}^{r-2m} / \sim_{k\Sigma_\mu} \text{ is a partial (bottom) arc configuration, corresponding to } \pi; \\ w \in V_{u-l}^{r-2l} / \sim_{k\Sigma_\lambda} \text{ is a partial (top) arc configuration, corresponding to } \sigma; \\ \xi(\hat{\sigma}) \text{ is the Schur algebra element corresponding to the double coset } \Sigma_\rho \hat{\sigma} \Sigma_\nu. \end{array}$$

In the third datum, $\Sigma_\rho = \Sigma_{r-2u} \cap \sigma \Sigma_\lambda \sigma^{-1}$. The element $\hat{\sigma}$ has been defined in [24, Notation 8.3] as the restriction of σ to the 'free' vertices not attached to horizontal arcs. This way of writing the basis uses [24, Section 8], which asserts that the classical Schur algebra $S(n, r-2l)$, for each l, is a non-unital subalgebra of $S_B(n,r)$.

For the proof of Proposition 3.8 we need the following formula that expresses the dimension of $\mathrm{Hom}_{B_r}(M(l,\lambda), M(m,\mu))$ in terms of generalised Brauer diagrams by indexing and counting basis elements of $S_B(n,r)$ as explained above.

Proposition 2.1. *Fix a partition λ of $r-2l$ and a partition μ of $r-2m$. Denote by X_{r-2m}^{r-2l} the set of all Brauer diagrams with $r-2l$ vertices in the top row and $r-2m$ vertices in the bottom row. Let the group $\Sigma_\lambda \times \Sigma_\mu$ act on X_{r-2m}^{r-2l} by the first component*

of its elements permuting the vertices in the top row and the second component permuting the vertices in the bottom row.

Then the dimension of $\mathrm{Hom}_{B_r}(M(l,\lambda), M(m,\mu))$ *equals the number of orbits in* $\Sigma_\lambda \backslash X^{r-2l}_{r-2m} / \Sigma_\mu := X^{r-2l}_{r-2m}/(\Sigma_\lambda \times \Sigma_\mu)$.

Proof. By the description above, a homomorphism $\phi_{u,\pi,\sigma} : M(l,\lambda) \longrightarrow M(m,\mu)$ is represented by a triple consisting of a top arc configuration, a bottom arc configuration and a permutation defining the through strings, that is, by a Brauer diagram, modulo the action of Σ_λ on the top vertices and of Σ_μ on the bottom vertices. □

3. Proof of Main Theorem, and consequences

3.1. **Outline.** The proof of the Main Theorem 1.1 occupies the following seven subsections. Subsection 3.2 recalls basic material on symmetric powers and fixes notation. Subsection 3.3 collects several abstract results from the literature, on categories of representations of classical groups and on restriction from general linear to orthogonal or symplectic groups. We are going to use these results to show that certain dimensions of homomorphism spaces do not depend on the characteristic of the underlying field k. In Section 4, an alternative combinatorial proof of this fact and a direct combinatorial description of these morphism spaces will be given in case of characteristic zero or large prime characteristic.

Subsection 3.4 introduces Schur functors and defines the algebra homomorphism $\phi : S_B(n,r) \to C$ that will be shown to be an isomorphism. Moreover, an alternative description of tensor products of symmetric powers will be given, as images of an inverse Schur functor. In Subsection 3.5, a characteristic free combinatorial description will be given for the space of G-module homomorphisms from tensor space to tensor products of symmetric powers. This is used in Subsection 3.6 to describe permutation modules of Brauer algebras as images under a Schur functor, providing a counterpart to the result in Subsection 3.5. In Subsection 3.7, injectivity of Φ is shown, and in Subsection 3.8, proving surjectivity finishes the proof of the Main Theorem 1.1. Subsections 3.9 and 3.10 prove and explain Corollaries 1.2 and 1.3. Finally, Subsection 3.11 discusses connections between several Schur functors and puts the information together.

3.2. **Symmetric powers.** Let E be an n-dimensional k-vector space and $\lambda = (\lambda_1, \ldots, \lambda_n)$ a composition of r into n parts some of which possibly are zero. For a natural number m, define the mth symmetric power

$$\mathrm{Sym}^m E = E^{\otimes m}/I_m$$

with $I_m = \langle x_1 \otimes \cdots \otimes x_m - x_{\tau(1)} \otimes \cdots \otimes x_{\tau(m)} \mid \tau \in \Sigma_m, x_i \in E \rangle$. The symmetric power $\mathrm{Sym}^m E$ can be identified with the vector space of all polynomials in n variables that are homogeneous of degree m. Denoting by $\{v_1, \ldots, v_n\}$ a basis of E, the space $\mathrm{Sym}^m E$ has basis

$$\{v_1^{i_1} \cdots v_n^{i_n} \mid \sum_{j=1}^{n} i_j = m\}.$$

We write π_r for the natural projection $\pi_r : E^{\otimes r} \to \mathrm{Sym}^r E$. For the composition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of r, define $\mathrm{Sym}^\lambda E = \mathrm{Sym}^{\lambda_1} E \otimes \cdots \otimes \mathrm{Sym}^{\lambda_n} E$. We will refer to $\mathrm{Sym}^\lambda E$ as *symmetric powers*. Let $\pi_\lambda = \pi_{\lambda_1} \otimes \cdots \otimes \pi_{\lambda_n}$. For $x = x_1 \otimes \cdots \otimes x_r$ write x^λ for the element

$$x^\lambda = x_1 \cdots x_{\lambda_1} \otimes x_{\lambda_1+1} \cdots x_{\lambda_1+\lambda_2} \otimes \cdots \otimes \cdots x_r \in \mathrm{Sym}^\lambda E.$$

Then $\pi_\lambda(x) = x^\lambda$. More generally, we use the following notation: Let $x = x_1 \otimes \cdots \otimes x_{r-2l}$ and $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a composition of $r - 2l$. Write x^λ for the element

$$x^\lambda = x_1 \cdots x_{\lambda_1} \otimes x_{\lambda_1+1} \cdots x_{\lambda_1+\lambda_2} \otimes \cdots \otimes \cdots x_{r-2l} \in \mathrm{Sym}^\lambda E.$$

Define the G-homomorphism $\pi_{\lambda,l}$ as composition

$$\pi_{\lambda,l} : E^{\otimes r} \to E^{\otimes r-2l} \otimes \vartheta^l \to \mathrm{Sym}^\lambda E \otimes \vartheta^l$$

where the first map is given by l contractions on adjacent places on the last $2l$ places in the tensor product, and the second map is given by the natural projection $x \mapsto x^\lambda$, tensored with the identity map on the last $2l$ places of the tensor product. We say that x_i and x_j are in the same λ-component of the tensor x, if for some $t \geq 0$,

$$\sum_{s=1}^{t} \lambda_s < i, j \leq \sum_{s=1}^{t+1} \lambda_s.$$

Then the kernel of the map π_λ is spanned by elements of the form

$$x_{ij} := x(id - (i,j)) = \cdots \otimes x_i \otimes \cdots \otimes x_j \otimes \cdots - \cdots \otimes x_j \otimes \cdots \otimes x_i \otimes \cdots$$

where x_i and x_j lie in the same λ-component.

3.3. **Categories of filtered modules.** The endomorphism ring C to be determined in this article depends, by definition, on the underlying field k and its characteristic. The Schur algebra $S_B(n,r)$ of the Brauer algebra, which will be shown to be isomorphic to C, has been shown in [24] not to depend on k, in the sense that it has a combinatorially defined basis that is independent of k. The structure constants of this basis and the ring structure of $S_B(n,r)$ - for instance, whether it is semisimple or not - do however heavily depend on k. The dimensions of the Brauer algebras and of the generalised Schur algebras of classical groups also do not depend on k. Hence the dimensions of the endomorphism rings occuring in Schur-Weyl duality are independent of k, too. In this subsection we recall results from representation theory of classical groups that imply such characteristic independence and we collect facts to be used later on to show that also the dimension of C does not depend on k.

Rational representations of classical groups G form highest weight categories. Therefore, Donkin's generalised Schur algebras [11] are quasi-hereditary algebras as defined by Cline, Parshall and Scott in [6]. Their projective modules are filtered by *standard modules* $\Delta(\lambda)$ and their injective modules are filtered by *co-standard* modules $\nabla(\lambda)$. The standard modules are precisely the *Weyl modules* and the co-standard modules are dual Weyl modules, where dual refers to the contravariant duality in the category of rational representations.

The category $\mathcal{F}(\Delta)$ is the full subcategory of the category of rational representations consisting of the modules that admit a filtration whose sections are Weyl modules. The

category $\mathcal{F}(\nabla)$ is defined dually, using costandard modules. Crucial homological information is provided by the following *orthogonality property*:

$$\operatorname{Ext}^j_G(\Delta(\lambda), \nabla(\mu)) = \begin{cases} k, & \text{if } j = 0 \text{ and } \lambda = \mu, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Using long exact cohomology sequences, a similar Ext-orthogonality is obtained between objects in $\mathcal{F}(\Delta)$ and objects in $\mathcal{F}(\nabla)$. An important consequence is that objects in the intersection of these two categories have no self-extensions. More precisely, $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ equals $add(T)$, the category of direct summands of direct sums of Ringel's characteristic tilting module T, see [32] or, for instance, [14]. Up to a choice of multiplicities of direct summands, the equality $add(T) = \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ can be taken as definition of T. The characteristic tilting module T is an injective object in $\mathcal{F}(\Delta)$ and a projective one in $\mathcal{F}(\nabla)$. By the orthogonality property (5), the functors $\operatorname{Hom}_G(-, \nabla(\mu))$ are exact on short exact sequences in $\mathcal{F}(\Delta)$ and the functors $\operatorname{Hom}_G(\Delta(\lambda), -)$ are exact on short exact sequences in $\mathcal{F}(\nabla)$. Inductively, it follows that the dimension of $\operatorname{Hom}_G(X, Y)$ for X in $\mathcal{F}(\Delta)$ and Y in $\mathcal{F}(\nabla)$ only depends on the multiplicities in the Δ-filtration of X and in the ∇-filtration of Y. Such filtrations, and the multiplicities, are preserved under modular reduction from characteristic zero to prime characteristic. Therefore, dimensions of $\operatorname{Hom}_G(X, Y)$ are characteristic independent: More precisely, when X is a module with standard filtration and Y is a module with co-standard filtration, then the space of homomorphisms $\operatorname{Hom}_G(X, Y)$ has dimension $\sum_\lambda a_\lambda b_\lambda$, where a_λ is the multiplicity of $\Delta(\lambda)$ in any standard filtration of X and b_λ is the multiplicity of $\nabla(\lambda)$ in any co-standard filtration of Y. These multiplicities are well-defined, by general theory of quasi-hereditary algebras, and independent of k. Hence the dimension of $\operatorname{End}_G(E^{\otimes r})$ does not depend on k or its characteristic. In order to establish characteristic independence of dimensions of certain morphism spaces, we will use that the relevant objects are in the subcategories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$, respectively, see Proposition 3.2 below.

An example is Schur-Weyl duality for general linear groups. Over $G = GL_n$ with $n \geq r$, tensor space $E^{\otimes r}$ is projective and injective and therefore a direct summand of the characteristic tilting module. When dropping the assumption $n \geq r$, tensor space is not projective any more, but still a direct summand of the characteristic tilting module. Even in this general case, Schur-Weyl duality can be derived using such arguments, see [26] for details.

Proposition 3.1. *Let G be a classical group. Let $n \geq r$ when $G = GL_n$ is a general linear group and let n and r be as in Theorem 1.1 when G is orthogonal or symplectic. Then tensor space $E^{\otimes r}$ is relative injective in $\mathcal{F}(\Delta)$ and relative projective in $\mathcal{F}(\nabla)$.*

Here, relative projective or injective means exactness of the respective Hom-functor on short exact sequences in the subcategory, and thus vanishing of first extension groups. For example, $P \in \mathcal{F}(\nabla)$ is relative projective in $\mathcal{F}(\nabla)$ if and only if $\operatorname{Ext}^1_G(P, -)$ vanishes on $\mathcal{F}(\nabla)$, which is an extension closed subcategory.

Proof. When $G = GL_n$ and $n \geq r$, then tensor space $E^{\otimes r}$ is a projective module over the classical Schur algebra. Since it is self-dual, it is also injective. See [19] for details. Projective modules are Δ-filtered and injective modules are ∇-filtered. Therefore, $E^{\otimes r}$

has both filtrations and must be a direct summand of a direct sum of copies of the characteristic tilting module T, which is relative injective in $\mathcal{F}(\Delta)$ and relative projective in $\mathcal{F}(\nabla)$.

Now let G be the symplectic or the orthogonal group acting on tensor space by restricting the GL_n-action. When restricting representations from general linear to symplectic or orthogonal groups, the categories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ are mapped into the corresponding categories for the smaller groups by results of Donkin and Brundan, see [13, 4]. More precisely, Proposition 3.3 in [4] states in particular that the pairs (SL_n, SP_n) and (SL_n, SO_n) (the latter only in case of characteristic different from two) are good pairs. This means restricting from the first group to the second sends modules with ∇-filtrations over the first group to modules with ∇-filtrations over the second group. The first case, involving the symplectic group, follows from a result of Donkin, in Appendix A of [13].

Hence, tensor space is a direct summand of a characteristic tilting module over every classical group, and thus it is relative injective in $\mathcal{F}(\Delta)$ and relative projective in $\mathcal{F}(\nabla)$. □

Over $G = GL_n$, the full tensor powers of the symmetric powers are injective and therefore they are objects in $\mathcal{F}(\nabla)$. Because of the compatibility with restriction just quoted this implies:

Proposition 3.2. *Let G be a classical group, n and r as in Proposition 3.1 and λ a partition of some $s \leq r$. Then the dimension of* $\mathrm{Hom}_G(E^{\otimes r}, \mathrm{Sym}^\lambda E)$ *does not depend on the ground field k.*

When working with subcategories and using cohomology it is important to know that kernels of certain surjective maps belong to the given subcategory. We will need:

Lemma 3.3. *In the short exact sequence*

$$0 \to \textit{kernel} \to E^{\otimes|\lambda|} \xrightarrow{\pi_\lambda} \mathrm{Sym}^\lambda E \to 0$$

the kernel is in $\mathcal{F}(\nabla)$.

This short exact sequence gives the relative projective cover of $\mathrm{Sym}^\lambda E$ in the subcategory $\mathcal{F}(\nabla)$. Lemma 3.3 has been shown by Donkin in [14, claim 2.1.(15((ii)(b)] in the case of quantum general linear groups. As remarked there, the proof given there works for reductive algebraic groups as well.

3.4. **Schur functors.** From now on, G is a symplectic or orthogonal group and the assumptions of Theorem 1.1 are valid. Following [15], we define the Schur functor f_0 and the inverse Schur functor g_0 for the symplectic and orthogonal groups as follows:

$$\begin{aligned} f_0 :&\quad \text{mod-}G \to \text{mod-}B_r, &\quad f_0(-) &= \mathrm{Hom}_G(E^{\otimes r}, -), \\ g_0 :&\quad \text{mod-}B_r \to \text{mod-}G, &\quad g_0(-) &= - \otimes_{B_r} E^{\otimes r}. \end{aligned}$$

with $G = \mathrm{Sp}_n$ or $G = \mathrm{O}_n$ respectively. Here, as throughout, module categories are categories of finite dimensional right modules. Unlike in [15], we assume here that the action of the Brauer algebra B_r on the tensor space is without a twist by a sign (since we

are using $B_r = \mathfrak{A}_r$ as explained above). Moreover, we use G-modules instead of modules over a generalised Schur algebra, to simplify notation.

Lemma 3.4. *For all l and all $\lambda \vdash r-2l$ there is an isomorphism of G-modules*

$$g_0(M(l,\lambda)) \simeq \mathrm{Sym}^\lambda E \otimes \vartheta^l.$$

This has been shown in [15, Prop 2.2]. In our notation it can be seen as follows:

Proof. Let $\pi_\lambda = \pi_{\lambda,0}$. In the following, $g : \text{mod-}\Sigma_r \to \text{mod-}S(n,r)$, defined by $g(-) = -\otimes_{k\Sigma_r} E^{\otimes r}$ denotes the usual classical (type A) inverse Schur functor associated to GL_n. Then there is a chain of G-module isomorphisms, whose composition we denote by κ:

$$\begin{aligned} g_0(M(l,\lambda)) &= M(l,\lambda)\otimes_{B_r} E^{\otimes r} \\ &\simeq M^\lambda \otimes_{k\Sigma_{r-2l}} e_l B_r \otimes_{B_r} E^{\otimes r} \\ &\simeq M^\lambda \otimes_{k\Sigma_{r-2l}} e_l E^{\otimes r} \\ &\simeq M^\lambda \otimes_{k\Sigma_{r-2l}} (E^{\otimes r-2l}\otimes \vartheta^l) \\ &\simeq (M^\lambda \otimes_{k\Sigma_{r-2l}} E^{\otimes r-2l})\otimes \vartheta^l \\ &\simeq g(M^\lambda)\otimes\vartheta^l \\ &\simeq \mathrm{Sym}^\lambda E\otimes\vartheta^l. \end{aligned}$$

The latter is isomorphic to $\mathrm{Sym}^\lambda E$ as G-module since ϑ is the trivial G-module.

Here, as G-modules,

$$\widehat{\ }: e_l E^{\otimes r} = E^{\otimes r-2l}\otimes\vartheta^l \simeq E^{\otimes r-2l}, \qquad x \mapsto \widehat{x},$$

given by $\widehat{z\otimes\vartheta^l} = z$, and Σ_{r-2l} operates by place permutations (without sign) on the tensor space $E^{\otimes r-2l}$. Given $x \in E^{\otimes r}$, the isomorphism κ above is realised by mapping

$$\Sigma_\lambda\sigma\otimes e_l b\otimes x \mapsto \pi_\lambda(\widehat{\sigma e_l b x})\otimes\vartheta^l,$$

with well-defined inverse map given by

$$\pi_\lambda(\widehat{x})\otimes\vartheta^l \mapsto \Sigma_\lambda id\otimes e_l\otimes x.$$

□

The inverse Schur functor g_0 induces an algebra homomorphism

$$\Phi : S_B(n,r) \to C, \qquad \alpha \mapsto g_0(\alpha) \tag{6}$$

where $S_B(n,r) = \mathrm{End}_{B_r}(\oplus M(l,\lambda))$ and

$$C = \mathrm{End}_G(\bigoplus_{l\le\frac{r}{2},\lambda\vdash r-2l} (\mathrm{Sym}^\lambda E\otimes\vartheta^l)) \simeq \mathrm{End}_G(\bigoplus_{l\le\frac{r}{2},\lambda\vdash r-2l} (\mathrm{Sym}^\lambda E)).$$

It is this map Φ that will be shown to be an isomorphism, when proving Theorem 1.1.

3.5. **Maps from tensor space to symmetric powers.** By Proposition 3.2, the dimension of $\mathrm{Hom}_G(E^{\otimes s}\otimes\vartheta^l, \mathrm{Sym}^\mu E\otimes\vartheta^m)$ does not depend on k. We will now give a combinatorial description of these homomorphisms in terms of Brauer diagrams.

Proposition 3.5. *Let $s=r-2l$, $t=r-2m$ and μ a partition of t. Then in the diagram*

$$\begin{array}{ccc} E^{\otimes s}\otimes\vartheta^l & \xrightarrow{\ \gamma\ } & E^{\otimes t}\otimes\vartheta^m \\ & {\scriptstyle\beta}\searrow & \downarrow{\scriptstyle\pi_\mu} \\ & & \mathrm{Sym}^\mu E\otimes\vartheta^m \end{array},$$

composition with π_μ provides a surjective map

$$\pi_\mu\circ - : \gamma\mapsto\pi_\mu\circ\gamma=\beta$$

inducing an isomorphism

$$\alpha : \mathrm{Hom}_G(E^{\otimes s}\otimes\vartheta^l, E^{\otimes t}\otimes\vartheta^m)/\Sigma_\mu \simeq \mathrm{Hom}_G(E^{\otimes s}\otimes\vartheta^l, \mathrm{Sym}^\mu E\otimes\vartheta^m).$$

By Schur-Weyl duality (as formulated in [35]), the maps γ are linear combinations of (s,t)-Brauer diagrams, whose rows have s and t vertices, respectively. Each β is of the form $\beta=\pi_\mu\circ\gamma$. Moreover, γ_1 and γ_2 define the same β if and only if there exists $\sigma\in\Sigma_\mu$ such that $\gamma_2=\sigma\circ\gamma_1$.

Proof. Composition with π_μ defines a map $\pi_\mu\circ -$ as stated. This map is surjective: Indeed, the map $\beta : E^{\otimes s}\otimes\vartheta^l\to\mathrm{Sym}^\mu E\otimes\vartheta^m$ starts and ends in objects of $\mathcal{F}(\nabla)$ and the surjective map $\pi_\mu : E^{\otimes t}\otimes\vartheta^m\to\mathrm{Sym}^\mu E\otimes\vartheta^m$ is part of a short exact sequence in $\mathcal{F}(\nabla)$, by Lemma 3.3. By Proposition 3.1, module $E^{\otimes t}\simeq E^{\otimes s}\otimes\vartheta^l$ is relative projective in $\mathcal{F}(\nabla)$. Being relative projective is equivalent to having the lifting property:

$$\begin{array}{ccccc} & & & & E^{\otimes s}\otimes\vartheta^l \\ & & & {\scriptstyle\exists\gamma}\swarrow & \vdots{\scriptstyle\forall\beta} \\ kernel(\pi_\mu) & \longrightarrow & E^{\otimes t}\otimes\vartheta^m & \xrightarrow{\ \pi_\mu\ } & \mathrm{Sym}^\mu E\otimes\vartheta^m \end{array}$$

Thus, $\beta=\pi_\mu\circ\gamma$ for some γ. Note that the lifting property requires the kernel of the surjective map π_μ to belong to the subcategory $\mathcal{F}(\nabla)$.

Certainly, γ_1 and γ_2 define the same β if there exists $\sigma\in\Sigma_\mu$ such that $\gamma_2=\sigma\circ\gamma_1$. We have to show the converse, which implies injectivity of α. By Proposition 3.2, the dimension of $\mathrm{Hom}_G(E^{\otimes s}\otimes\vartheta^l, \mathrm{Sym}^\mu E\otimes\vartheta^m)$ does not depend on the choice of the ground field k.Therefore, it is enough to check injectivity of α in characteristic zero.

In that case, $\mathrm{Sym}^\mu E\otimes\vartheta^m$ is a direct summand of $E^{\otimes t}\otimes\vartheta^m$ through the split epimorphism π_μ. More precisely, this provides an isomorphism $\mathrm{Sym}^\mu E\otimes\vartheta^m\simeq(E^{\otimes t}\otimes\vartheta^m)/\Sigma_\mu$. Over GL_n, Schur-Weyl duality implies an isomorphism

$$\begin{aligned} \mathrm{Hom}_{GL_n}(E^{\otimes t}\otimes\vartheta^m, \mathrm{Sym}^\mu E\otimes\vartheta^m) &\simeq \mathrm{Hom}_{GL_n}(E^{\otimes t}\otimes\vartheta^m, (E^{\otimes t}\otimes\vartheta^m)/\Sigma_\mu) \\ &\simeq \mathrm{Hom}_{GL_n}(E^{\otimes t}\otimes\vartheta^m, (E^{\otimes t}\otimes\vartheta^m))/\Sigma_\mu \\ &\simeq k\Sigma_t/\Sigma_\mu. \end{aligned}$$

Indeed, by Schur-Weyl duality the GL_n-maps between the tensor spaces are linear combinations of group elements in Σ_t. Thus the GL_n-maps into the symmetric power are the compositions of these maps with the split epimorphism π_μ, which identifies the elements in each coset of Σ_t/Σ_μ. This shows that $\mathrm{Hom}_{GL_n}(E^{\otimes t} \otimes \vartheta^m, \mathrm{Sym}^\mu E \otimes \vartheta^m) \subset k\Sigma_t/\Sigma_\mu$. Since tensor space is isomorphic to a full direct sum of copies of symmetric powers (the contragredient dual of the Schur algebra, which is a full set of injective modules), the inclusion must be equality. The multiplication map

$$\mathrm{Hom}_G(E^{\otimes s} \otimes \vartheta^l, E^{\otimes t} \otimes \vartheta^m) \otimes_{k\Sigma_t} \mathrm{Hom}_{GL_n}(E^{\otimes t} \otimes \vartheta^m, X) \to \mathrm{Hom}_G(E^{\otimes s} \otimes \vartheta^l, X)$$

is an isomorphism for $X = E^{\otimes t} \otimes \vartheta^m$, again by Schur-Weyl duality. Thus it is an isomorphism for X a direct summand of $E^{\otimes t} \otimes \vartheta^m$, too, hence in particular for $X = \mathrm{Sym}^\mu E \otimes \vartheta^m$. This assertion is a special case of [15, Lemma 2.3(ii)]. This proves injectivity of α. □

3.6. **The image of $\mathrm{Sym}^\lambda E$ under the Schur functor.** We next apply the Schur functor f_0 to symmetric powers. The following result restates parts of [15, Theorem 2.1 and Theorem 4.1] in our notation:

Lemma 3.6. *For all l and all $\lambda \vdash r - 2l$ there is a right B-module isomorphism*

$$f_0(\mathrm{Sym}^\lambda E \otimes \vartheta^l) \simeq M(l, \lambda).$$

Proof. First we show that the two vector spaces have the same dimension, and then we provide an explicit G-module isomorphism. By [22, 24], the vector space dimension of $M(l, \lambda)$ does not depend on the choice or characteristic of k. More precisely, by definition $M(l, \lambda) = M^\lambda \otimes e_l B_r$ has a basis consisting of Σ_λ-orbits on $e_l B_r$; so, the basis elements are represented by Σ_λ-orbits of Brauer diagrams with rows of r and $r - 2l$ vertices, respectively, the remaining $2l$ vertices being reserved for l fixed arcs.

By Proposition 3.2 in Subsection 3.3, the dimension of

$$f_0(\mathrm{Sym}^\lambda E \otimes \vartheta^l) = \mathrm{Hom}_G(E^{\otimes r}, \mathrm{Sym}^\lambda E \otimes \vartheta^l)$$

does not depend on k either. By Proposition 3.5, this vector space has a basis consisting of Σ_λ-orbits of Brauer diagrams also having rows of r and $r - 2l$ vertices with l fixed arcs on the remaining $2l$ vertices; hence this basis is in bijection with the above basis of $M(l, \lambda)$.

An explicit isomorphism $\psi : M(l, \lambda) \to f_0(\mathrm{Sym}^\lambda E \otimes \vartheta^l)$ with $x \mapsto \psi_x$ is given by the following map: Given an element $x = \Sigma_\lambda \sigma \otimes e_l b \in M^\lambda \otimes e_l B = M(l, \lambda)$, then x is mapped to

$$\psi_x : E^{\otimes r} \xrightarrow{b\cdot} E^{\otimes r} \xrightarrow{e_l \cdot} E^{\otimes r-2l} \otimes \vartheta^l \xrightarrow{\sigma\cdot} E^{\otimes r-2l} \otimes \vartheta^l \xrightarrow{\pi_\lambda} \mathrm{Sym}^\lambda E \otimes \vartheta^l,$$

that is, $\psi_x(v) = \pi_\lambda(\sigma e_l b v)$. By Schur-Weyl duality, see (1), this is a right G-module homomorphism. Map ψ sends the above basis of $M(l, \lambda)$ to the above basis of $f_0(\mathrm{Sym}^\lambda E \otimes \vartheta^l)$; hence ψ is an isomorphism. □

3.7. **Injectivity of $\Phi : S_B(n,r) \to C$.** By Lemma 3.4, the inverse Schur functor sends permutation modules $M(l,\lambda)$ to symmetric powers $\mathrm{Sym}^\lambda E \otimes \vartheta^l$. Moreover, there exists a homomorphism of algebras

$$\Phi : S_B(n,r) \longrightarrow C, \qquad \phi_{u,\pi,\sigma} \mapsto g_0(\phi_{u,\pi,\sigma}).$$

Under the homomorphism Φ, the basis element $\phi_{u,\pi,\sigma} : M(l,\lambda) \to M(m,\mu)$ of $S_B(n,r)$, see (4), is mapped to a G-homomorphism

$$g_0(\phi_{u,\pi,\sigma}) = \phi_{u,\pi,\sigma} \otimes id : \mathrm{Sym}^\lambda E \otimes \vartheta^l \to \mathrm{Sym}^\mu E \otimes \vartheta^m$$

in the algebra C.

Proposition 3.7. *The map Φ is injective, that is, $S_B(n,r)$ is a subalgebra of C.*

Proof. By Lemma 3.6 and Lemma 3.4, there are the following two isomorphisms of B-modules:

$$M(l,\lambda) \to \mathrm{Hom}_G(E^{\otimes r}, \mathrm{Sym}^\lambda E \otimes \vartheta^l), \qquad \Sigma_\lambda \mathrm{id} \otimes e_l b \mapsto \pi_\lambda \circ m_{e_l} \circ m_b,$$

and $\mathrm{Hom}_G(E^{\otimes r}, \mathrm{Sym}^\lambda E \otimes \vartheta^l) \to \mathrm{Hom}_G(E^{\otimes r}, M(l,\lambda) \otimes_B E^{\otimes r})$ with

$$\pi_\lambda \circ m_{e_l} \circ m_b \mapsto \kappa^{-1} \circ \pi_\lambda \circ m_{e_l} \circ m_b.$$

Here m_a denotes multiplication by the element a from left, and

$$\kappa : M(l,\lambda) \otimes_B E^{\otimes r} \to \mathrm{Sym}^\lambda E \otimes \vartheta^l, \qquad \Sigma_\lambda \mathrm{id} \otimes e_l b \otimes x \mapsto \pi_\lambda(\widehat{e_l b x}) \otimes \vartheta^l.$$

The composition of these two isomorphisms is denoted as $\alpha(l,\lambda)$, that is

$$\alpha(l,\lambda) : M(l,\lambda) \to \mathrm{Hom}_G(E^{\otimes r}, M(l,\lambda) \otimes_B E^{\otimes r}), \qquad z \mapsto (x \mapsto z \otimes x).$$

Let $\varphi : M(l,\lambda) \to M(m,\mu)$ be some B-homomorphism. Then applying the inverse Schur functor g_0 and the Schur functor f_0, we obtain:

$$\begin{aligned} \Phi(\varphi) = g_0(\varphi) = \varphi \otimes \mathrm{id} \quad &: \quad M(l,\lambda) \otimes E^{\otimes r} \to M(m,\mu) \otimes_B E^{\otimes r}, \\ f_0(g_0(\varphi)) = (\varphi \otimes \mathrm{id}) \circ - \quad &: \quad \mathrm{Hom}_G(E^{\otimes r}, M(l,\lambda) \otimes_B E^{\otimes r}) \to \mathrm{Hom}(E^{\otimes r}, M(m,\mu) \otimes_B E^{\otimes r}). \end{aligned}$$

We check that the following diagram is commutative:

$$\begin{array}{ccc} M(l,\lambda) & \xrightarrow{\alpha(l,\lambda)} & \mathrm{Hom}(E^{\otimes r}, M(l,\lambda) \otimes E^{\otimes r}) \\ \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle f_0(g_0(\varphi))} \\ M(m,\mu) & \xrightarrow{\alpha(m,\mu)} & \mathrm{Hom}(E^{\otimes r}, M(m,\mu) \otimes E^{\otimes r}). \end{array}$$

Indeed, it is enough to check commutativity by evaluating the maps on a generator $\Sigma_\lambda \mathrm{id} \otimes e_l$ of $M(l,\lambda)$. By the definition of $M(l,\lambda)$,

$$f_0(g_0(\varphi)) \circ \alpha(l,\lambda)(\Sigma_\lambda \mathrm{id} \otimes e_l) : \quad x \mapsto \varphi(\Sigma_\lambda \mathrm{id} \otimes e_l) \otimes x.$$

Similarly,

$$\alpha(m,\mu)(\varphi(\Sigma_\lambda \mathrm{id} \otimes e_l)) : \quad x \mapsto \varphi(\Sigma_\lambda \mathrm{id} \otimes e_l) \otimes x.$$

Assume that $\Phi(\varphi) = 0$, that is $g_0(\varphi) = 0$. Then $f_0(g_0(\varphi)) = 0$ and since $\alpha(m,\mu)$ is an isomorphism, it follows that

$$\varphi = \alpha(m,\mu)^{-1} \circ f_0(g_0(\varphi)) \circ \alpha(l,\lambda) = 0.$$

This implies that the map $\Phi : S_B(n,r) \to C$ is injective. □

Composing the Schur functor f_0 with the isomorphisms κ^{-1} and α^{-1} defines an algebra homomorphism

$$\Psi : C \longrightarrow S_B(n,r), \qquad \beta \mapsto \beta \circ -,$$

and the commutative diagram in the proof shows that $\Psi \circ \Phi$ is the identity.

The map $\alpha(l,\lambda)$ can be produced as an adjunction unit. In fact, the natural isomorphism

$$\mathrm{Hom}_G(M(l,\lambda) \otimes_B E^{\otimes r}, M(l,\lambda) \otimes_B E^{\otimes r}) \simeq \mathrm{Hom}_B(M(l,\lambda), \mathrm{Hom}_G(E^{\otimes r}, M(l,\lambda) \otimes E^{\otimes r})$$

sends the identity map on $M(l,\lambda)$ to the map $\alpha(l,\lambda) : m \mapsto [x \mapsto m \otimes x]$. Thus, commutativity of the diagram in the above proof also follows from adjunction being natural.

3.8. **Surjectivity of $\Phi : S_B(n,r) \to C$.** In Proposition 3.5, a basis of $\mathrm{Hom}_G(E^{\otimes s} \otimes \vartheta^l, \mathrm{Sym}^\mu E \otimes \vartheta^m)$ has been given combinatorially, in terms of Brauer diagrams. Next we produce from this basis a combinatorial basis of $\mathrm{Hom}_G(\mathrm{Sym}^\lambda E \otimes \vartheta^l, \mathrm{Sym}^\mu E \otimes \vartheta^m)$. Counting basis elements yields surjectivity of Φ, finishing the proof of Theorem 1.1.

Proposition 3.8. *Fix $s = r - 2l$, $t = r - 2m$, λ a partition of s and μ a partition of t. Then in the diagram*

$$\begin{array}{ccc} E^{\otimes s} \otimes \vartheta^l & & \\ {\scriptstyle \pi_\lambda}\downarrow & \overset{\beta}{\searrow} & \\ \mathrm{Sym}^\lambda E \otimes \vartheta^l & \xrightarrow[\alpha]{} & \mathrm{Sym}^\mu E \otimes \vartheta^m \end{array} ,$$

pre-composition with π_λ provides an injective map

$$- \circ \pi_\lambda : \alpha \mapsto \alpha \circ \pi_\lambda = \beta$$

inducing an isomorphism

$$\mathrm{Hom}_G(\mathrm{Sym}^\lambda E \otimes \vartheta^l, \mathrm{Sym}^\mu E \otimes \vartheta^m) \simeq \Sigma_\lambda \backslash \mathrm{Hom}_G(E^{\otimes s} \otimes \vartheta^l, \mathrm{Sym}^\mu E \otimes \vartheta^m)$$

Thus Φ is surjective, and hence an isomorphism.

Proof. Given α, we can define $\beta := \alpha \circ \pi_\lambda$. Conversely β factors in this way if and only if its kernel is contained in the kernel of π_γ, which means $\beta \circ \sigma = \beta$ for all $\sigma \in \Sigma_\lambda$. This gives an upper bound for the number of maps α: The vector space dimension of $\mathrm{Hom}_G(\mathrm{Sym}^\lambda E \otimes \vartheta^l, \mathrm{Sym}^\mu E \otimes \vartheta^m)$ is bounded above by the dimension of $\Sigma_\lambda \backslash \mathrm{Hom}_G(E^{\otimes s} \otimes \vartheta^l, \mathrm{Sym}^\mu E \otimes \vartheta^m)$. By Proposition 3.5, the dimensions of the vector spaces $\mathrm{Hom}_G(E^{\otimes s} \otimes \vartheta^l, \mathrm{Sym}^\mu E \otimes \vartheta^m)$ and $\mathrm{Hom}_G(E^{\otimes s} \otimes \vartheta^l, E^{\otimes t} \otimes \vartheta^m)/\Sigma_\mu$ are equal. Hence

$$\dim \mathrm{Hom}_G(\mathrm{Sym}^\lambda E \otimes \vartheta^l, \mathrm{Sym}^\mu E \otimes \vartheta^m) \leq \dim \Sigma_\lambda \backslash \mathrm{Hom}_G(E^{\otimes s} \otimes \vartheta^l, E^{\otimes t} \otimes \vartheta^m)/\Sigma_\mu.$$

The latter space has a basis consisting of orbits of Brauer diagrams with $r-2s$ vertices in the top row and $r-2t$ vertices in the bottom row, under the action of the group $\Sigma_\lambda \times \Sigma_\mu$ on the top vertices through projection of group elements on the first component, and on the bottom vertices through the second component. By Proposition 2.1, this is exactly the

dimension of $\mathrm{Hom}_{B_r}(M(s,\lambda), M(t,\mu))$. Hence $\dim C \leq \dim S_B(n,r)$. By Proposition 3.7 the map Φ is injective, implying the converse inequality. □

Summarising the combinatorial description of maps obtained so far, we get the following commutative diagram:

$$\begin{array}{ccc} E^{\otimes s}\otimes \vartheta^l & \xrightarrow{\gamma} & E^{\otimes t}\otimes \vartheta^m \\ \downarrow{\scriptstyle \pi_\lambda} & \searrow^{\beta} & \downarrow{\scriptstyle \pi_\mu} \\ \mathrm{Sym}^\lambda E\otimes \vartheta^l & \xrightarrow[\alpha]{} & \mathrm{Sym}^\mu E\otimes \vartheta^m \end{array} \quad ,$$

The maps α have now been determined in terms of maps β, which in turn have been determined in terms of maps γ. In all cases, combinatorial descriptions have been found that show that the dimensions of these morphism spaces do not depend on the characteristic of the underlying field. Moreover, the maps α have been shown to correspond to double cosets of Brauer diagrams, similar to the description in Proposition 2.1.

3.9. **Proof of Corollary 1.2.** Recall the definition of $M := \oplus_{l,\lambda\vdash n-2l}\mathrm{Sym}^\lambda E$. There are two claims:

$$C = \mathrm{End}_{S_{env}(G)}(M) \qquad \text{and} \qquad S_{env}(G) = \mathrm{End}_{S_B(n,r)}(M)$$

The first claim is true by definition; we are going to prove the second claim.

Proof. The group algebra of the orthogonal or symplectic group G acts on the vector space $M = \oplus_{l,\lambda}\mathrm{Sym}^\lambda E$ via its finite dimensional quotient algebra $S_{env}(G)$. The algebra $S_{env}(G)$ acts faithfully on tensor space and thus a fortiori on M. Since the actions of G on M and of $S_B(n,r) = \mathrm{End}_G(M)$ on M commute, those of $S_{env}(G)$ on M and of $S_B(n,r)$ on M commute as well. Hence $S_{env}(G) \subset \mathcal{E} := \mathrm{End}_{S_B(n,r)}(M)$ and we have to show the converse inclusion.

Let e be the projection from the G-module M to its G-direct summand $E^{\otimes r}$. Viewed as an endomorphism of M, the element e is an idempotent in $S_B(n,r)$ and it commutes with the elements of $\mathcal{E}$. This implies that $E^{\otimes r} = Me$ is an $\mathcal{E}$-module and the action of $\mathcal{E}$ on $E^{\otimes r}$ commutes with the action of $eS_B(n,r)e$. Since tensor space $E^{\otimes r}$ is the image of $M(0,1^r) = B_r$ under the inverse Schur functor g_0, the centraliser algebra $eS_B(n,r)e$ coincides with the Brauer algebra B_r. By definition, $\mathcal{E}$ acts faithfully on $M = \oplus_{l,\lambda\vdash n-2l}\mathrm{Sym}^\lambda E$. As each $\mathrm{Sym}^\lambda E$ is a quotient of the tensor space $E^{\otimes r}$, the action of $\mathcal{E}$ on $E^{\otimes r}$ is faithful. Thus $\mathcal{E}$ is contained in $\mathrm{End}_{B_r}(E^{\otimes r}) = S_{env}(G)$ by Equation (2). □

3.10. **Proof of Corollary 1.3.** Here, we give additional information on, and a proof of Corollary 1.3.

Proof. (a) The algebra C has an integral form with an explicit basis, which is independent of the ground field k and its characteristic.

The basis in assertion (a) corresponds under the isomorphism $C \simeq S_B(n,r)$ to the basis of $S_B(n,r)$ mentioned in Section 2.2 and described in Equation (4); it has been shown

to be a basis in [24, Theorem 5.3]. This basis is indexed by certain double cosets of symmetric groups. Hence it does not depend on the characteristic of the field. In fact, the ground ring need not even be a field.

(b) The algebra C carries a quasi-hereditary structure, that is $C-mod$ is a highest weight category.

The quasi-heredity claimed in (b) follows from [24, Theorem 7.1], which states the corresponding result for the algebra $S_B(n,r)$ in general. The category of finite dimensional modules over a quasi-hereditary algebra always is a highest weight category.

(c) The global (cohomological) dimension of C is finite.

Cline, Parshall and Scott, and Dlab and Ringel have shown that quasi-hereditary algebras over fields have finite global dimension, which implies (c). See [7, Theorem 4.4] and [10, Appendix, Statement 9]. More precisely, Dlab and Ringel have shown that the global dimension is bounded above by $2s-2$, where s is the number of simple modules up to isomorphism. By statement (f) below, s equals the number of all partitions of all numbers $r-2l \geq 0$.

(d) There is a Schur-Weyl duality between C and the Brauer algebra $B_r(\pm n)$.

Schur-Weyl duality between $S_B(n,r,\delta)$ and $B_r(\delta)$ on the bimodule $\oplus M(l,\lambda)$ has been shown in [24, Theorem 11.4(a)] for any parameter δ. It uses $n \geq 2r$.

Note that the claims on C are just special cases of known results for $S_B(n,r,\delta)$. In fact, the assertions (a), (b) and (d) are all true for $S_B(n,r,\delta)$ over any ground ring, and (c) is true over any ground field, and for any choice of the parameter δ. The assertion (e), however, needs the ground ring to be a field, and n to be at least greater than or equal to r.

(e) When the characteristic is different from two or three, the algebra C satisfies a universal property that makes it unique up to Morita equivalence: It is the quasi-hereditary 1-cover of the Brauer algebra in the sense of Rouquier.

The claim is [24, Theorem 11.4 (b) and (c)]. Under these assumptions, the Brauer algebra B_r is of the form $eS_B(n,r)e$ for some idempotent $e \in S_B(n,r)$ and the two algebras B_r and $S_B(n,r)$ are in Schur–Weyl duality on the bimodule $e \cdot S_B(n,r)$. According to Rouquier's definition [33], the algebra $S_B(n,r)$ is a 0-cover of B_r. For a quasi-hereditary 1-cover, an additional condition is required: The exact Schur functor $e\cdot -$ has to identify extension spaces between modules with standard filtration over $S_B(n,r)$ with extension spaces over B_r:

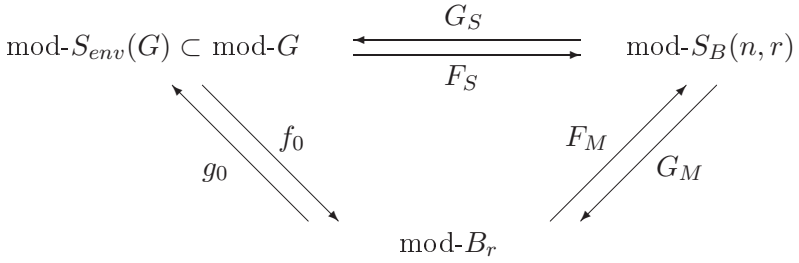
$$\mathrm{Ext}^1_{S_B(n,r)}(X,Y) \simeq \mathrm{Ext}^1_{B_r}(eX,eY).$$

These latter isomorphisms hold by [22], needing the characteristic being different from two and three. See [21, Sections 11, 12 and 13] for explicit statements, and for more information.

(f) The simple C-modules are parametrised by the disjoint union of all partitions $\lambda \vdash r-2l$ of the non-negative integers of the form $r, r-2, r-4, \ldots$.

This is a consequence of the quasi-hereditary structure of the algebra $S_B(n,r)$ as exhibited in [24]. The isomorphism classes of simple modules of a quasi-hereditary algebra correspond bijectively to the standard modules or equivalently to the ideals in a heredity chain. The heredity chain is constructed in [24] by first forming a coarse chain of ideals, indexed by non-negative integers of the form $r-2l$, and then refining this into a heredity chain. The coarse chain imitates the chain of ideals in the Brauer algebra obtained by counting horizontal arcs in top and bottom row. Within the coarse layer indexed by $r-2l$ the heredity chain is indexed by all the partitions of $r-2l$. Indeed, in a sense made precise in [24] this part of the heredity chain is 'induced up' from a heredity chain of the classical Schur algebra $S(n, r-2l)$. □

3.11. **A triangle of Schur functors.** Finally, we summarise the current situation with respect to Schur functors for orthogonal and symplectic groups, which shows a marked difference to the type A situation. In type A, Green's Schur algebra is both a generalised Schur algebra and an endomorphism ring of permutation modules over $k\Sigma_r$. In types B, C and D, tensor space is different from the sum of permutation modules over B_r, see [25]. We get the following triangle of functors with non-trivial functors between mod-G (or mod-$S_{env}(G)$) and mod-$S_B(n,r)$:



with

$$\begin{aligned} g_0 &= - \otimes_{B_r} E^{\otimes r}, & f_0 &= \mathrm{Hom}_G(E^{\otimes r}, -), \\ G_S &= - \otimes_{S_B(n,r)} (\oplus \mathrm{Sym}^\lambda E), & F_S &= \mathrm{Hom}_G(\oplus \mathrm{Sym}^\lambda E, -), \\ G_M &= - \otimes_{S_B(n,r)} (\oplus M(l,\lambda)), & F_M &= \mathrm{Hom}_{B_r}(\oplus M(l,\lambda), -). \end{aligned}$$

This triangle commutes in the sense that $G_S = g_0 \circ G_M$ and similarly for the adjoints. Indeed, there are isomorphims of left $S_B(n,r)$-modules

$$M(l,\lambda) \otimes_{B_r} E^{\otimes r} \simeq M^\lambda \otimes_{k\Sigma_{r-2l}} e_l B_r \otimes_{B_r} E^{\otimes r} \simeq \mathrm{Sym}^\lambda E \otimes \vartheta^l \simeq \mathrm{Sym}^\lambda E$$

as in Section 3.4. Uniqueness of adjoints then implies $F_S = F_M \circ f_0$.

When the ground field k has characteristic different from two and three, the functors F_M and G_M are mutually inverse equivalences between the exact categories of Δ-filtered $S_B(n,r)$-modules and cell filtered B_r-modules, by [22]; this uses and extends a similar equivalence, due to Hemmer and Nakano [23], between Weyl filtered modules of GL_n and Specht filtered modules of $k\Sigma_r$. See also [21, 24] for further structural properties and relations between mod-$S_B(n,r)$ and mod-B_r. It is not known how well these two

functors compare or identify cohomology in higher degrees. By [17] this is equivalent to saying that the dominant dimension of $S_B(n,r)$ is not known. The other two pairs of functors are not known to restrict to equivalences between corresponding categories of filtered modules. The dominant dimensions of generalised Schur algebras or of enveloping algebras $S_{env}(G)$ are not known.

4. A direct combinatorial description of morphisms between symmetric powers

In this Section, we give a direct combinatorial description of the morphism spaces between symmetric powers assuming that *the underlying field has characteristic zero or bigger than r.* Under this assumption Theorem 1.1 can be shown without using the results collected in Subsection 3.3.

We use Schur-Weyl duality for symplectic and orthogonal Schur algebras (stated above in Equations (1) and (2)), which in particular implies that every G-endomorphism of the tensor space $E^{\otimes r}$ is given by multiplication with a Brauer algebra element $\sum_b \lambda_b b$ where b runs through Brauer diagrams and $\lambda_b \in k$. Let $\pi_\lambda = \pi_{\lambda,l}$.

For a composition μ of $r-2m$, the G-module homomorphism ι_μ is defined to be the composition:

$$\iota_\mu : \mathrm{Sym}^\mu E \otimes \vartheta^m \to E^{\otimes r-2m} \otimes \vartheta^m \to E^{\otimes r}, \qquad x^\mu \mapsto \frac{1}{\mid \Sigma_\mu \mid} \cdot \sum_{\sigma \in \Sigma_\mu} (x\sigma) \otimes \vartheta^m.$$

Here the symmetric group acts by place permutation.

Under the assumption of the underlying field k having characteristic zero or larger than r, map ι_μ is a split monomorphism, composing with the split epimorphim π_μ to the identity on $\mathrm{Sym}^\mu E \otimes \vartheta^m$. For the following proof, the factor $\frac{1}{|\Sigma_\mu|}$ may as well be omitted. The crucial point is that under our assumptions, ι_μ is injective, which is not true in general.

Using the notation introduced in Section 3.2, the result is as follows:

Proposition 4.1. *Fix λ, μ and π_λ, ι_μ as above. Let $\psi : E^{\otimes r} \to E^{\otimes r}$ be a G-module homomorphism. Then ψ factors as $\psi = \iota_\mu \circ \varphi \circ \pi_\lambda = \pi_\lambda \varphi \iota_\mu$,*

$$\begin{array}{ccc} E^{\otimes r} & \xrightarrow{\quad\psi\quad} & E^{\otimes r} \\ {\scriptstyle \pi_\lambda}\downarrow & & \uparrow{\scriptstyle \iota_\mu} \\ \mathrm{Sym}^\lambda E \otimes \vartheta^l & \xrightarrow[\varphi]{} & \mathrm{Sym}^\mu E \otimes \vartheta^m \end{array},$$

for some G-homomorphism $\varphi : \mathrm{Sym}^\lambda E \otimes \vartheta^l \to \mathrm{Sym}^\mu E \otimes \vartheta^m$, if and only if $\psi = \sum_D \lambda_D D$ with $\lambda_D \in k$. Here the sum runs over some elements D of the Brauer algebra, which are of the form

$$D_b = \sum_{b' \in T_b} b', \qquad \text{with } T_b = \{\sigma b \tau \mid \sigma \in \Sigma_\lambda, \tau \in \Sigma_\mu\} \tag{7}$$

where each b is a Brauer diagram with l horizontal arcs of adjacent vertices on the last $2l$ vertices in the top row and with m horizontal arcs of adjacent vertices on the last $2m$ vertices in the bottom row. The factorisation of ψ is unique, if it exists.

This proposition says in particular that there is a k-linear bijection between the space of maps $\psi = \sum_D \lambda_D D$ and the maps φ. With the elements D being linearly independent, and their number being independent of the field k, it follows:

Corollary 4.2. *The dimension of the space* $\mathrm{Hom}_G(\mathrm{Sym}^\lambda E, \mathrm{Sym}^\mu E)$ *does not depend on the characteristic of k, as long as this is zero or larger than r, more precisely, as long as the map ι_μ is injective.*

Proof of Proposition4.1. Since the Brauer algebra acts from the right, we write $\pi_\lambda \varphi \iota_\mu$ for $\psi = \iota_\mu \circ \varphi \circ \pi_\lambda$.

(a) By Schur-Weyl duality for the tensor space (see (1)), every G-endomorphism ψ of the tensor space $E^{\otimes r}$ is given by multiplication with a Brauer algebra element, say $\sum_b \lambda_b b$, where the sum runs through some Brauer diagrams $b \in B_r(\delta)$. Assume such a homomorphism $\psi = \sum_b \lambda_b b$ of the tensor space $E^{\otimes r}$ factors through a homomorphism $\mathrm{Sym}^\lambda E \otimes \vartheta^l \to \mathrm{Sym}^\mu E \otimes \vartheta^m$, that is $\psi = \sum_b \lambda_b b = \pi_\lambda \varphi \iota_\mu$ for some $\varphi : \mathrm{Sym}^\lambda E \otimes \vartheta^l \to \mathrm{Sym}^\mu E \otimes \vartheta^m$.

(i) Since e_l and e_m are the identity maps on $e_l E^r = E^{\otimes r-2l} \otimes \vartheta^l$ and $e_m E^r = E^{\otimes r-2m} \otimes \vartheta^m$, respectively, it follows that
$$e_l(\sum_b \lambda_b b)e_m = (e_l \pi_\lambda)\varphi(\iota_\mu e_m) = \pi_\lambda \varphi \iota_\mu = \sum_b \lambda_b b.$$
Since Brauer diagrams form a basis of the Brauer algebra, the diagrams b all have l arcs on adjacent vertices on the last $2l$ vertices in the top row, and m arcs on adjacent vertices on the last $2m$ vertices in the bottom row.

(ii) An arbitrary vector in the image of ι_μ is a linear combination of vectors of the form
$$\sum_{\sigma \in \Sigma_\mu} (x\sigma) \otimes \vartheta^m.$$
These vectors are invariant under the action of Σ_μ. Let m_τ be multiplication with a permutation $\tau \in \Sigma_\mu$. Then $\iota_\mu m_\tau = \iota_\mu$, and hence
$$(\sum_b \lambda_b b)m_\tau = \pi_\lambda \varphi \iota_\mu m_\tau = \pi_\lambda \varphi \iota_\mu = \sum_b \lambda_b b.$$
Note that
$$\sum_b \lambda_b b = \sum_b \lambda_b b\tau = \sum_b \lambda_{b\tau^{-1}} b,$$
and hence $\lambda_b = \lambda_{b\tau^{-1}}$ for all $\tau \in \Sigma_\mu$, that is the coefficients λ_b are constant on Σ_μ-orbits.

(iii) Similarly as in the previous step, we can postcompose with multiplication by $\sigma \in \Sigma_\lambda$. Then $m_\sigma \pi_\lambda = \pi_\lambda$ for any $\sigma \in \Sigma_\lambda$. Hence
$$m_\sigma(\sum_b \lambda_b b) = \sum_b \lambda_b b.$$

It follows that

$$\sum_b \lambda_b b = \sigma \sum_b \lambda_b b = \sum_b \lambda_{\sigma^{-1}b} b,$$

and hence $\lambda_b = \lambda_{\sigma^{-1}b}$ for all $\sigma \in \Sigma_\lambda$. That is, the coefficients are constant on Σ_λ-orbits.

It follows that we can write ψ as a linear combination of elements D as claimed.

(b) Conversely, given $\psi = \sum \lambda_D D$ as defined in the proposition. We show that $\ker \pi_\lambda \subseteq \ker \psi$ and $\mathrm{im}\psi \subseteq \mathrm{im}\iota_\mu$. If so, then ψ factors as $\psi = \pi_\lambda g \iota_\mu$ for some G-homomorphism $\varphi : \mathrm{Sym}^\lambda E \otimes \vartheta^l \to \mathrm{Sym}^\mu E \otimes \vartheta^m$.

By definition, π_λ is a composition $\pi_\lambda : E^{\otimes r} \to E^{\otimes r-2l} \to \mathrm{Sym}^\lambda E$ of first multiplication by the idempotent e_l and then canonical projection onto $\mathrm{Sym}^\lambda E$.

The elements D by definition satisfy $D = De_l$. Hence ψ annihilates the kernel of multiplication by e_l, and thus factors through $E^{\otimes r-2l}$; denote the induced map on residue classes by $\overline{\psi} : E^{\otimes r-2l} \otimes \vartheta^l \to \mathrm{Sym}^\lambda E \otimes \vartheta^l$. We have to check that the kernel of the canonical projection $E^{r-2l} \to \mathrm{Sym}^\lambda E$ gets annihilated by $\overline{\psi}$. This kernel is generated by elements of the form $x_{ij} := x(id-(i,j)) = \cdots \otimes x_i \otimes \cdots \otimes x_j \otimes \cdots - \cdots \otimes x_j \otimes \cdots \otimes x_i \otimes \cdots$. Let b be a Brauer diagram with l adjacent horizontal arcs on the last $2l$ vertices in the top row of b. By the definition of ψ, we have $\overline{\psi} = \sigma\overline{\psi}$ for all $\sigma \in \Sigma_\lambda$. Hence, $x_{i,j}\overline{\psi} = x_{i,j} \cdot \sigma\overline{\psi}$.

Choose $\sigma = (i,j)$ to be the transposition exchanging the positions of x_i and x_j. Then $x_{i,j}\sigma = -x_{i,j}$ and hence $x_{i,j}\overline{\psi} = -x_{i,j}\overline{\psi}$. If $char(k) \neq 2$, it follows that $x_{i,j}\overline{\psi} = 0$. In case $char(k) = 2$, use that

$$(x_{i,j} + x_{j,i})\overline{\psi} = x_{i,j}\overline{\psi} + x_{i,j}\sigma\overline{\psi} = 2x_{i,j}\overline{\psi} = 0.$$

Hence, the kernel of π_λ is contained in that of ψ.

Next, note that

$$\mathrm{im}\iota_\mu = \{x \mid x\tau = x \text{ for all } \tau \in \Sigma_\mu\} \otimes \vartheta^m.$$

Let $D_b = \sum b'$ be as defined in the proposition. Then for a tensor $x \in E^{\otimes r}$ and $\tau' \in \Sigma_\mu$,

$$x(\sum b')\tau' = x(\sum b')$$

and thus

$$x\psi\tau' = x\psi.$$

By definition of ψ, for $x \in E^{\otimes r}$, there exists $y \in E^{\otimes r-2m}$ with $x\psi = y \otimes \vartheta^m$. As

$$y\tau' \otimes \vartheta^m = (y \otimes \vartheta^m)\tau' = x\psi\tau' = x\psi = y \otimes \vartheta^m$$

it follows that $y\tau' = y$. Hence $x\psi \in \mathrm{im}\iota_\mu$, that is $\mathrm{im}\psi \subseteq \mathrm{im}\iota_\mu$.

(c) Finally, for uniqueness, assume that $\pi_\lambda\varphi_1\iota_\mu = \pi_\lambda\varphi_2\iota_\mu$. Since ι_μ is injective, it follows that $\pi_\lambda\varphi_1 = \pi_\lambda\varphi_2$. Since π_λ is surjective, $\varphi_1 = \varphi_2$. It is in this last step, where we use the assumption on the characteristic, ensuring that ι_μ is injective. □

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