SCHUR ALGEBRAS OF BRAUER ALGEBRAS I

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Abstract. Schur algebras of Brauer algebras are defined as endomorphism algebras of certain direct sums of 'permutation modules' over Brauer algebras. Explicit combinatorial bases of these new Schur algebras are given; in particular, these Schur algebras are defined integrally. The new Schur algebras are related to the Brauer algebra by Schur-Weyl dualities on the above sums of permutation modules. Moreover, they are shown to be quasi-hereditary. Over fields of characteristic different from two or three, the new Schur algebras are quasi-hereditary 1-covers of Brauer algebras, and hence the unique 'canonical' Schur algebras of Brauer algebras.

1. Introduction

Richard Brauer defined the algebras now carrying his name more than seventy years ago [1] in order to describe the centraliser algebras of symplectic or orthogonal groups acting on tensor space. By their construction, Brauer algebras play a crucial role in representation theory and invariant theory of these classical groups. This role is strengthened further by a Schur-Weyl duality, due to Brauer in characteristic zero, and recently also established in prime characteristic (see Dipper, Doty and Hu [5, 9] for final results). More generally, the family of Brauer algebras $B_r(\delta)$ depending on a natural number r and a continuous parameter δ (that has to be specialised to certain integers to get Brauer's original algebras) have been investigated and used also in knot theory and in mathematical physics. Brauer algebras are now often considered as the paradigmatic example of what is called diagram algebras. Typically, difficulties met in this class of algebras are present already in Brauer algebras, and results for Brauer algebras often extend to many or all diagram algebras without causing further difficulties. All of these algebras share the fundamental structure of being cellular, defined by Graham and Lehrer [11].

A frequently taken point of view is to consider Brauer algebras as orthogonal or symplectic analogues of symmetric groups; Schur-Weyl duality is an instance of such an analogy. Recently, Hartmann and Paget [14] have carried the analogy further by defining 'permutation modules' for Brauer algebras. Moreover, they managed to extend a surprising result of Hemmer and Nakanao [15] from symmetric groups to Brauer algebras: In characteristic different from two or three the category of modules with cell filtrations (=Specht filtrations in the symmetric group case) behaves like a highest weight category in algebraic Lie theory; in the case of symmetric groups it is actually equivalent to the category of Weyl filtered modules (of a fixed polynomial degree) of a general linear group. This has been carried further in [13], where a deeper structure of Brauer algebras and other diagram algebras has been established; when the parameter δ is not zero, these algebras are cellularly stratified. In the case of Brauer algebras this provides another formulation of the observation (made already in [2] and used in [11, 17] for proving cellularity) that Brauer algebras are made up of group algebras of symmetric groups of varying size. The article [13] also extends the results of [15, 14] and it implies, in characteristic different from two or three, the existence of a new 'Schur algebra' associated with the Brauer algebra. In [13] the existence of this Schur algebra is based on a new, abstract Schur-Weyl duality, involving the Brauer algebra and the new Schur algebra under the assumptions made there.

The aim of this article is to investigate this 'new Schur algebra' $S_B(n, r, \delta)$. We define it for any Brauer algebra, any choice of parameter $\delta \neq 0$ and any commutative domain K as ground ring (and moreover for any choice of non-zero multiplicities of permutation modules). We construct a combinatorial basis (Theorem 5.3) and thus show that $S_B(n, r, \delta)$ is defined integrally, as a Schur algebra should be; its dimension, various filtrations by ideals and other basic pieces of structure do not depend on the choice or the characteristic of the ground ring nor on the value of the parameter $\delta \neq 0$. Over a field of characteristic not two or three we recover the algebras of [13].

The second main result (Theorem 7.1) provides a structure that is accepted as a fundamental property of any Schur algebra. It states that $S_B(n, r, \delta)$ always is quasi-hereditary, that is, its representations form a highest weight category. Moreover, there is a Schur-Weyl duality with the Brauer algebra $B_r(\delta)$, again for any choice of parameter and ground ring (Corollary 11.4). Here, one may speculate about the existence of a Lie theoretic object behind these new Schur algebras; this object would have to carry orthogonal and symplectic features and it would have a 'rational' representation theory given by the modules over the new Schur algebras (of varying degrees). Under the assumptions made in [14, 13], the algebra $S_B(n, r, \delta)$ is what Rouquier calls a faithful quasi-hereditary 1-cover of the Brauer algebra and as such it is unique, and hence canonically associated with $B_r(\delta)$.

Another structural feature of the new Schur algebras is that, in a sense we will make precise, they are made up of classical (type A) Schur algebras $S_A(n, r - 2u)$ of varying size; this quite resembles the property of Brauer algebras $B_r(\delta)$ to be made up of group algebras of symmetric groups $K\Sigma_{r-2u}$ of varying size. So, like the Brauer algebras, the new Schur algebras are built from type A objects, despite their close connections to the other classical types.

An interesting perspective for the new Schur algebras may be provided by a connection with invariant theory. In the forthcoming second part of this series of articles we are going to prove that (for certain choices of parameters) the new Schur algebras also come up as endomorphism rings of sums of tensor powers of symmetric powers of the natural representations of orthogonal or symplectic groups, in any characteristic. This result also provides a direct link of the new Schur algebras with representation theory of orthogonal and symplectic groups.

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2. Preliminaries

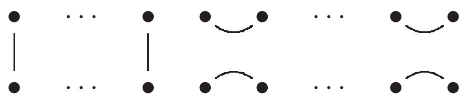
After recalling some basic material, we are going to define the main objects of this article, the Schur algebras $S_B(n, r, \delta)$. Throughout, we fix a ground ring K, which is assumed to be a commutative domain. When considering free K-modules of finite rank, we will use the terms 'K-basis' for a set of free generators, and 'dimension' for the K-rank of the module.

2.1. **Brauer algebras, diagrams and a filtration.** The Brauer algebra $B = B_r(\delta)$ depends on the ground ring K, a parameter $\delta \in K$ and a natural number $r \in \mathbb{N}$. It has K-basis the set of diagrams of the following form: A diagram has $2r$ vertices arranged in two rows of r vertices each, and r edges such that each vertex is incident to precisely one edge. Diagrams are considered up to homotopy, thus the dimension of $B_r(\delta)$ is $(2r-1)!! = (2r-1) \cdot (2r-3) \cdot \ldots$. To multiply two diagrams, the diagrams are concatenated and any closed loops appearing are removed, to give a diagram d. The result of the multiplication then is, by definition, $\delta^c d$, where c is the number of closed loops removed. See e.g. [1, 2, 17, 14, 16, 13] for more details and examples.

Edges in Brauer diagrams can be horizontal arcs, connecting two vertices in the top row or two vertices in the bottom row, or through strings, connecting a top vertex with a bottom vertex. The set of through strings is closed under concatenation and forms a group isomorphic to the symmetric group on r letters. The group algebra $K\Sigma_r$ is both a subalgebra and a quotient of $B_r(\delta)$.

Let $B = B_r(\delta)$ be the Brauer algebra on $2r$ vertices with parameter $\delta \in K$. Let n be a natural number with $2n \leq r$. Define $B^{\geq n}$ to be the K-vector space spanned by all Brauer algebra diagrams with at least n arcs in both top and bottom row. Note that $B^{\geq n}$ is a two-sided ideal in B, and this defines a chain of two-sided ideals in B, that is, a filtration of B by B-B-bimodules. We refer to this filtration as the 'coarse' filtration of the Brauer algebra. This filtration has already appeared in work of W.Brown, for example see [2]. It can be refined into a cell filtration, see [11, 17]. The coarse filtration is the basic datum for the cellularly stratified structure of $B_r(\delta)$ exhibited and used in [13].

2.2. **Substructures and induction functors.** Let $\delta \neq 0$. For $0 \leq l \leq [r/2]$, define the following idempotents in $B = B_r(\delta)$:

(1) $$e_l = \frac{1}{\delta^{\frac{r-l}{2}}} \cdot$$  .

The centraliser subalgebra $e_l B_r(\delta) e_l$ is isomorphic to the Brauer algebra $B_{r-2l}(\delta)$, and $e_l K\Sigma_r e_l$ is isomorphic to $K\Sigma_{r-2l}$.

A Brauer diagram d has, say, l arcs in the top row and then also l arcs in the bottom row. Each row then has $r-2l$ 'free' vertices, and there is a permutation $\sigma \in \Sigma_{r-2l}$ connecting the free top vertices with the free bottom vertices. This means, d is described by a triple (σ, v, w), consisting of the free permutation σ and the two arc configurations on the top and the bottom row, v and w respectively. This point of view has been used in [11, 17] to construct the cellular structure of the Brauer algebra from the cellular structures of the group algebras $K\Sigma_{r-2l}$ sitting inside the centraliser subalgebras $e_l B_{r-2l}(\delta) e_l$. In [13] this description has been strengthened and extended further by showing that in this way Brauer algebras become cellularly stratified. We will not use this terminology, but the following results are derived in [13] from the concept of cellularly stratified: As above, the Brauer algebra $B = B_r(\delta)$ has a chain of two-sided ideals of the form

(2) $$B = Be_0B =: J_0 \supset Be_1B =: J_1 \supset Be_2B =: J_2 \supset \ldots$$

where $Be_nB = J_n = B^{\geq n}$ are the ideals in the above 'coarse filtration' of B. The subquotients $(B/J_l)e_l$ and $e_l(B/J_l)$ are left or right projective modules over $e_l(B/J_l)e_l$, respectively. Hence there is an exact induction functor

(3) $$G_l : e_l(B/J_l)e_l\text{-mod} \to \text{mod-}B, \qquad M \mapsto M \otimes_{e_lBe_l} e_l(B/J_l)$$

where mod-B denotes the category of right B-modules. Moreover, there is an isomorphism of K-algebras

$$e_l(B/J_l)e_l \simeq K\Sigma_{r-2l},$$

and hence the induction functor G_l turns representations of the 'small' symmetric group Σ_{r-2l} into representations of the Brauer algebra B.

For a natural number l, define $V_l = V_l^r$ to be the set consisting of all partial diagrams on r dots with l arcs. Here is an example of a partial diagram v on $r = 8$ dots with three horizontal arcs and two free dots:

$v =$  .

By abuse of notation, we denote both the set of partial diagrams and the vector space spanned by the set of partial diagrams by V_l^r. The precise meaning will be clear from the context.

Let v be a partial diagram with l horizontal arcs and d a Brauer diagram in B. The algebra B acts from the right on the vector space V_l^r by

$$v \cdot d = \begin{cases} \delta^l \cdot w & \text{if } w \text{ has exactly } l \text{ edges,} \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

with w the partial diagram obtained by the concatenation of v above d, reading off the horizontal arcs in the bottom row of the concatenated diagram. For an example, see [16]. By [13, Section 4] the following can be taken as an alternative definition of the induction functor:

Lemma 2.1. *The induction functor G_l defined in* (3) *sends a $K\Sigma_{r-2l}$-module M to the B-module $M \otimes_K V_l$ with*

$$(m \otimes v) \cdot d \quad = \quad \begin{cases} m \cdot \pi(d, v) \otimes vd & \text{if } vd \text{ has } l \text{ arcs,} \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

where $\pi(d, v)$ is the permutation induced on the free dots of the diagram vd.

This definition has been used in [14, Section 5], where more details can be found. In particular, the cell modules of B arise in this way (see [13, Proposition 4.2]): they are parametrised by labels (l, λ) with λ a partition of $r - 2l$ for some $0 \leq l \leq [r/2]$, and they are defined with action as above by

$$S(l, \lambda) = S^\lambda \otimes V_l, \tag{6}$$

where S^λ denotes the Specht module of $K\Sigma_{r-2l}$. The modules $S(l, \lambda)$ have been studied in characteristic zero in [18].

2.3. **Wreath products.** For a natural number l, write $H_l = \mathbb{Z}/2\mathbb{Z} \wr \Sigma_l$ for the wreath product of $\mathbb{Z}/2\mathbb{Z}$ by Σ_l. In particular, H_l has $2^l \cdot l!$ elements. Let v_l be a partial Brauer diagram on r dots with l horizontal arcs given by

$$v_l = \bullet \quad \cdots \quad \bullet \quad \overset{\frown}{\bullet\;\bullet} \quad \cdots \quad \overset{\frown}{\bullet\;\bullet}\ . \tag{7}$$

The symmetric group $\Sigma_r \subseteq B$ acts on (partial) diagrams as defined above. A permutation fixing v_l (or e_l) can interchange the beginning and end vertex of a horizontal arc. Since there are l horizontal arcs in the partial diagram v_l, this provides us with l copies of $\mathbb{Z}/2\mathbb{Z}$. Moreover, permutations fixing v_l can swap any two horizontal arcs in v_l. These permutations permute the l copies of $\mathbb{Z}/2\mathbb{Z}$, and hence

$$\mathrm{Stab}_{\Sigma_{2l}}(v_l) = \mathbb{Z}/2\mathbb{Z} \wr \Sigma_l = H_l. \tag{8}$$

In the same way it follows that any (partial) diagram with l horizontal arcs in the row(s) has the same stabiliser (via the embedding of Σ_{2l} in Σ_r induced by the vertices defining arcs). Moreover, the dimension of V_l^r is

$$\mid V_l^r \mid = \frac{r(r-1)\cdots(r-2l+1)}{2^l\, l!}.$$

Here, for the first arcs we have r choices for the first vertex, and $r-1$ choices for the second vertex. For the second of these arcs, we then have $r-2$ choices for the first vertex, and $r-3$ choices for the second vertex. And so on. To avoid multiple counting, divide by the order of the stabilizer of an arc configuration in V_l^r. Alternatively, a basis diagram is determined by first choosing $2l$ vertices and then connecting arcs. There are $\binom{r}{2l}$ choices for the $2l$ vertices and $(2l-1)!! = (2l-1)\cdot(2l-3)\cdots 3 \cdot 1$ choices for the arcs. Thus the dimension of V_l^r is $\binom{r}{2l}(2l-1)!!$.

2.4. **Cosets and permutation modules.** We recall a few facts about permutation modules. Let G be a finite group. We denote the permutation module associated to some G-set M by $K[M]$. If the group G acts transitively on the set M, and H is the stabilizer of some element $m \in M$, then $K[M] = K[H\backslash G]$ where $H\backslash G$ denotes the set of right cosets of G modulo H.

Let H and U be subgroups of G. There is a one-to-one correspondence between the set of left cosets G/H and the set of right cosets $H\backslash G$, given by $gH \mapsto Hg^{-1}$. In particular, $\dim K[G/H] = \dim K[H\backslash G]$. Moreover, there is a one-to-one correspondence between the following double cosets:

$$U\backslash G/H \longrightarrow H\backslash G/U, \qquad \text{given by } UgH \mapsto Hg^{-1}U.$$

Let V be a left KG-module. Taking the vector space dual, V^*, we obtain a right KG-module structure on V^* by defining $(f \cdot x)(v) = f(xv)$ for all $x \in G$, $v \in V$ and $f \in V^*$. Denote by $K[G/H]$ the left KG-permutation module on the left cosets of G modulo H, and similarly, by $K[H\backslash G]$ the right KG-permutation module on the right cosets of G modulo H. Then the right KG-modules $K[G/H]^*$ and $K[H\backslash G]$ are isomorphic. Moreover, there is an isomorphism of vector spaces

$$K[U\backslash G] \otimes_{KG} K[G/H] \simeq K[U\backslash G/H]. \tag{9}$$

2.5. **Schur algebras of Brauer algebras.** Recall that we always assume $\delta \neq 0$. Denote by M^λ the permutation module of $K\Sigma_{r-2l}$ on the cosets of the Young subgroup Σ_λ of Σ_{r-2l}. In [14, Section 6], for each label of a cell module, a *permutation module* $M(l,\lambda)$ and a *Young module* $Y(l,\lambda)$ has been constructed. These permutation modules are defined as

$$M(l,\lambda) = M^\lambda \otimes_{K\Sigma_{r-2l}} e_l B. \tag{10}$$

For more details on these modules see Section 3, which refines and extends Proposition 23 in [14]. The main object of study in this article are the following endomorphism rings of particular direct sums of these permutation modules:

Definition 2.2. Fix a commutative domain K, a non-zero parameter $\delta \in K^*$ and natural numbers n and r. Then the *Schur algebra of the Brauer algebra* $B_r(\delta)$ is the K-algebra

$$S_B(n,r,\delta) := End_{B_r(\delta)}\Big(\bigoplus_{0 \le l \le [\frac{r}{2}]} \bigoplus_{\lambda} M(l,\lambda)\Big).$$

We often will suppress the parameter δ and just write $S_B(n,r)$, since none of our results will depend on the choice of $\delta \neq 0$. In the definition, λ runs through the set $\Lambda(r-2l)$ of all compositions of $r-2l$, or alternatively, through all partitions of $r-2l$; whether we use compositions or partitions in the definition results in a Morita equivalent version of $S_B(n,r,\delta)$. This difference between compositions or partitions does not matter for any of our results here, and therefore we suppress it in notation. It may of course matter for symmetry properties not considered here. More generally, we could define in this setting an algebra by any set of compositions that contains for each partition at least one of the compositions of this partition type; the resulting endomorphism algebra then will be Morita equivalent to $S_B(n,r,\delta)$. Indeed, when λ and μ are two compositions that belong to the same partition of some $r-2l$, then the permutation modules M^λ and M^μ of the symmetric group Σ_{r-2l} are isomorphic. Hence, the permutation modules $M(l,\lambda)$ and $M(\mu)$ of the Brauer algebra are isomorphic, too. So, any such choice of index set amounts to a choice of non-zero multiplicities n_λ in $\bigoplus_{0\le l\le[\frac{r}{2}]} \bigoplus_\lambda M(l,\lambda)^{n_\lambda}$, that is, to applying a Morita equivalence to the endomorphism ring $S_B(n,r,\delta)$.

By applying the induction functor G_l to a permutation module M^λ of the symmetric group, a quotient module of $M(l,\lambda)$ is produced that is, in general, much smaller than $M(l,\lambda)$ itself. Indeed, it coincides precisely with the lth layer of the coarse filtration of the permutation module $M(l,\lambda)$, see Section 3.3. In general, the induction functor G_l always produces a module that lives in the l-th layer of the coarse filtration, while a module produced by the tensor functor $-\otimes_{K\Sigma_{r-2l}} e_l B$ will, in general, have subquotients in each m-th layer of the coarse filtration with $m \geq l$.

3. Filtrations

Filtrations of the Brauer algebra, of certain modules and of the Schur algebra of the Brauer algebra will be crucial tools in all what follows.

3.1. **The bimodules B_l and B_l^n.** Let $B = B_r(\delta)$ be the Brauer algebra on $2r$ vertices with parameter $\delta \in K$ and denote as above by $B^{\geq n} = Be_nB$ the ideals in the coarse filtration of B. For $0 \leq n \leq [r/2]$, define the quotient space

$$B^n = B^{\geq n}/B^{\geq n+1}.$$

These quotient spaces are B-B-bimodules.

We say that a partial diagram $\alpha \in V_l^r$ is *contained* in a partial diagram $\beta \in V_n^r$ with $n \geq l$, if β contains all arcs of α. We typically will choose $\alpha = v_l \in V_l^r$ as defined in (7), that is, α has l arcs between adjacent vertices on the last $2l$ vertices. In this example, with $n \geq l$, a partial diagram $\beta \in V_n^r$ contains α if β has l adjacent arcs on the last $2l$ vertices and $n-l$ arcs somewhere on the first $r-2l$ vertices.

We now fix an arc configuration $\alpha \in V_l^r$. Let $B_l = B_l(\alpha)$ be the K-vector space with basis all Brauer diagrams with top and bottom arc configuration in V_m^r for some $m \geq l$, such that the top arc configuration contains α. Let now $n \geq l$. We define $B_l^{\geq n} = B_l^{\geq n}(\alpha)$ as the K-vector space with basis all Brauer diagrams with top and bottom arc configuration in V_m^r for some $m \geq n$, such that the top arc configuration contains α. Then B_l is a right B-module, filtered by the right B-modules $B_l^{\geq n}$:

$$B_l = B_l^{\geq l} \supset B_l^{\geq l+1} \supset \ldots \supset B_l^{[\geq r/2]} \supset B_l^{[\geq r/2]+1} = \{0\}.$$

Define

$$B_l^n = B_l^{\geq n}/B_l^{\geq n+1},$$

which is the K-vector space with basis all Brauer diagrams with top and bottom arc configuration in V_n^r such that the top arc configuration contains α. The projective module e_lB is a special case of this construction, and the right B-modules e_lB and $B_l(\alpha)$ are isomorphic.

3.2. **Permutation modules B_l^n.** For $n \geq l$, the vector spaces $B_l^{\geq n}$ are also left $K\Sigma_{r-2l}$-modules where the symmetric group Σ_{r-2l} operates precisely on those $r-2l$ vertices which do not belong to an arc in α. Hence also $B_l^n = B_l^{\geq n}/B_l^{\geq n+1}$ is a left $K\Sigma_{r-2l}$-module. It follows that $B_l^{\geq n}$ and B_l^n are $\Sigma_{r-2l} - B$-bimodules. Moreover,

$$B_l \simeq \bigoplus_{n \geq l} B_l^n$$

as $K\Sigma_{r-2l}$-module. Since the Σ_{r-2l}-action on B_l^n permutes the basis elements of B_l^n, this is a permutation module for Σ_{r-2l}, and hence can be written as a direct sum of transitive permutation modules: Fix a bottom arc configuration $\beta \in V_n^r$. Then the left Σ_{r-2l}-action on B_l^n permutes transitively the top arc configurations of diagrams in B_l^n (with fixed α and β, but $n-l$ free arcs on the top row). The stabilizer of this action is a subgroup isomorphic to H_{n-l}, and hence we obtain a module isomorphic to $K[\Sigma_{r-2l}/H_{n-l}]$. So for each fixed bottom arc configuration β in V_n^r, we obtain one copy of a permutation module isomorphic to $K[\Sigma_{r-2l}/H_{n-l}]$ and hence:

$$B_l^n \simeq \bigoplus_{\beta \in V_n^r} K[\Sigma_{r-2l}/H_{n-l}]. \tag{11}$$

3.3. **The coarse filtration of** $M(l,\lambda)$**.** By above, the following is an exact sequence of Σ_{r-2l}-B-bimodules,

$$0 \to B_l^{\geq n+1} \to B_l^{\geq n} \to B_l^n \to 0,$$

and it is split-exact as sequence of left Σ_{r-2l}-modules. Let n be a natural number and λ a composition of r in at most n parts. Then tensoring with $M^\lambda \otimes_{K\Sigma_{r-2l}} -$ gives the sequence

$$0 \longrightarrow M^\lambda \otimes_{K\Sigma_{r-2l}} B_l^{\geq n+1} \longrightarrow M^\lambda \otimes_{K\Sigma_{r-2l}} B_l^{\geq n} \longrightarrow M^\lambda \otimes_{K\Sigma_{r-2l}} B_l^n \longrightarrow 0$$

which is exact as sequence of B-modules. In particular, there is a filtration of $M(l,\lambda)$ by B-modules: write

$$\begin{aligned} M^{\geq n}(l,\lambda) &:= M^\lambda \otimes_{K\Sigma_{r-2l}} B_l^{\geq n}, \\ M^n(l,\lambda) &:= M^\lambda \otimes_{K\Sigma_{r-2l}} B_l^n, \end{aligned}$$

then the right B-module $M(l,\lambda)$ is filtered by

$$M(l,\lambda) = M^\lambda \otimes_{K\Sigma_{r-2l}} B_l = M^{\geq l}(l,\lambda) \supset M^{\geq l+1}(l,\lambda) \supset \ldots \supset M^{\geq [r/2]}(l,\lambda) \supset \{0\}$$

with subquotients $M^{\geq n}(l,\lambda)/M^{\geq n+1}(l,\lambda) \simeq M^n(l,\lambda)$ for $l \leq n \leq [r/2]$.

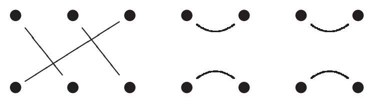
3.4. **Dimension of the layers** $M^n(l,\lambda)$**.** Now we determine the K-dimension of the right B-module $M^n(l,\lambda) = M^\lambda \otimes_{K\Sigma_{r-2l}} B_l^n$. To do so we use adjointness of functors and the decomposition of B_l^n as given in (11). Then we have with $H = H_{n-l}$:

$$\begin{aligned} \mathrm{Hom}_K(M^n(l,\lambda),K) &= \mathrm{Hom}_K(M^\lambda \otimes_{K\Sigma_{r-2l}} B_l^n, K) \\ &= \mathrm{Hom}_{K\Sigma_{r-2l}}(M^\lambda, \mathrm{Hom}_K(B_l^n, K)) \\ &= \mathrm{Hom}_{K\Sigma_{r-2l}}(M^\lambda, (\oplus K[\Sigma_{r-2l}/H])^*) \\ &= \mathrm{Hom}_{K\Sigma_{r-2l}}(M^\lambda, \oplus K[H\backslash\Sigma_{r-2l}]) \\ &= \bigoplus_{\beta \in V_n^r} \mathrm{Hom}_{K\Sigma_{r-2l}}(K \otimes_{K\Sigma_\lambda} K\Sigma_{r-2l}, K[H\backslash\Sigma_{r-2l}]) \\ &= \bigoplus_{\beta \in V_n^r} \mathrm{Hom}_{K\Sigma_\lambda}(K, \mathrm{Hom}_{K\Sigma_{r-2l}}(K\Sigma_{r-2l}, K[H\backslash\Sigma_{r-2l}])) \\ &= \bigoplus_{\beta \in V_n^r} \mathrm{Hom}_{K\Sigma_\lambda}(K, K[H\backslash\Sigma_{r-2l}]). \end{aligned}$$

Any Σ_λ-homomorphism from the trivial module K to the permutation module $K[H\backslash\Sigma_{r-2l}]$ is given by mapping $1 \in K$ to a sum of Σ_λ-orbits of elements in $K[H\backslash\Sigma_{r-2l}]$. Thus, a basis of $\mathrm{Hom}_{K\Sigma_\lambda}(K, K[H\backslash\Sigma_{r-2l}])$ can be chosen to consist of the maps sending 1 to a Σ_λ-orbit sum of the left cosets of the permutation module. Such orbits are in one-to-one correspondence with the double cosets $H\backslash\Sigma_{r-2l}/\Sigma_\lambda$. So $\dim \mathrm{Hom}_K(M^n(l,\lambda),K) = \mid V_n^r \mid \cdot \#(H\backslash\Sigma_{r-2l}/\Sigma_\lambda)$, and hence, with $H = H_{n-l}$,

$$\begin{aligned} \dim M^n(l,\lambda) &= \dim(M^n(l,\lambda))^* \\ &= \mid V_n^r \mid \cdot \#(H\backslash\Sigma_{r-2l}/\Sigma_\lambda) \\ &= \mid V_n^r \mid \cdot \#(\Sigma_\lambda\backslash\Sigma_{r-2l}/H). \end{aligned} \tag{12}$$

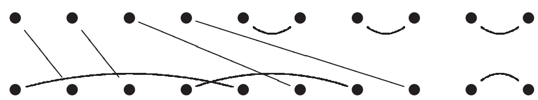
3.5. A basis of the permutation module $M(l, \lambda)$, compatible with the coarse filtration. The permutation module M^λ has permutation basis consisting of cosets $m_j = \Sigma_\lambda \pi_j$ where the element $\pi_j \in \Sigma_{r-2l}$ runs through a set of representatives of right cosets of Σ_λ in Σ_{r-2l}, say, indexed by $j \in I_\lambda$. We embed the elements π_j into the Brauer algebra $B = B_r(\delta)$. Here we embed Σ_{r-2l} on the first $r-2l$ vertices, while on the last $2l$ vertices we choose the fixed arc configuration consisting of l horizontal arcs in both the top and bottom row, each arc joining two adjacent vertices. For example, for $l = 2$ we identify $\pi_j = (1, 2, 3)$ with the diagram

 $\in B_7(\delta)$.

The module $M(l, \lambda) = M^\lambda \otimes_{K\Sigma_{r-2l}} e_l B$ is generated as K-vector space as follows:

$$M(l, \lambda) = \text{Span}_K\{m_j \otimes e_l b \mid j \in I_\lambda, b \in B \text{ Brauer diagram}\},$$

where the tensor product is over $K\Sigma_{r-2l}$. Let $b \in B$ be any Brauer diagram. Note that $e_l b$ is a diagram with bottom configuration in V_n^r for some $n \in \mathbb{N}$, while its top configuration has $n - l$ horizontal arcs somewhere on the first $r-2l$ vertices and then l horizontal arcs of adjacent vertices on the last $2l$ vertices. For every $b \in B$, we always find a permutation $\pi \in \Sigma_{r-2l}$ (on the first $r - 2l$ vertices) such that the diagram $d_t := \pi e_l b$ is a diagram with bottom configuration in V_n^r, while its top configuration has n arcs between adjacent vertices on the last $2n$ vertices, and the induced permutation on the free dots (not involving horizontal arcs) of d_t is the identity, that is, has no crossing lines. For example, choosing $r = 10$ and $n = 3$, a possible such diagram is

$d_t =$  .

So d_t is entirely determined by its bottom configuration $\text{bot}(d_t)$. Using that

$$m_i \otimes e_l b = m_i \pi^{-1} \otimes \pi e_l b = m_j \otimes d_t,$$

for a certain m_j, it follows that

$$\begin{aligned} M(l, \lambda) &= \text{Span}_K\{m_j \otimes e_l b \mid j \in I_\lambda, b \in B \text{ Brauer diagram}\} \\ &= \text{Span}_K\{m_j \otimes d_t \mid \exists n \geq l\colon \text{bot}(d_t) \in V_n^r,\ j \in I_\lambda\ \}. \end{aligned}$$

It then also follows that

$$\begin{aligned} M^{\geq n}(l, \lambda) &= \text{Span}_K\{m_j \otimes d_t \mid \exists m \geq n\colon \text{bot}(d_t) \in V_m^r,\ j \in I_\lambda\ \}, \\ M^n(l, \lambda) &= \text{Span}_K\{m_j \otimes d_t \mid \text{bot}(d_t) \in V_n^r,\ j \in I_\lambda\ \}. \end{aligned}$$

Define the stabilizer

$$H_{n-l}(d_t) = \{\pi \in \Sigma_{r-2l} \mid \pi d_t = d_t\}.$$

The group $H_{n-l}(d_t) \subseteq \Sigma_{r-2l}$ (on the first $r - 2l$ vertices) only depends from the top row configuration of the elements $d_t = \pi e_l b$ – which is the same for each diagram $d_t \in V_n^r$: It consists of $r - 2n$ free vertices, and then n horizontal arcs of adjacent vertices on the last $2n$ vertices. The elements of $H_{n-l}(d_t)$ only affect the vertices between $r - 2n + 1$ and $r - 2l$, and on these vertices $H_{n-l}(d_t) = \mathbb{Z}/2\mathbb{Z} \wr \Sigma_{n-l}$ by (8). Since the group is independent of the element d_t, we henceforth write H_{n-l} instead of $H_{n-l}(d_t)$.

We next construct a basis for the quotient module $M^n(l, \lambda)$. Note that two elements $m_j \otimes d_t$ and $m_i \otimes d_t$ in the above spanning set of $M^n(l, \lambda)$ are equal if there exists a permutation $\pi \in H_{n-l}$ with $m_i \pi = m_j$ since

$$m_i \otimes d_t = m_i \otimes \pi d_t = m_i \pi \otimes d_t = m_j \otimes d_t.$$

Consider the H_{n-l}-action on the right cosets $\Sigma_\lambda \backslash \Sigma_{r-2l}$. Then the above shows that in order to span $M^n(l, \lambda)$, it is enough to take a representative of each H_{n-l}-orbit on $\Sigma_\lambda \backslash \Sigma_{r-2l}$.

Proposition 3.1. *As a K-vector space, $M^n(l,\lambda)$ has a basis*

$$\{m \otimes d_t \mid d_t \in V_n^r,\ m \in \Sigma_\lambda \backslash \Sigma_{r-2l}/H_{n-l}\ \}.$$

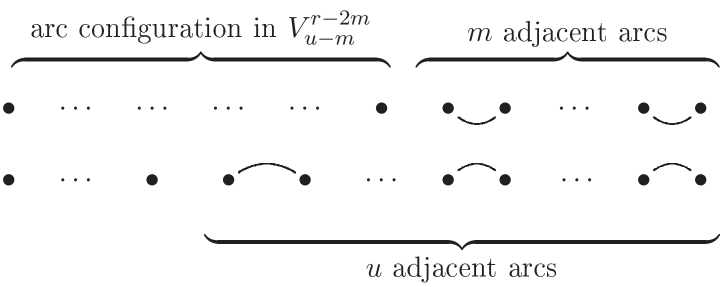
Proof. As just explained, the set $\{m \otimes d_t \mid d_t \in V_n^r,\ m \in \Sigma_\lambda \backslash \Sigma_{r-2l}/H_{n-l}\ \}$ spans $M^n(l,\lambda)$. It follows from (12) that the cardinality of this spanning set equals the dimension of $M^n(l,\lambda)$. Thus it is in fact a basis. □

4. The structure of the layers of the coarse filtration of the permutation modules

The coarse filtration of the Brauer algebra consists of sections (= layers) that are described and controlled, in a precise way, by various symmetric groups. Here, we describe the layers of the corresponding filtration of the permutation modules.

4.1. **The module $M^u(m,\mu)$ as induced module.** Let $B = B_r(\delta)$ be the Brauer algebra of degree r. Let $u \geq m$ be natural numbers and $H = \mathbb{Z}/2\mathbb{Z} \wr \Sigma_{u-m}$. We write $B^u = B^{\geq u}/B^{\geq u+1}$. Recall that when multiplying a Brauer diagram with l arcs by one with $t \geq l$ arcs, the result is a diagram with $\geq t$ arcs.

Remark 4.1. We repeatedly need to determine vector space bases or dimensions of some of the objects introduced. Here is an example. Let $m \leq u$ and $H = \mathbb{Z}/2\mathbb{Z} \wr \Sigma_{u-m}$. The vector space $e_m B^u e_u$ has a basis consisting of Brauer diagrams in B^u of the following form: the top row contains m arcs between adjacent vertices on the last $2m$ vertices; the bottom row contains u arcs between adjacent vertices on the last $2u$ vertices. Moreover, there are $u-m$ further arcs on the first $r-2m$ vertices in the top row:



.

On the remaining $r-2u$ free vertices there is an element of Σ_{r-2u}. So the dimension of $e_m B^u e_u$ is $\mid V_{u-m}^{r-2m} \mid \cdot \mid \Sigma_{r-2u} \mid$.

In the following, we assume, without loss of generality, that Σ_{r-2m} operates on the vertices $\{1,\ldots,r-2m\}$ and that Σ_{r-2u} is the subgroup acting on the vertices $\{1,\ldots,r-2u\}$. The group $H = \mathbb{Z}/2\mathbb{Z} \wr \Sigma_{u-m}$ is defined as stabilizer of a partial diagram with $u-m$ arcs on adjacent vertex in the range $\{r-2u+1,\ldots,r-2m\}$. In this way, $\Sigma_{r-2u} \times H$ is a subgroup of Σ_{r-2m}.

Lemma 4.2. *There is an isomorphism of $K\Sigma_{r-2m}$-$K\Sigma_{r-2u}$-bimodules*

$$e_m B^u e_u \simeq K[\Sigma_{r-2m}/H].$$

Proof. We define a map $\psi : K[\Sigma_{r-2m}/H] \to e_m B^u e_u$ as follows: Given a basis element πH in $K[\Sigma_{r-2m}/H]$, define a Brauer diagram $e_{\pi,u} = \psi(\pi H)$ with top row containing m arcs between adjacent vertices on the last $2m$ vertices, and with the bottom row containing u arcs between adjacent vertices on the last $2u$ vertices; moreover, the top row contains $u-m$ further arcs, given by arcs from $\pi(r-2u+2i-1)$ to $\pi(r-2u+2i)$ where $1 \leq i \leq u-m$. The through strings of

$\psi(\pi H)$ are from $\pi(i)$ to i for $1 \leq i \leq r-2u$. This map is well-defined. Either check that this map is injective and surjective, or define the inverse map in a similar way. Note that ψ defines a $K\Sigma_{r-2m}$-$K\Sigma_{r-2u}$-bimodule isomorphism. □

Lemma 4.3. *The following $K\Sigma_{r-2m}$-B-bimodules are isomorphic:*

$$e_m B^u \simeq G_u(e_m B^u e_u).$$

Proof. (i) Let $b \in e_u B^{\geq u+1}$ be a Brauer diagram and denote by $\alpha = \mathrm{top}(b)$ the top row of b. Define $e(\alpha)$ to be the (semi-)idempotent with bottom and top row equal to α and through strings given as an identity. By definition, $B^u = B^{\geq u}/B^{\geq u+1}$, and so, multiplying B^u, or $B^u e_u \subset B^u$, with anything with strictly more than u arcs gives zero. Since α has at least $u+1$ arcs, it follows in particular that $(B^u e_u)\cdot e(\alpha) = 0$. Note that $e(\alpha)$ lies in $e_u B e_u$. Hence

$$B^u e_u \otimes_{e_u B e_u} b = B^u e_u \otimes_{e_u B e_u} e(\alpha)\cdot b \quad = \quad B^u e_u e(\alpha) \otimes_{e_u B e_u} b = 0.$$

This implies that $e_m B^u e_u \otimes_{e_u B e_u} e_u B^{\geq u+1} = 0$.

(ii) By looking at a basis of $e_m B^u e_u$, it follows that $e_m B^u e_u$ as a right $K\Sigma_{r-2u}$-module is free of rank $\mid V^{r-2m}_{u-m} \mid$. By the definition in (3) and using part (i),

$$\begin{aligned} G_u(e_m B^u e_u) &= e_m B^u e_u \otimes_{e_u B e_u} e_u B \\ &\simeq e_m B^u e_u \otimes_{e_u B e_u} e_u B^u \\ &= e_m B^u e_u \otimes_{K\Sigma_{r-2u}} e_u B^u \\ &\simeq (K\Sigma_{r-2u})^{|V^{r-2m}_{u-m}|} \otimes_{K\Sigma_{r-2u}} e_u B^u \\ &\simeq (e_u B^u)^{|V^{r-2m}_{u-m}|} \end{aligned}$$

where $e_u B e_u$ is acting on the left on $e_u B^u$ and on the right on $e_m B^u e_u$, both times via its quotient $K\Sigma_{r-2u} \simeq e_u B^u e_u$. Writing down bases for $e_m B^u$ and $e_u B^u$, it follows

$$\begin{aligned} \dim e_m B^u &= \mid V^{r-2m}_{u-m} \mid \cdot \mid V^r_u \mid \cdot \mid \Sigma_{r-2u} \mid \\ &= \dim (e_u B^u)^{|V^{r-2m}_{u-m}|} \\ &= \dim e_m B^u e_u \otimes_{e_u B e_u} e_u B^u. \end{aligned}$$

The map $e_m B^u e_u \otimes_{e_u B e_u} e_u B^u \longrightarrow e_m B^u$, given by multiplication, is surjective. As both vector spaces have the same dimension, it is an isomorphism of $K\Sigma_{r-2m}$-B-bimodules. Hence $G_u(e_m B^u e_u) \simeq e_m B^u e_u \otimes_{e_u B e_u} e_u B^u \simeq e_m B^u$. □

Proposition 4.4. *Let $H = \mathbb{Z}/2\mathbb{Z} \wr \Sigma_{u-m}$ such that $\Sigma_{r-2u} \times H$ is a subgroup of Σ_{r-2m}. Let μ be a composition of $r-2m$. The following B-modules are isomorphic:*

$$M^u(m,\mu) = G_u(K[\Sigma_\mu \backslash \Sigma_{r-2m}/H]).$$

Proof. It follows from Lemma 4.3 and Lemma 4.2 that

$$e_m B^u \quad = \quad G_u(e_m B^u e_u) = G_u(K[\Sigma_{r-2m}/H]). \tag{13}$$

The functor G_u is defined by tensoring from the right. Hence $M \otimes G_u(V) = G_u(M \otimes V)$ for any module V. Now use Equations (13) and (9), then

$$\begin{aligned} M^\mu \otimes_{K\Sigma_{r-2m}} e_m B^u &= K[\Sigma_\mu \backslash \Sigma_{r-2m}] \otimes_{K\Sigma_{r-2m}} G_u(K[\Sigma_{r-2m}/H]) \\ &= G_u(K[\Sigma_\mu \backslash \Sigma_{r-2m}] \otimes_{K\Sigma_{r-2m}} K[\Sigma_{r-2m}/H]) \\ &= G_u(K[\Sigma_\mu \backslash \Sigma_{r-2m}/H]). \end{aligned}$$

□

4.2. **The decomposition of the B-module $M^u(m,\mu)$ into direct summands.** Let U and $H\times\Sigma$ be subgroups of a finite group G. Then the set of double cosets $\{UgH \mid g\in G\}$ is a Σ-set with action from the right, and hence gives rise to a $K\Sigma$-permutation module $K[U\backslash G/H]$. Writing this permutation module as direct sum of transitive permutation module, we have the following correspondence:

$$\text{transitive direct summands} \leftrightarrow \Sigma\text{-orbits on } U\backslash G/H \leftrightarrow U\backslash G/H\times\Sigma$$

where the second correspondence is given by the map $UgH\times\Sigma\mapsto UgH$. We apply this in the following situation: Let $u\geq m$ be natural numbers and $\Lambda(r-2u)$ be the set of compositions of $r-2u$. Let μ be a composition of $r-2m$. Choose $H=\mathbb{Z}/2\mathbb{Z}\wr\Sigma_{u-m}$ such that $\Sigma_{r-2u}\times H$ is a subgroup of Σ_{r-2m}. Then $\Sigma_\mu\backslash\Sigma_{r-2m}/H$ is a Σ_{r-2u}-set, with operation of Σ_{r-2u} on the double cosets from the right.

Lemma 4.5. *There exists a function $\varphi:\Sigma_\mu\backslash\Sigma_{r-2m}/H\times\Sigma_{r-2u}\to\Lambda(r-2u)$ such that the $K\Sigma_{r-2u}$-module $K[\Sigma_\mu\backslash\Sigma_{r-2m}/H]$ decomposes as follows:*

$$K[\Sigma_\mu\backslash\Sigma_{r-2m}/H]=\bigoplus_{\pi\in\Sigma_\mu\backslash\Sigma_{r-2m}/(\Sigma_{r-2u}\times H)} M^{\varphi(\pi)}.$$

Proof. Writing the permutation module $K[\Sigma_\mu\backslash\Sigma_{r-2m}/H]$ as a direct sum of transitive permutation modules, the direct summands correspond by above to representatives of double cosets $\Sigma_\mu\backslash\Sigma_{r-2m}/(\Sigma_{r-2u}\times H)$. We determine the transitive permutation module corresponding to a double coset $\Sigma_\mu\pi H$ for $\pi\in\Sigma_{r-2m}$. By assumption, Σ_{r-2u} commutes with H. Hence

$$\begin{aligned}\mathrm{Stab}_{\Sigma_{r-2u}}(\Sigma_\mu\pi H) &= \mathrm{Stab}_{\Sigma_{r-2u}}(\Sigma_\mu\pi)=\Sigma_{r-2u}\cap\mathrm{Stab}_{\Sigma_{r-2m}}(\Sigma_\mu\pi)\\ &= \Sigma_{r-2u}\cap\pi^{-1}\Sigma_\mu\pi=:\Sigma_\nu.\end{aligned}$$

for some composition ν. So the permutation module corresponding to the double coset representative π is the permutation module M^ν. The claim now follows with the function

$$\varphi:\Sigma_\mu\backslash\Sigma_{r-2m}/(\Sigma_{r-2u}\times H)\to\Lambda(r-2u),\qquad \pi\mapsto\varphi(\pi)=\nu. \tag{14}$$

□

Proposition 4.6. *Let $u\geq m$ be natural numbers. Let μ be a composition of $r-2m$. We have the following right B-module decomposition:*

$$M^u(m,\mu)=\bigoplus_{\pi\in\Sigma_\mu\backslash\Sigma_{r-2m}/(\Sigma_{r-2u}\times H)} G_u(M^{\varphi(\pi)}).$$

Here the composition $\varphi(\pi)$ is defined as in (14).

Proof. By Lemma 4.5, the $K\Sigma_{r-2u}$-module $K[\Sigma_\mu\backslash\Sigma_{r-2m}/H]$ decomposes as a direct sum of transitive permutation modules: $K[\Sigma_\mu\backslash\Sigma_{r-2m}/H_u]=\bigoplus_{\pi\in\Sigma_\mu\backslash\Sigma_{r-2m}/(\Sigma_{r-2u}\times H)}M^{\varphi(\pi)}$, where the composition $\varphi(\pi)$ is defined as in (14). The claim then follows from Proposition 4.4. □

4.3. **A combinatorial description of $\varphi(\pi)$.** The Young subgroup Σ_λ operates on the right on the set V_{u-m}^{r-2m} of partial diagrams of length $r-2m$ with $u-m$ horizontal arcs. We say partial diagrams $v,w\in V_{u-m}^{r-2m}$ are *equivalent*, $v\sim_{\Sigma_\mu}w$, iff there exists $\sigma\in\Sigma_\mu$ with $v\sigma=w$. Note that $\sim$ is an equivalence relation on V_{u-m}^{r-2m}, and we denote the equivalence class of v by $[v]$.

Lemma 4.7. *There are two bijections:*

$$\begin{aligned}\Sigma_{r-2m}/(\Sigma_{r-2u}\times H_{u-m}) &\leftrightarrow V_{u-m}^{r-2m},\\ \Sigma_\mu\backslash\Sigma_{r-2m}/(\Sigma_{r-2u}\times H_{u-m}) &\leftrightarrow V_{u-m}^{r-2m}/\sim_{\Sigma_\mu}.\end{aligned}$$

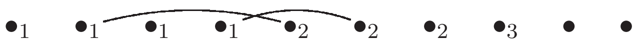
Proof. The group Σ_{r-2m} acts transitively on the set V_{u-m}^{r-2m} of arc configurations. The free vertices are stabilised by Σ_{r-2u}, while any given arc configuration has a stabiliser H_{u-m}. This yields the bijections. □

Let μ be a composition of $r-2m$ and v a partial diagram on $r-2m$ vertices with $u-m$ horizontal arcs. We introduce a particular composition $\mu_{[v]}$ associated to the partial diagram v: Let i be the ordered multi-index with $\lambda(i) = \mu$, where $\lambda(i)$ denotes the weight of the multi-index i. Write i on the arc configuration v on $r-2m$ vertices, from left to right. Then read off the multi-index i_v on the free vertices of v, from left to right. Define $\mu_v = \lambda(i_v)$. Given two partial diagrams $v, w \in V_{u-m}^{r-2m}$ which are Σ_μ-conjugate, it follows that $\mu_v = \mu_w$, since Σ_μ permutes same numbers of multi-index i only. Therefore, define $\mu_{[v]} = \mu_v$.

For example, if $\mu = (4,3,1)$ with $l = 1$ and v is the partial arc configuration

$v =$  ,

then form the diagram

$v =$  ,

and read off that $i_v = (1,1,2,3)$, and hence $\mu_v = \lambda(i_v) = (2,1,1)$.

Corollary 4.8. *Let $[v] \in V_{u-m}^{r-2m}/\sim_{\Sigma_\mu}$ be the equivalence class of arc configurations corresponding (by Lemma 4.7) to the double coset $\pi \in \Sigma_\mu \backslash \Sigma_{r-2m}/(\Sigma_{r-2u} \times H_{u-m})$. Then $\varphi(\pi) = \mu_{[v]}$.*

5. An integral basis of $S_B(n,r,\delta)$

We will determine a basis of $\mathrm{Hom}_B(M(l,\lambda), M(m,\mu))$ that only depends on combinatorial data, but not on the ground ring K or its characteristic, nor on the choice of $\delta \neq 0$.

5.1. **From Brauer algebras to symmetric groups.** First we rewrite the space of homomorphisms with source $M(l,\lambda)$ as follows:

$$
\begin{aligned}
\mathrm{Hom}_B(M(l,\lambda),-) &\overset{(a)}{\simeq} \mathrm{Hom}_B(M^\lambda \otimes_{K\Sigma_{r-2l}} e_l B, -)\\
&\overset{(b)}{\simeq} \mathrm{Hom}_{K\Sigma_{r-2l}}(M^\lambda, \mathrm{Hom}_B(e_l B, -))\\
&\overset{(c)}{\simeq} \mathrm{Hom}_{K\Sigma_{r-2l}}(M^\lambda, - \cdot e_l)\\
&\overset{(d)}{\simeq} \mathrm{Hom}_{K\Sigma_{r-2l}}(K \otimes_{K\Sigma_\lambda} K\Sigma_{r-2l}, - \cdot e_l)\\
&\overset{(e)}{\simeq} \mathrm{Hom}_{K\Sigma_\lambda}(K, \mathrm{Hom}_{K\Sigma_{r-2l}}(K\Sigma_{r-2l}, - \cdot e_l))\\
&\overset{(f)}{\simeq} \mathrm{Hom}_{K\Sigma_\lambda}(K, - \cdot e_l)\\
&\overset{(g)}{\simeq} (- \cdot e_l)^{\Sigma_\lambda}
\end{aligned}
$$

Here, the isomorphism (a) is just the definition of $M(l,\lambda)$, (b) is adjointness, (c) is a standard argument for maps starting in a projective module, (d) is the definition of the permutation module M^λ over the group algebra $K\Sigma_{r-2l}$, (e) is again adjointness, (f) is again the argument for maps starting in a projective module, and finally (g) is the definition of invariants as sums of elements in an orbit of the group Σ_λ.

Thus, the space of homomorphisms starting in $M(l,\lambda)$ and ending in some module, say X, is isomorphic to the space $(Xe_l)^{\Sigma_\lambda}$ of Σ_λ-invariants of X. The correspondence identifies a homomorphism φ with the image $\varphi(\Sigma_\lambda 1 \otimes_{K\Sigma_{r-2l}} e_l)$ of the standard generator of $M(l,\lambda)$ in the space of invariants. Therefore, the next step is to decompose the target space $M(m,\mu)$ over $K\Sigma_\lambda$.

5.2. **A decomposition of $M(m,\mu)$.** As we have seen in Section 3, the module $M(m,\mu)$ has a right B-module filtration:

$$M(m,\mu) = M^{\geq m}(m,\mu) \supset M^{\geq m+1}(m,\mu) \supset \ldots \supset M^{\geq [r/2]}(m,\mu) \supset \{0\}.$$

Multiplying this filtration from the right by e_l gives a filtration of right e_lBe_l-modules. Each e_lBe_l-module, by restriction of the action, is a $K\Sigma_{r-2l}$-module. Here Σ_{r-2l} operates by permutation on the free dots of the Brauer diagrams forming a basis of the respective modules. In particular, $M(m,\mu)e_l$ becomes a Σ_{r-2l}-permutation module. We define $M^{=u}(m,\mu)$ to be the K-vector space spanned by Brauer diagrams with precisely u horizontal arcs in the top and bottom row. So $M^{=u}(m,\mu)$ is a subspace of $M(m,\mu)$. The symmetric group Σ_{r-2l} operates on the free vertices of the Brauer diagrams in $M^{=u}(m,\mu)e_l$, making this into a $K\Sigma_{r-2l}$-module. Then

$$M(m,\mu)e_l \quad = \quad \bigoplus_{u\geq m} M^{=u}(m,\mu)e_l \tag{15}$$

is a decomposition as $K\Sigma_{r-2l}$-module. (Since $u \geq l$, every basis element in $M^{=u}(m,\mu)e_l$ has precisely u horizontal arcs in both the top and bottom row, where the bottom row has l arcs between adjacent vertices fixed on the right and further $u-l$ horizontal arcs on the first $r-2l$ (remaining) vertices. The symmetric group operates on the first $r-2l$ vertices, and hence permutes these $u-l$ arcs. So each $M^{=u}(m,\mu)e_l$ is a $K\Sigma_{r-2l}$-permutation module.)

By definition, $M^{=u}(m,\mu)e_l$ does not carry an e_lBe_l-module structure, and hence the latter decomposition is not an e_lBe_l-module decomposition. The module $M^u(m,\mu)$, however, by definition is a quotient module of $M(m,\mu)$, and as such typically not a subspace of $M(m,\mu)$. But $M^u(m,\mu)e_l$ is an e_lBe_l-module and, by restricting the action, a $K\Sigma_{r-2l}$-module. Note that $M^u(m,\mu)e_l \simeq M^{=u}(m,\mu)e_l$ as $K\Sigma_{r-2l}$-modules. And of course $M(m,\mu)e_l \simeq \bigoplus_{u\geq m} M^u(m,\mu)e_l$ is a decomposition as $K\Sigma_{r-2l}$-module, but not an equality.

5.3. **Decomposing the layers $M^\nu \otimes_{e_uBe_u} e_uBe_l$.** Proposition 4.6 gives a decomposition of each layer of $M^u(m,\mu)$; the summands there are $G_u(M^{\varphi(\pi)})$, where the compositions $\varphi(\pi)$ have been described in Lemma 4.5. We now pick one such summand, say $G_u(M^\nu) = M^\nu \otimes_{e_uBe_u} e_uB$.

In order to compute a summand $\mathrm{Hom}_{K\Sigma_{r-2l}}(M^\lambda, M^\nu \otimes_{e_uBe_u} e_uBe_l)$, we first describe the $K\Sigma_{r-2l}$-module $M^\nu \otimes_{e_uBe_u} e_uBe_l$ as a permutation module. The result depends on the relation between l and u. The homomorphism spaces then will be spanned by double cosets.

Proposition 5.1. *Let u and l be natural numbers and ν a composition of $r-2u$. Then there are isomorphisms of right $K\Sigma_{r-2l}$-modules:*

$$G_u(M^\nu)e_l \simeq \begin{cases} 0 & \text{if } l > u, \\ M^\nu & \text{if } l = u, \\ K[\Sigma_\nu \times H_{u-l}\backslash\Sigma_{r-2l}] & \text{if } l < u. \end{cases}$$

Proof. If $l = u$ then there are isomorphisms $G_u(M^\nu)e_l = M^\nu \otimes_{e_uBe_u} e_uBe_u \simeq M^\nu$. If $l > u$ then $e_l = \lambda e_u e_l e_u \in e_uBe_u$ with $\lambda \in F$ and the surjection $e_uBe_u \to K\Sigma_{r-2u}$ sends e_l to zero. Thus, e_l annihilates the e_uBe_u-module $K\Sigma_{r-2u}$ and hence $G_u(M^\nu)e_l$ vanishes.

It remains to consider the case when $l < u$. The space e_uBe_l has a vector space basis given by all diagrams which have at least u arcs in the top row, u of which are fixed by the configuration of e_u, and the same number of arcs in the bottom row, l of which are fixed by the configuration of e_l. Let b be such a basis element with at least $u+1$ horizontal arcs in the top and bottom row. Let $\alpha = \text{top}(b)$, and write $e(\alpha)$ for the semi-idempotent with top and bottom row given by α and through strings given by the identity permutation. Then $e(\alpha)\cdot b = \lambda \cdot b$ with λ some power of the parameter δ. Moreover $e(\alpha) \in e_u B e_u$, and hence it acts on M^ν via the quotient map $e_u B e_u \to K\Sigma_{r-2u}$. Since $e(\alpha)$ has strictly more than u horizontal arcs, it acts as zero. So after tensoring with M^ν, all diagrams with strictly more than u arcs disappear. Therefore, the module $G_u(M^\nu)e_l$ has generators (over K) of the form $m\otimes b$ where b runs through those of the above diagrams that have exactly u arcs in each row, with no intersection of through strings, and where m runs through a basis of M^ν, that is, the spanning set is as follows:.

$$G_u(M^\nu)e_l \quad = \quad \text{Span}\{m\otimes\alpha \mid \tilde{\alpha} \in V_{u-l}^{r-2l}, m \in \Sigma_{r-2u}/\Sigma_\nu\}.$$

where we write $\tilde{\alpha}$ for the arc configuration obtained when restricting α to the first $r-2l$ vertices. Linear independence of these generators is obtained by a counting argument in a similar way as above in Subsection 3.5.

The symmetric group Σ_{r-2l} acts on the right, that is, it permutes vertices on the bottom row of d. This action is transitive, since it can be used to permute the additional $u-l$ arcs in the top row and simultaneously to permute the generators of M^ν (the latter via the action on the through strings). One particular basis element has the form $1\otimes e_u$ where 1 is representing the coset containing the unit element of Σ_{r-2u}. The stabiliser of this element is the group $\Sigma_\nu \times H_{u-l}$, where Σ_ν acts on the free vertices and the wreath product acts on the additional arcs. It follows that there is an isomorphism of right $K\Sigma_{r-2l}$-modules $G_u(M^\nu)e_l \simeq K[\Sigma_\nu\times H_{u-l}\backslash\Sigma_{r-2l}]$. □

This implies the statement on homomorphisms that we need:

Corollary 5.2. *Let u and l be natural numbers, λ a composition of $r-2l$ and ν a composition of $r-2u$. Then there are isomorphisms of K-spaces:*

$$\text{Hom}_{K\Sigma_{r-2l}}(M^\lambda, G_u(M^\nu)e_l) \simeq \begin{cases} 0 & \text{if } l > u, \\ K[\Sigma_\nu\backslash\Sigma_{r-2l}/\Sigma_\lambda] & \text{if } l = u, \\ K[\Sigma_\nu\times H_{u-l}\backslash\Sigma_{r-2l}/\Sigma_\lambda] & \text{if } l < u. \end{cases}$$

5.4. **Invariants and double cosets.** Given are natural numbers u, l, m with $u \geq l$ and $u\geq m$, a composition μ of $r-2m$ and a composition λ of $r-2l$.

Theorem 5.3. *A K-vector space basis of $\text{Hom}_B(M(l,\lambda), M(m,\mu))$ is given by the set of all $\phi_{u,\pi,\sigma}$ with $u\geq l,m$ and*

$$\begin{array}{rll} \pi\in\Sigma_{r-2m} & \textit{a representative of} & \Sigma_\mu\backslash\Sigma_{r-2m}/(\Sigma_{r-2u}\times H_{u-m}), \\ \sigma\in\Sigma_{r-2l} & \textit{a representative of} & \Sigma_\nu\times H_{u-l}\backslash\Sigma_{r-2l}/\Sigma_\lambda, \end{array}$$

where $\Sigma_\nu = \Sigma_{r-2u}\cap\pi^{-1}\Sigma_\mu\pi$. The map $\phi_{u,\pi,\sigma}$ on the generator $\Sigma_\lambda\cdot id\otimes e_l$ is given by

$$\phi_{u,\pi,\sigma}(\Sigma_\lambda\cdot id\otimes e_l) = \sum_{\alpha\in(\Sigma_\lambda\cap\sigma^{-1}(\Sigma_\nu\times H_{u-l})\sigma)\backslash\Sigma_\lambda} (\Sigma_\mu\cdot id\otimes e_{\pi,u})\sigma\cdot\alpha. \tag{16}$$

Whenever it is necessary, we will use the more precise notation $\phi_{u,\pi,\sigma}^{(l,\lambda)(m,\mu)}$ for the map $\phi_{u,\pi,\sigma}\in \text{Hom}_B(M(l,\lambda),M(m,\mu))$, which also indicates the domain and the target of the map.

Proof. We construct a basis of $\text{Hom}_B(M(l,\lambda),M(m,\mu))$ by giving a basis of $\text{Hom}_{K\Sigma_\lambda}(K, M(m,\mu)e_l)$. We use (15): $M(m,\mu)e_l = \bigoplus_{u\geq m} M^{=u}(m,\mu)e_l$.

The assumption $u \geq l$ ensures that $\mathrm{Hom}_{K\Sigma_\lambda}(K, M(m,\mu)e_l)$ does not vanish. A homomorphism is determined by giving the image of 1, which must be some Σ_λ-orbit in $M(m,\mu)e_l$, and hence lies in

$$(M(m,\mu)e_l)^{\Sigma_\lambda} = \bigoplus_{u\geq m}(M(m,\mu)^{=u}e_l)^{\Sigma_\lambda}.$$

The latter is a decomposition of $K\Sigma_{r-2l}$-modules. When taking invariants, direct summands carry over, and by Proposition 4.6, we have the following decomposition of right $K\Sigma_{r-2l}$-modules:

$$\begin{aligned}(M(m,\mu)^{=u}e_l)^{\Sigma_\lambda} &= \bigoplus_{\pi}(G_u(M^{\varphi(\pi)}e_l))^{\Sigma_\lambda} \\ &= \bigoplus_{\pi}(K[\Sigma_{\varphi(\pi)} \times H_{u-l}\backslash\Sigma_{r-2l}])^{\Sigma_\lambda} \\ &= \bigoplus_{\pi} K[\Sigma_{\varphi(\pi)} \times H_{u-l}\backslash\Sigma_{r-2l}/\Sigma_\lambda].\end{aligned}$$

Here we define $\Sigma_{\varphi(\pi)} = \Sigma_\nu = \Sigma_{r-2u} \cap \pi^{-1}\Sigma_\mu\pi$, and the sum runs through a set of representatives π of the double cosets $\Sigma_\mu\backslash\Sigma_{r-2m}/\Sigma_{r-2u} \times H_{u-m}$. Fix the following double coset representatives:

$$\begin{aligned}\pi \in \Sigma_{r-2m} \quad &\text{a representative of} \quad \Sigma_\mu\backslash\Sigma_{r-2m}/(\Sigma_{r-2u} \times H_{u-m}),\\ \sigma \in \Sigma_{r-2l} \quad &\text{a representative of} \quad \Sigma_\nu \times H_{u-l}\backslash\Sigma_{r-2l}/\Sigma_\lambda.\end{aligned}$$

Choosing π means choosing a direct summand in the uth layer of $M(m,\mu)$, and for that direct summand we have chosen via σ a basis element of the permutation basis. The module $M(l,\lambda)$ is generated by $\Sigma_\lambda \cdot 1 \otimes e_l$. A basis of $\mathrm{Hom}_B(M(l,\lambda), M(m,\mu))$ is then given by the maps $\phi_{u,\pi,\sigma}$ with u,π and σ as above and where $\phi_{u,\pi,\sigma}$ maps the generator $\Sigma_\lambda \cdot 1 \otimes e_l$ to the Σ_λ-orbit of the chosen basis element. To understand the module $M(m,\mu)$, we applied various isomorphisms and now choose a basis element in $K[\Sigma_{\varphi(\pi)\times H_{u-l}}\backslash\Sigma_{r-2l}/\Sigma_\lambda]$. This needs to be translated back into an element of $M(m,\mu)$ using Section 2.4, Lemma 4.2, Proposition 4.4, Lemma 4.5, Proposition 4.6 and their proofs. Recall the definition of $e_{\pi,u}$ given in the second part of the proof of Lemma 4.2. Doing so, the B-homomorphism

$$\phi_{u,\pi,\sigma} : M(l,\lambda) \longrightarrow M(m,\mu)$$

is given by

$$\phi_{u,\pi,\sigma}(\Sigma_\lambda \cdot id \otimes e_l) = \sum_{\alpha\in\Sigma_\lambda\cap\sigma^{-1}(\Sigma_\nu\times H_{u-l})\sigma\backslash\Sigma_\lambda} (\Sigma_\mu \cdot id \otimes e_{\pi,u})\sigma \cdot \alpha$$

It should be noted that when taking the Σ_λ-orbit of the basis element this is the same as taking the sum over all elements in Σ_λ modulo those elements that stabilize the basis element. This stabilizer is $\Sigma_\lambda \cap \sigma^{-1}(\Sigma_\nu \times H_{u-l})\sigma$. □

Remark 5.4. (a) Note that the sum in (16) is a sum of diagrams, possibly with repetitions. The coefficients are non-negative integers.
(b) The basis $\{\phi_{u,\pi,\sigma}\}$ is independent of the underlying field and the parameter δ. Hence the Schur algebra $S_B(n,r,\delta)$ has dimension independent from the underlying field and the chosen parameter δ. This dimension is

$$\sum_{l=0}^{[r/2]}\sum_{m=0}^{[r/2]}\sum_{\lambda}\sum_{\mu}\sum_{u \text{ with } u\geq l, u\geq m}\sum_{\pi}|\Sigma_{\varphi(\pi)} \times H_{u-l}\backslash\Sigma_{r-2l}/\Sigma_\lambda|$$

where $\lambda \in \Lambda(n, r-2l)$, $\mu \in \Lambda(n, r-2m)$ and $\pi \in \Sigma_\mu\backslash\Sigma_{r-2l}/(\Sigma_{r-2u} \times H_{u-l})$.

Corollary 5.5. *The K-dimension of $S_B(n,r,\delta)$ and the K-dimensions of its coarse layers do not depend on K nor on δ. In particular, there is an integral version of $S_B(n,r,x)$, defined over the* *polynomial ring $\mathbb{Z}[x]$ such that for any choice of K and $\delta \in K$, the algebra $S_B(n, r, \delta)$ is obtained from the integral version by changing scalars and specialising the variable x to δ.*

When K is a field not of characteristic two and three, the Schur algebras implicitly occurring in [13] are part of the above family of algebras. We will see later, when proving quasi-heredity, that even the dimensions of the layers in the heredity chain (that refines the coarse ideal chain) do not depend on K nor on δ.

6. Classical Schur algebras on the diagonal of $S_B(n, r, \delta)$

Each layer in the coarse filtration of a Brauer algebra $B_r(\delta)$ is associated with some symmetric group Σ_{r-2l}. Analogously we are going to show that each layer in the coarse filtration of $S_B(n, r, \delta)$ is associated with some classical Schur algebra $S_A(n, r-2l)$ of type A.

6.1. **The classical Schur algebra.** Let n, r be natural numbers, let E be a vector space with basis e_i for $1 \leq i \leq n$. Denote by $I = I(n, r)$ the set of multi-indices of length r with entries in the set $\{1, \ldots, n\}$. Then $E^{\otimes r}$ has basis consisting of all elements $e_i = e_{i_1} \otimes \ldots \otimes e_{i_r}$, for $i \in I$. The symmetric group Σ_r operates on the set of multi-indices I by place permutation:

$$i \cdot \sigma = (i_{\sigma 1}, \ldots, i_{\sigma r})$$

for $\sigma \in \Sigma_r$ and multi-index $i \in I$. This action extends to the tensor space $E^{\otimes r}$, making $E^{\otimes r}$ into a $K\Sigma_r$-module with $e_i \cdot \sigma = e_{\sigma i}$ for $\sigma \in \Sigma_r$. The Schur algebra $S_A(n, r)$ is defined to be $\mathrm{End}_{K\Sigma_r}(E^{\otimes r})$. The action of the symmetric group on the set of multi-indices extends diagonally to an action of Σ_r on the set $I \times I$: for $\sigma \in \Sigma_r$, define $(i, j)\sigma = (i\sigma, j\sigma)$. We write $(i, j) \sim (k, l)$ if both pairs lie in the same Σ_r-orbit of $I \times I$. Let

$$\Omega = I(n, r) \times I(n, r)/\sim,$$

then the Schur algebra $S_A(n, r)$ has K-basis $\{\xi_{i,j} \mid (i, j) \in \Omega\}$. On these basis elements, the multiplication of the Schur algebra is as follows: Let $i, j, k, l \in I$, then

$$\xi_{i,j} \cdot \xi_{k,l} = \sum_{(p,q)\in\Omega} Z(i, j, k, l, p, q)\xi_{p,q} \tag{17}$$

with

$$Z(i, j, k, l, p, q) = \#\{s \in I \mid (i, j) \sim (p, s),\ (s, q) \sim (k, l)\}.$$

For details we refer the reader to the monograph by Green [12], Chapter 2.

We need the following notation. Given a multi-index $i \in I$, denote by $\lambda(i)$ the composition $\lambda \in \Lambda(n, r)$ with λ_t equal to the number of i_ρ in i with $i_\rho = t$, for $1 \leq \rho \leq r$. The multi-index corresponding to a composition $\lambda \in \Lambda(n, r)$ is defined to be the multi-index $i = (i_1, \ldots, i_r)$, ordered increasingly, containing λ_k times the natural number k with $1 \leq k \leq n$. We denote this multi-index by $\mathrm{ind}(\lambda)$. Denote by M^λ the permutation module of $K\Sigma_r$ on the cosets Σ_r/Σ_λ. This permutation module can be realised as a direct summand of the tensor space $E^{\otimes r}$,

$$M^\lambda = \mathrm{Span}\{e_i \mid i \in I(n, r),\ \lambda(i) = \lambda\}.$$

The elements $\xi_{i,j}$ can then be interpreted as $K\Sigma_r$-homomorphisms,

$$\xi_{i,j} : M^{\lambda(j)} \to M^{\lambda(i)}.$$

Then the action of $S_A(n, r)$ on tensor space translates into the following action on M^λ (see formula 2.6a in [12], which we reformulate in terms of stabilisers)

$$(\Sigma_\lambda \otimes 1)\xi_{i,j} = \sum_{\alpha\in S}(\Sigma_\mu \otimes \sigma\alpha),$$

where j is the increasing index of weight λ, σ represents the double coset corresponding to $\xi_{i,j}$ and $S = (\mathrm{Stab}(i) \cap \Sigma_\lambda)\backslash\Sigma_\lambda$.

6.2. **Subquotients of $S_B(n, r, \delta)$.** We will show that the basis elements $\xi_{i,j}$ of the classical Schur algebra $S_A(n, r-2u)$ (for each u) can be identified with certain basis elements of the Brauer Schur algebra $S_B(n, r, \delta)$, giving rise to subquotients of $S_B(n, r, \delta)$ being isomorphic to Schur algebras $S_A(n, r-2u)$ for $r - 2u \geq 0$. The prototype is the subalgebra $S_A(n, r)$ of $S_B(n, r, \delta)$, which equals the top quotient in the coarse filtration. Here, the B-module $M(0, \lambda)$ is just the inflated $K\Sigma_r$-module M^λ. The Schur algebra $S_A(n, r)$ is K-spanned by all $S_B(n, r, \delta)$-basis elements

$$\phi_{0,id,\sigma} : M(0, \lambda) \longrightarrow M(0, \mu)$$

with parameter $\sigma \in \Sigma_\mu\backslash\Sigma_r/\Sigma_\lambda$.

Let u be some natural number such that $r - 2u \geq 0$. Consider the set $B^A_{r,u}$ of all basis elements of $S_B(n, r, \delta)$ which map the generator $\Sigma_\lambda \cdot 1 \otimes e_u$ of $M(u, \lambda)$ into the uth layer of $M(u, \mu)$,

$$B^A_{r,u} = \{\phi^{(u,\lambda)(u,\mu)}_{u,id,\sigma} \mid \sigma \in \Sigma_\mu\backslash\Sigma_{r-2u}/\Sigma_\lambda,\ \lambda, \mu \in \Lambda(n, r-2u)\}.$$

Here, as before, the upper indices (u, λ) and (u, μ) indicate the domain $M(u, \lambda)$ and the target $M(u, \mu)$ of the map $\phi_{u,id,\sigma}$.

Proposition 6.1. *The vector space* $\mathrm{Span}\{B^A_{r,u}\}$ *is an algebra subquotient of* $S_B(n, r, \delta)$ *isomorphic to the Schur algebra* $S_A(n, r-2u)$.

Proof. We can identify basis elements in $B^A_{r,u}$ with basis elements $\xi_{i,j}$ of the Schur algebra $S_A(n, r-2u)$. This identification is given as follows:

$$\phi^{(u,\lambda)(u,\mu)}_{u,id,\sigma} \longleftrightarrow \xi_{ij'} \tag{18}$$

with $j' = \mathrm{ind}(\lambda)$ and $i = \mathrm{ind}(\mu)\sigma$. Moreover, the proof of Theorem 5.3 identifies a map $\varphi : M^\lambda \to M^\mu$ with $\varphi \otimes id : M^\lambda \otimes_{e_u B e_u} e_u B \to M^\mu \otimes_{e_u B e_u} e_u B$ where

$$(\varphi \otimes id)(\Sigma_\lambda \cdot 1 \otimes e_l) \quad = \quad \varphi(\Sigma_\lambda \cdot 1) \otimes e_l = (\Sigma_\mu \cdot 1 \otimes e_l)x_\varphi$$

with $\Sigma_\mu x_\varphi = \varphi(\Sigma_\lambda \cdot 1)$ where $x_\varphi \in K\Sigma_{r-2l}$. This assignment is compatible with composition, and the claim follows. ☐

Remark 6.2. Every double coset $\Sigma_\mu\tau\Sigma_\lambda$ with $\tau \in \Sigma_{r-2u}$, corresponds to a Schur algebra element $\xi(\tau) \in S_A(n, r-2u)$. Define $i = \mathrm{ind}(\mu) \cdot \tau$ and $j = \mathrm{ind}(\lambda)$. Then the double coset of τ corresponds to the Schur algebra element $\xi(\tau) = \xi^A_{ij}$. Conversely, a Schur algebra element $\xi^A_{ij'}$ with ordered multi-index j' corresponds to double coset $\Sigma_{\lambda(i)}\tau\Sigma_{\lambda(j')}$ with $i = i'\tau$.

7. Quasi-hereditary structure

Like the Brauer algebra $B_r(\delta)$, the Schur algebra $S_B(n, r, \delta)$ has a coarse filtration that is defined in the following way: For $0 \leq u \leq [r/2]$, let $f_u := id_{\bigoplus_\lambda M(u,\lambda)}$ where the sum runs through all partitions (or compositions) of $r - 2u$. Then the f_u are pairwise orthogonal idempotents summing up to the identity of $S_B(n, r, \delta)$. The coarse filtration of $S_B(n, r, \delta)$ by definition is the following chain of ideals:

$$S_B(n, r, \delta) \supset S_B(n, r, \delta)(f_1 + f_2 + \ldots)S_B(n, r, \delta) \supset S_B(n, r, \delta)(f_2 + f_3 + \ldots)S_B(n, r, \delta) \supset \ldots.$$

The algebra $\mathrm{Span}\{B^A_{r,u}\}$ that in the previous section has been shown to be isomorphic to the classical Schur algebra $S_A(n, r-2u)$ can be rewritten as algebra subquotient

$$f_u S_B(n, r, \delta) f_u / f_u(S_B(n, r, \delta)(f_{u+1} + f_{u+2} + \ldots)S_B(n, r, \delta)) f_u.$$

We can now state the main structural property of the algebras $S_B(n,r,\delta)$. The proof will occupy this section and the subsequent three sections.

Theorem 7.1. *The Schur algebra $S_B^{\mathbb{Z}[X]}(n,r,X)$ defined over $\mathbb{Z}[X]$ is quasi-hereditary, and tensoring its heredity chain by $K\otimes_{\mathbb{Z}[X]}-$ (specialising X to δ) provides $S_B^K(n,r,\delta)$ with a quasi-hereditary structure.*

This provides an integral quasi-hereditary structure on the Schur algebras $S_B^K(n,r,\delta)$, in the sense of Cline, Parshall and Scott [3].

To show that $S_B(n,r,\delta)$ is quasi-hereditary, we will use a characterisation of quasi-hereditary algebras due to Dlab and Ringel [8, Theorem 1]. To state it, we need to introduce some notation. Let Λ be a quasi-hereditary algebra. Denote by Δ_Λ the set of standard modules of the algebra Λ. Denote by $\mathcal{F}(\Delta_\Lambda)$ the full subcategory of Λ-modules that are filtered by the standard modules in Δ_Λ.

Theorem 7.2. *(Dlab and Ringel [8]) Let Λ be an algebra and $f\in\Lambda$ an idempotent. Then there exists a heredity chain for Λ containing the ideal $\Lambda f\Lambda$ (and thus Λ is quasi-hereditary) if and only if the following conditions are satisfied:*

(i) *Both algebras $f\Lambda f$ and $\Lambda/\Lambda f\Lambda$ are quasi-hereditary;*
(ii) *$\Lambda f\in\mathcal{F}(\Delta_{f\Lambda f})$ and $f\Lambda\in\mathcal{F}({}_{f\Lambda f}\Delta)$;*
(iii) *the multiplication map $\Lambda f\otimes_{f\Lambda f} f\Lambda\to\Lambda f\Lambda$ is bijective.*

The strategy of the proof of Theorem 7.1 goes as follows: We use the coarse filtration of $S_B(n,r,\delta)$ defined by the idempotents f_u and consider the quotient algebras

$$\Lambda_u := S_B(n,r,\delta)/S_B(n,r,\delta)(f_{u+1}+\ldots)S_B(n,r,\delta).$$

Proceeding by induction on u, we prove that Λ_u is quasi-hereditary by the criterion in Theorem 7.2. The induction start is given by $\Lambda_0\simeq S_A(n,r)$ being quasi-hereditary. In the induction step, we check conditions (i) - (iii) in Theorem 7.2 for $\Lambda=\Lambda_u$ and $f=f_u$. We obtain quasi-heredity of $\Lambda_u/\Lambda_u f_u\Lambda_u=\Lambda_{u-1}$ from the induction assumption. Moreover, $f_u\Lambda_u f_u\simeq S_A(n,r-2u)$ is quasi-hereditary. This gives condition (i). To conclude the induction step, conditions (ii) and (iii) will be verified in the subsequent three sections.

8. The $f_u\Lambda_u f_u$-module $f_u\Lambda_u$ is projective

Let u and l be natural numbers with $0\le l\le u\le[r/2]$, and fix a partition or composition λ of $r-2l$. The space $f_u\Lambda_u$ consists of residue classes of maps from $M(l,\lambda)$ (with varying λ and l) to $M(u,\mu)$ (with varying μ) - note that according to our conventions, multiplication is composition, and composition with f_u forces the image to be contained in a sum of modules $M(u,\mu)$ with varying μ. Residue classes are taken modulo layers below u, that is, a non-zero element has to have non-trivial image in the top layer of $M(u,\mu)$. Since the integral basis constructed is compatible with layers, we just have to restrict indices of basis elements appropriately to get a basis of $f_u\Lambda_u$. We will see in this section that the vector space spanned by all basis elements in $f_u\Lambda_u$ with a *fixed bottom arc configuration* is a projective $S_A(n,r-2u)$-module, and hence $f_u\Lambda_u$ is projective, too.

We start by defining what we mean by basis elements with a fixed bottom arc configuration. A Brauer diagram d in $B_{r-2l}(\delta)$ consists of a top and a bottom partial diagram (arc configuration) in V_{u-l}^{r-2l} and an element of a symmetric group Σ_{r-2u}. Denote the bottom arc configuration of d

by bot(d). Consider the basis element $\phi_{u,id,\sigma}^{(l,\lambda),(m,\mu)}$ where the second parameter has been chosen to be id. By the definition in (16),

$$\phi_{u,id,\sigma}^{(l,\lambda),(m,\mu)}(\Sigma_\lambda \cdot 1 \otimes e_l) = \sum_\alpha \Sigma_\mu \cdot 1 \otimes e_u \sigma\alpha$$

where the sum runs through certain elements $\alpha \in \Sigma_\lambda$. Hence all Brauer diagrams occurring on the right hand side have Σ_λ-equivalent (see Section 4.3) bottom arc configuration:

$$[\text{bot } (e_u\sigma\alpha)] = [\text{bot } (e_u\sigma)].$$

We therefore can associate to each basis element $\phi_{u,id,\sigma}^{(l,\lambda),(m,\mu)}$ the bottom arc configuration $[\text{bot } (e_u\sigma)]$. To do so, we define for a partial diagram $v \in V_{u-l}^{r-2l}$ the following subset of basis vectors:

$$B_{[v]} = B_{[v]}^{(l,\lambda)} = \{\phi_{u,id,\sigma}^{(l,\lambda),(u,\mu)} \mid \mu \models r-2u, \sigma \in \Sigma_\mu \times H_{u-l}\backslash\Sigma_{r-2l}/\Sigma_\lambda \text{ with } [\text{bot}(e_u\sigma)] = [v]\}.$$

This is the set of those basis elements of $S_B(n,r,\delta)$ which are maps from some permutation module $M(l,\lambda)$ into the uth layer of $M(u,\mu)$. (So, here $u = m$ and there is no choice of direct summand in the uth layer of $M(u,\mu)$, since the top layer is indecomposable.) Define $P_{[v]}$ as the vector space spanned by the set $B_{[v]}$. We show that the module $P_{[v]}$ is a projective $S_A(n,r-2u)$-module. Define the left $S_A(n,r-2u)$-module

$$_AP(\lambda) = \text{Span}(\{\xi_{i,j}^A \mid (i,j) \in \Omega, \lambda(j) = \lambda\}).$$

Note that $_AP(\lambda)$ equals the projective module $S_A(n,r-2u)\cdot\xi_{j,j}^A$ for any multi-index j with $\lambda(j) = \lambda$.

Theorem 8.1. *Let $P_{[v]}$ be the span of the basis elements $B_{[v]}$. Then $P_{[v]}$ is a projective $S_A(n,r-2u)$-module isomorphic to the projective $S_A(n,r-2u)$-module $_AP(\lambda_v)$ of index λ_v.*

We will need the remainder of this section to prove this result, and we start with showing that we indeed have an $S_A(n,r-2u)$-module.

Lemma 8.2. *The vector space $P_{[v]}$ is a left $S_A(n,r-2u)$-module, with action induced by multiplication in $S_B(n,r,\delta)$, that is, by composition of maps between permutation modules.*

Proof. By the proof of Proposition 6.1, a basis of the Schur algebra $S_A(n,r-2u)$ is given by the elements $\phi_{u,id,\tilde{\sigma}}^{(u,\tilde{\lambda})(u,\tilde{\mu})}$ with $\tilde{\sigma} \in \Sigma_{\tilde{\mu}}\backslash\Sigma_{r-2u}/\ \Sigma_{\tilde{\lambda}}$, where $\tilde{\lambda},\tilde{\mu}$ are compositions of $r-2u$. There is the following left action of the Schur algebra $S_A(n,r-2u)$ on the span of $B_{[v]}$: for $\phi_{u,id,\sigma}^{(l,\lambda),(u,\mu)} \in B_{[v]}$, let

$$\phi := \phi_{u,id,\tilde{\sigma}}^{(u,\tilde{\lambda})(u,\tilde{\mu})} \cdot \phi_{u,id,\sigma}^{(l,\lambda),(u,\mu)} := \phi_{u,id,\tilde{\sigma}}^{(u,\tilde{\lambda})(u,\tilde{\mu})} \circ \phi_{u,id,\sigma}^{(l,\lambda),(u,\mu)}. \tag{19}$$

The right hand side is just composition of maps or multiplication inside $S_B(n,r,\delta)$. We need to show that ϕ in (19) can be written as a linear combination of elements in $B_{[v]}$. If $\mu \neq \tilde{\lambda}$, then ϕ is zero, and we are done. We therefore assume that $\mu = \tilde{\lambda}$. Since ϕ is a homomorphism from $M(l,\lambda)$ into the uth layer of $M(u,\tilde{\mu})$, it can be written as a linear combination of basis elements $\phi_{u,\pi_i,\sigma_i}^{(l,\lambda),(u,\tilde{\mu})}$, say

$$\phi_{u,id,\tilde{\sigma}}^{(u,\tilde{\lambda})(u,\tilde{\mu})} \cdot \phi_{u,id,\sigma}^{(l,\lambda),(u,\mu)} = \sum_i \gamma_i \cdot \phi_{u,\pi_i,\sigma_i}^{(l,\lambda),(u,\tilde{\mu})} \tag{20}$$

for scalars γ_i. There is only one possible double coset corresponding to π_i, and hence we can choose $\pi_i = \text{id}$. Evaluate ϕ on the generator $\Sigma_\lambda \cdot 1 \otimes e_l$ of $M(l,\lambda)$. Then, by the definition in (16),

$$\phi(\Sigma_\lambda \cdot 1 \otimes e_l) = \sum_{\alpha,\beta} \Sigma_{\tilde{\mu}} \cdot 1 \otimes e_u\tilde{\sigma}\beta\sigma\alpha \tag{21}$$

where the sum runs through some elements $\alpha \in \Sigma_\lambda$, and some element $\beta \in \Sigma_\mu \subseteq \Sigma_{r-2u}$. Here $\tilde{\sigma} \in \Sigma_{r-2u}$. The elements β and $\tilde{\sigma}$ do not move the arcs from e_u, and so

$$[\mathrm{bot}(e_u\tilde{\sigma}\beta\sigma\alpha)] = [\mathrm{bot}(e_u\sigma\alpha)] = [\mathrm{bot}(e_u\sigma)],$$

which means that all diagrams on the right hand side in (21) have bottom arc configurations in the equivalence classes $[\mathrm{bot}(e_u\sigma)]$. Similarly, on the right-hand-side of (20), the elements have bottom arc configuration in the equivalence classes $[\mathrm{bot}(e_u\sigma_i)]$. By the above, it follows that

$$[\mathrm{bot}(e_u\sigma_i)] = [\mathrm{bot}(e_u\sigma)] = [v] \qquad \text{for all } i,$$

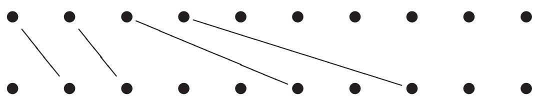
and hence $\phi^{(l,\lambda),(u,\tilde{\mu})}_{u,\pi_i,\sigma_i} \in B_{[v]}$. So (19) shows that the vector space spanned by the set $B_{[v]}$ is indeed a right $S_A(n, r-2u)$-module. □

In the next step we prove that the module $P_{[v]}$ is projective, identifying it with some projective $S_A(n, r-2u)$-module. Before we can do this, we need a multiplication formula for some basis elements in $S_B(n, r, \delta)$.

Notation 8.3. Fix natural numbers u and l with $u - l \geq 0$. Given a permutation $\rho \in \Sigma_{r-2l}$, define the ordered sets

$$\begin{aligned} \hat{M}_\rho &= \{1 \leq i \leq r-2l \mid i \text{ is a free vertex of } \mathrm{bot}(e_{u-l}\rho)\} =: \{x_1 < x_2 < \ldots < x_{r-2u}\}, \\ \tilde{M}_\rho &= \{1, \ldots, r-2l\} \backslash \hat{M}_\rho =: \{y_1 < y_2 < \ldots < y_{2u-2l}\}. \end{aligned}$$

We define an element $\hat{\rho} \in \Sigma_{r-2u}$ by $\hat{\rho}(i) = \rho(x_i)$ for $1 \leq i \leq r-2u$, and an element $\tilde{\rho} \in \Sigma_{r-2l}$ by $\tilde{\rho}(y_j) = \rho(y_j)$ and $\tilde{\rho}(x_i) = i$. Thinking in terms of Brauer diagrams, note that $\hat{\rho}$ is given on the free vertices of $\mathrm{bot}(e_{u-l}\rho)$ by non-intersecting edges, that is, by a slanted version of the identity diagram, such as


.

As usual, at times, we identify the permutation $\hat{\rho} \in \Sigma_{r-2u}$ with the element $\hat{\rho} \in \Sigma_{r-2l}$ by setting $\hat{\rho}(j) = j$ for all $r - 2u < j \leq r - 2l$. With this notation ρ factors as $\rho = \hat{\rho}\tilde{\rho}$. Moreover, taking $v = \mathrm{bot}(e_{u-l}\rho)$, then $\Sigma_{\lambda_v} = \Sigma_{r-2u} \cap \tilde{\rho}\Sigma_\lambda\tilde{\rho}^{-1}$ by construction of λ_v and $\tilde{\rho}$.

Proposition 8.4. *Let u, l be natural numbers with $u - l \geq 0$. Fix compositions λ of $r - 2l$ and ν of $r - 2u$.*

(1) *Let $\tau \in \Sigma_{r-2u}$ and $\sigma \in \Sigma_{r-2l}$. Define μ by setting $\Sigma_\mu = \Sigma_{r-2u} \cap \sigma\Sigma_\lambda\sigma^{-1}$. Then*

$$\phi^{(u,\mu),(u,\nu)}_{u,id,\tau} \circ \phi^{(l,\lambda),(u,\mu)}_{u,id,\sigma} = \phi^{(l,\lambda),(u,\nu)}_{u,id,\tau\sigma}.$$

(2) *Let $\rho \in \Sigma_{r-2l}$ and define μ by setting $\Sigma_\mu = \Sigma_{r-2u} \cap \tilde{\rho}\Sigma_\lambda\tilde{\rho}^{-1}$. Then*

$$\phi^{(u,\mu),(u,\nu)}_{u,id,\hat{\rho}} \circ \phi^{(l,\lambda),(u,\mu)}_{u,id,\tilde{\rho}} = \phi^{(l,\lambda),(u,\nu)}_{u,id,\rho}.$$

Proof. The second assertion is a special case of the first one, which we are going to prove.

Step 1. To prove the claim we first compute the effect of the left and of the right hand side on the generating element $\Sigma_\lambda id \otimes e_l$. In the second step we identify the indices coming up on both sides.

On the right hand side we have $\phi^{(l,\lambda),(u,\nu)}_{u,id,\tau\sigma}(\Sigma_\lambda id \otimes e_l) = \sum_\gamma (\Sigma_\nu id \otimes e_u)\tau\sigma\gamma$ where we sum over all elements

$$\gamma \in (\Sigma_\lambda \cap (\tau\sigma)^{-1}(\Sigma_\nu \times H_{u-l})\tau\sigma)\backslash\Sigma_\lambda.$$

On the left hand side we have

$$\begin{aligned}\phi_{u,id,\tau}^{(u,\mu),(u,\nu)} \circ \phi_{u,id,\sigma}^{(l,\lambda),(u,\mu)}(\Sigma_\lambda id \otimes e_l) &= \sum_{\alpha,\beta'} (\Sigma_\nu id \otimes e_u)\tau\beta'\sigma\alpha \\ &= \sum_{\alpha,\beta'} (\Sigma_\nu id \otimes e_u)\tau\sigma(\sigma^{-1}\beta'\sigma)\alpha \\ &= \sum_{\alpha,\beta} (\Sigma_\nu id \otimes e_u)\tau\sigma\beta\alpha\end{aligned}$$

where we sum over all elements

$$\begin{aligned}\alpha &\in \Sigma_\lambda \cap \sigma^{-1}(\Sigma_\mu \times H_{u-l})\sigma \backslash \Sigma_\lambda \\ \beta &\in \sigma^{-1}(\Sigma_\mu \cap \tau^{-1}\Sigma_\nu\tau)\sigma\backslash\sigma^{-1}\Sigma_\mu\sigma.\end{aligned}$$

We are left with showing that all the products $\beta\alpha$ exactly represent the same cosets as all the elements γ. So, we have to rewrite the various index sets occurring above. This will be done in the second step.

Step 2. We will apply the following general observation: Given finite groups $H \supseteq V \supseteq U$ with

$$V\backslash H = \{\alpha_i \mid i\} \quad \text{and} \quad U\backslash V = \{\beta_j \mid j\}.$$

It follows that $U\backslash H = \{\alpha_i\beta_j \mid i,j\} = \{\gamma_t \mid t\}$. We will rewrite the above index sets so that the coset representatives α_i correspond to α's above, the β_j correspond to β's above, and the γ_t correspond to γ's above. We take $H = \Sigma_\lambda$.

(a) First we note that, by definition, $\Sigma_\mu = \Sigma_{r-2u} \cap \sigma\Sigma_\lambda\sigma^{-1}$ and hence $\sigma^{-1}\Sigma_\mu\sigma \subset \Sigma_\lambda$.

(b) If Y is a Young subgroup of some symmetric group and U and V are subgroups of the same symmetric group, but with disjoint support, then there is an equality $Y \cap (U \times V) = (Y \cap U) \times (Y \cap V)$. Indeed, if an element $(u,v) \in U \times V$ is in Y, then each of the factors u and v must fix the support of Y, since the other element cannot correct any move out of the prescribed support.

(c) Rewriting the index-set of β, we obtain:

$$\begin{aligned}&\sigma^{-1}(\Sigma_\mu \cap \tau^{-1}\Sigma_\nu\tau)\sigma\backslash\sigma^{-1}\Sigma_\mu\sigma \\ =\ &\sigma^{-1}(\Sigma_\mu \cap \tau^{-1}\Sigma_\nu\tau)\sigma \times [\sigma^{-1}H_{u-l}\sigma \cap \Sigma_\lambda]\backslash\sigma^{-1}\Sigma_\mu\sigma \times [\sigma^{-1}H_{u-l}\sigma \cap \Sigma_\lambda] \\ \overset{(a)}{=}\ &\sigma^{-1}(\Sigma_\mu \cap \tau^{-1}\Sigma_\nu\tau)\sigma \times [\sigma^{-1}\tau^{-1}H_{u-l}\tau\sigma \cap \Sigma_\lambda]\backslash[\sigma^{-1}\Sigma_\mu\sigma \cap \Sigma_\lambda] \times [\sigma^{-1}H_{u-l}\sigma \cap \Sigma_\lambda] \\ \overset{(b)}{=}\ &\sigma^{-1}(\Sigma_\mu \cap \tau^{-1}\Sigma_\nu\tau)\sigma \times [\sigma^{-1}\tau^{-1}H_{u-l}\tau\sigma \cap \Sigma_\lambda]\backslash[\sigma^{-1}(\Sigma_\mu \times H_{u-l})\sigma \cap \Sigma_\lambda =: U\backslash V.\end{aligned}$$

In this calculation we used that $\tau^{-1}H_{u-l}\tau = H_{u-l}$ since $\tau \in \Sigma_{r-2u}$ which in turn has trivial intersection with H_{u-l}.

(d) The inclusion $\tau^{-1}\Sigma_\nu\tau \subset \Sigma_{r-2u}$ implies $\sigma(\Sigma_\lambda \cap \sigma^{-1}\tau^{-1}\Sigma_\nu\tau\sigma)\sigma^{-1} \subset \Sigma_{r-2u}$ and hence $\Sigma_\lambda \cap \sigma^{-1}\tau^{-1}\Sigma_\nu\tau\sigma \subset \sigma^{-1}\Sigma_{r-2u}\sigma$.

(e) There is an equality

$$\sigma^{-1}\Sigma_\mu\sigma \cap \sigma^{-1}\tau^{-1}\Sigma_\nu\tau\sigma = \Sigma_\lambda \cap \sigma^{-1}\tau^{-1}\Sigma_\nu\tau\sigma.$$

Indeed, by definition of Σ_μ, we have $\sigma^{-1}\Sigma_\mu\sigma = \sigma^{-1}\Sigma_{r-2u}\sigma \cap \Sigma_\lambda$. This implies

$$\sigma^{-1}\Sigma_\mu\sigma \cap \sigma^{-1}\tau^{-1}\Sigma_\nu\tau\sigma = \sigma^{-1}\Sigma_{r-2u}\sigma \cap \Sigma_\lambda \cap \sigma^{-1}\tau^{-1}\Sigma_\nu\tau\sigma.$$

By the second inclusion in (d), the first of the three sets in the last intersection contains the intersection of the other two sets and thus can be omitted, which yields the equality claimed.

(f) Using first (e) and then (b) we can reformulate the denominator U coming up in (c) as follows:

$$\begin{aligned} U &= \sigma^{-1}(\Sigma_\mu \cap \tau^{-1}\Sigma_\nu\tau)\sigma \times [\sigma^{-1}\tau^{-1}H_{u-l}\tau\sigma \cap \Sigma_\lambda] \\ &= [\Sigma_\lambda \cap \sigma^{-1}\tau^{-1}\Sigma_\nu\tau\sigma] \times [\sigma^{-1}\tau^{-1}H_{u-l}\tau\sigma \cap \Sigma_\lambda] \\ &= \Sigma_\lambda \cap \sigma^{-1}\tau^{-1}[\Sigma_\nu \times H_{u-l}]\tau\sigma. \end{aligned}$$

Using the general observation explained above, the index set for γ now has been rewritten in such a way that it is clear how to partition the coset represented by γ by using α and β. □

We are now ready to complete the proof of Theorem 8.1, which is the following assertion:

Theorem. *Let $P_{[v]}$ be the span of the basis elements $B_{[v]}$. Then $P_{[v]}$ is a projective $S_A(n, r-2u)$-module isomorphic to the projective $S_A(n, r-2u)$-module ${}_AP(\lambda_v)$ of index λ_v.*

Proof. (i) We will make repeated use of the notation introduced in 8.3. Recall that for the given double coset representative $\sigma \in \Sigma_\mu \times H_{u-l}\backslash\Sigma_{r-2l}/\Sigma_\lambda$, we have the corresponding partial diagram $v = \mathrm{bot}(e_{u-l}\sigma) = \mathrm{bot}(e_{u-l}\tilde{\sigma})$. Our aim is to show that the following map

$$\psi : P_{[v]} \to {}_AP(\lambda_v), \qquad \phi^{(l,\lambda),(u,\mu)}_{u,id,\sigma} \mapsto \xi(\hat{\sigma})$$

is an isomorphism of $S_A(n, r-2u)$-modules. Here $\xi(\hat{\sigma})$ is defined as in Remark 6.2.

The map ψ is linear by definition, and it sends basis elements to basis elements. Given a bottom arc configuration v, by the construction of $\hat{\sigma}$ given above, every double coset $\Sigma_\mu\tau\Sigma_{\lambda_v}$ determines a unique double coset $(\Sigma_\mu \times H_{u-l})\sigma\Sigma_\lambda$ with $\hat{\sigma} = \tau$. Therefore, ψ is an isomorphism of vector spaces. We have to prove it is a homomorphism of left modules over the Schur algebra $S_A(n, r-2u)$.

(ii) Given double coset representative $\sigma \in \Sigma_\mu \times H_{u-l}\backslash\Sigma_{r-2l}/\Sigma_\lambda$, then $\Sigma_{\lambda_v} = \Sigma_{r-2u} \cap \tilde{\sigma}\Sigma_\lambda\tilde{\sigma}^{-1}$ and by Proposition 8.4(2) we obtain the factorisation

$$\phi^{(l,\lambda),(u,\mu)}_{u,id,\sigma} = \phi^{(u,\lambda_v),(u,\mu)}_{u,id,\hat{\sigma}} \circ \phi^{(l,\lambda),(u,\lambda_v)}_{u,id,\tilde{\sigma}}. \tag{22}$$

We translate this formula to Equation (23) below. By construction, the basis element $\phi^{(l,\lambda),(u,\mu)}_{u,id,\sigma}$ corresponds to $\hat{\sigma} \in \Sigma_\mu\backslash\Sigma_{r-2u}/\Sigma_{\lambda_v}$, which in turn by Remark 6.2 corresponds to a pair of multi-indices (i, j) with

$$\begin{aligned} i &:= \mathrm{ind}(\mu) \cdot \hat{\sigma}, \\ j &:= \mathrm{ind}(\lambda_v). \end{aligned}$$

We hence make the following definition:

$$\xi^B_{i,j} := \phi^{(l,\lambda),(u,\mu)}_{u,id,\sigma}.$$

Using the notation introduced in 8.3, the free points of σ and $\tilde{\sigma}$ coincide, $\hat{M}_{\tilde{\sigma}} = \hat{M}_\sigma$. (Similarly, $\tilde{M}_{\tilde{\sigma}} = \tilde{M}_\sigma$.) It follows that

$$\hat{\tilde{\sigma}}(j) = \tilde{\sigma}(x_j) = j$$

for all $1 \le j \le r-2u$. Hence $\hat{\tilde{\sigma}}$ is the identity on Σ_{r-2u}. Hence, under the defined correspondence in (22),

$$\phi^{(l,\lambda),(u,\lambda_v)}_{u,id,\tilde{\sigma}} = \xi^B_{jj}.$$

Finally, with the multi-indices introduced, $\phi^{(u,\lambda_v),(u,\mu)}_{u,id,\hat{\sigma}} = \xi^A_{ij}$, using Remark 6.2. We hence translated Equation (22) to the formula:

$$\xi^B_{i,j} = \xi^A_{i,j} \cdot_B \xi^B_{j,j}. \tag{23}$$

Here and below we use the subscripts B and A at multiplication to distinguish multiplication in $S_B(n, r, \delta)$ from that in $S_A(n, r-2u)$.

(iii) We are now ready to check that ψ is an $S_A(n, r-2u)$-module homomorphism:

$$\begin{aligned}
\psi(\xi^A_{h,i}\cdot_B \xi^B_{i,j}) &= \psi(\xi^A_{h,i}\cdot_B \xi^A_{i,j}\cdot_B \xi^B_{j,j}) && \text{by Equation (23),}\\
&= \psi((\xi^A_{h,i}\cdot_A \xi^A_{i,j})\cdot_B \xi^B_{j,j}) && \text{by associativity and } S_A \le S_B(n,r,\delta) \text{ subalgebra}\\
&= \psi(\textstyle\sum_{(p,q)} Z\,\xi^A_{p,q}\cdot_B \xi^B_{j,j}) && \text{by the multiplication formula in } S_A\\
&= \textstyle\sum_{(p,j)} Z\psi(\xi^A_{p,j}\cdot_B \xi^B_{j,j}) && \text{by the multiplication formula in } S_A \text{ and } \psi \text{ linear}\\
&= \textstyle\sum_{(p,j)} Z\psi(\xi^B_{p,j}) && \text{by Equation (23),}\\
&= \textstyle\sum_{(p,j)} Z\xi^A_{p,j} && \text{by the definition of } \psi,\\
&= \xi^A_{h,i}\cdot_A \xi^A_{i,j} && \text{by the multiplication formula in } S^A,\\
&= \xi^A_{h,i}\cdot_A \psi(\xi^B_{i,j}) && \text{by the definition of } \psi.
\end{aligned}$$

The coefficients $Z = Z(h,i,i,j,p,q)$ are as defined in (17). This proves that ψ is a module homomorphism over $S_A(n, r-2u)$. □

As a consequence of the proof of Theorem 8.1 we have:

Corollary 8.5. *The left $f_u\Lambda_u f_u$-module $f_u\Lambda_u$ is projective. More precisely, it decomposes into a direct sum:*

$$f_u\Lambda_u = \bigoplus_{(l\le u,\lambda)} \bigoplus_{[v]\in V^{r-2l}_{u-l}/\sim_{\Sigma_\lambda}} P(\lambda_{[v]}).$$

Since projective modules are automatically Δ-filtered, we have verified the first half of condition (ii) in Theorem 7.2.

9. The $f_u\Lambda_u f_u$-module $\Lambda_u f_u$ is projective

To verify the second half of condition (ii) in Theorem 7.2 we again prove more strongly that the module in question is projective. This is a consequence of Proposition 5.1. Let $l = u \ge m$ be fixed natural numbers. In particular H_{u-l} is the trivial group. Fix a composition μ of $r-2m$. Choose a double coset $\pi \in \Sigma_\mu\backslash\Sigma_{r-2m}/(\Sigma_{r-2u}\times H_{u-m})$, and define the set

$$B^{(m,\mu)}_\pi = \{\phi^{(u,\lambda)(m,\mu)}_{u,\pi,\sigma} \mid \lambda \models r-2u, \sigma\in\Sigma_\nu\backslash\Sigma_{r-2u}/\Sigma_\lambda\}.$$

By the proof of Proposition 6.1, the vector space spanned by the set $B^{(m,\mu)}_\pi$ is a right $S_A(n, r-2u)$-module. Denote the projective right $S_A(n,r-2u)$-module $\xi^A_\lambda\cdot S_A(n,r-2u)$ of index λ by $P_A(\lambda)$.

Proposition 9.1. *The right $f_u\Lambda_u f_u$-module $\Lambda_u f_u$ is projective. More precisely, it decomposes as a direct sum:*

$$\Lambda_u f_u = \bigoplus_{(m\le u,\mu)}\bigoplus_{\pi} \mathrm{Span}(B^{(m,\mu)}_\pi) = \bigoplus_{(m\le u,\mu)}\bigoplus_{\pi} P_A(\varphi(\pi)) = \bigoplus_{(m\le u,\mu)}\bigoplus_{[w]\in V^{r-2m}_{u-m}/\sim_{\Sigma_\mu}} P_A(\mu_{[w]}).$$

Here, the map φ is as defined in Lemma 4.5 and described in Section 4.3. Note that at this point we have checked condition (ii) in Theorem 7.2.

Proof. The Schur algebra $f_u\Lambda_u f_u$ can be identified with the set of Σ_{r-2u}-endomorphisms of $\oplus_\lambda M^\lambda$. By Section 5, the basis elements $\phi^{(u,\lambda)(m,\mu)}_{u,\pi,\sigma}$ correspond to basis elements in

$$\mathrm{Hom}_{\Sigma_{r-2u}}(M^\lambda, G_u(M^{\varphi(\pi)})e_u).$$

Hence, the maps in $B^{(m,\mu)}_\pi$ start in $\oplus_\lambda M^\lambda$ and end in $G_u(M^{\varphi(\pi)})e_u$. By Proposition 5.1, $G_u(M^{\varphi(\pi)})e_u$ as a Σ_{r-2u}-module is isomorphic to the permutation module $M^{\varphi(\pi)}$, since $u = l$.

Hence

$$\mathrm{Span}B_{\pi}^{(m,\mu)} \simeq \bigoplus_{\lambda \models r-2u} \mathrm{Hom}_{\Sigma_{r-2u}}(M^{\lambda}, M^{\varphi(\pi)}) = \xi_{\varphi(\pi)} S_A(n, r-2u) = P_A(\varphi(\pi)).$$

The right action of $f_u \Lambda_u f_u$ is by pre-composition. This is compatible with rewriting $G_u(M^{\varphi(\pi)})e_u$ as $M^{\varphi(\pi)}$ and with the claimed decomposition of $\Lambda_u f_u$. In particular, $Q_{\pi}^{(m,\mu)}$ is projective. By Lemma 4.7,

$$\Lambda_u f_u = \bigoplus_{(m \leq u, \mu)} \bigoplus_{[w] \in V_{u-m}^{r-2m}/\sim_{\Sigma_\mu}} P_A(\mu_{[w]}).$$

□

10. Multiplication provides a bijection

The last condition in Theorem 7.2 to be checked is (iii). The natural number u is fixed throughout this section, so we write $\Lambda = \Lambda_u$ and $f = f_u$. Recall that $f\Lambda f$ is the classical Schur algebra $S_A(n, r-2u)$. The left module decomposition in Corollary 8.5 states:

$$f\Lambda = \bigoplus_{(l \leq u, \lambda)} \bigoplus_{[v] \in V_{u-l}^{r-2l}/\sim_{\Sigma_\lambda}} {}_AP(\lambda_{[v]}).$$

We also have seen how to rephrase the right module decomposition using exactly the same indices:

$$\Lambda f = \bigoplus_{(m \leq u, \mu)} \bigoplus_{[w] \in V_{u-m}^{r-2m}/\sim_{\Sigma_\mu}} P_A(\mu_{[w]}),$$

where $P_A(\mu_{[w]})$ is a projective right module. Thus we can reformulate the tensor product as

$$\begin{aligned} \Lambda f \otimes_{f\Lambda f} f\Lambda &\simeq \bigoplus_{(l \leq u, \lambda)} \bigoplus_{[v] \in V_{u-l}^{r-2l}/\sim_{\Sigma_\lambda}} \bigoplus_{(m \leq u, \mu)} \bigoplus_{[w] \in V_{u-m}^{r-2m}/\sim_{\Sigma_\mu}} P_A(\mu_{[w]}) \otimes_{f\Lambda f} {}_AP(\lambda_{[v]}) \\ &= \bigoplus_{(l \leq u, \lambda)} \bigoplus_{[v] \in V_{u-l}^{r-2l}/\sim_{\Sigma_\lambda}} \bigoplus_{(m \leq u, \mu)} \bigoplus_{[w] \in V_{u-m}^{r-2m}/\sim_{\Sigma_\mu}} \xi_{\mu_{[w]}} S_A(n, r-2u) \xi_{\lambda_{[v]}} \\ &= \bigoplus_{(l \leq u, \lambda)} \bigoplus_{[v] \in V_{u-l}^{r-2l}/\sim_{\Sigma_\lambda}} \bigoplus_{(m \leq u, \mu)} \bigoplus_{[w] \in V_{u-m}^{r-2m}/\sim_{\Sigma_\mu}} \mathrm{Hom}_{\Sigma_{r-2u}}(M^{\lambda_{[v]}}, M^{\mu_{[w]}}) \end{aligned}$$

So the K-dimension is

$$\dim \Lambda f \otimes_{f\Lambda f} f\Lambda = \sum_{(l \leq u, \lambda)} \sum_{[v] \in V_{u-l}^{r-2l}/\sim_{\Sigma_\lambda}} \sum_{(m \leq u, \mu)} \sum_{[w] \in V_{u-m}^{r-2m}/\sim_{\Sigma_\mu}} \mid \Sigma_{\mu_{[w]}} \backslash \Sigma_{r-2u} / \Sigma_{\lambda_{[v]}} \mid . \tag{24}$$

Let μ be a composition of $r-2m$ and λ a composition of $r-2l$. Fix $u \geq m, l$ and an element $\pi \in \Sigma_{r-2m}$ representing a double coset in $\Sigma_\mu \backslash \Sigma_{r-2m} / (\Sigma_{r-2u} \times H_{u-m})$. As before, set $\Sigma_\nu = \Sigma_{r-2u} \cap \pi^{-1} \Sigma_\mu \pi$.

Proposition 10.1. *There is a bijection ψ given by sending $\sigma \in (\Sigma_\nu \times H_{u-l}) \backslash \Sigma_{r-2l} / \Sigma_\lambda$ to the pair $(\overline{\sigma}, \xi(\hat{\sigma}))$. Here $\overline{\sigma}$ is the image of σ in $(\Sigma_{r-2u} \times H_{u-l}) \backslash \Sigma_{r-2l} / \Sigma_\lambda$. Define the composition ρ by $\Sigma_\rho = \Sigma_{r-2u} \cap \sigma \Sigma_\lambda \sigma^{-1}$. Then $\xi(\hat{\sigma})$ is the element in the classical Schur algebra $S_A(n, r-2u)$ corresponding to the double coset in $\Sigma_\rho \backslash \Sigma_{r-2u} / \Sigma_\nu$ represented by $\hat{\sigma}$ (defined in 8.3).*

In other words, the map $(\Sigma_\nu \times H_{u-l}) \backslash \Sigma_{r-2l} / \Sigma_\lambda \longrightarrow (\Sigma_{r-2u} \times H_{u-l}) \backslash \Sigma_{r-2l} / \Sigma_\lambda$ given by sending σ to $\overline{\sigma}$ is surjective, and the fibre of $\overline{\sigma}$ is in bijection to the set $\Sigma_\rho \backslash \Sigma_{r-2u} / \Sigma_\nu$ with ρ defined as above. Note that ρ depends on $\overline{\sigma}$ but only formally on σ.

Proof. The map ψ is well-defined since for subgroups $U \leq V$ the projection $U\backslash G/H \to V\backslash G/H$, mapping UgH to VgH is well-defined. In the following, depending from the context, σ denotes either a permutation in Σ_{r-2l} or the corresponding double coset in $(\Sigma_\nu \times H_{u-l})\backslash\Sigma_{r-2l}/\Sigma_\lambda$.

The map ψ is injective: Suppose $\psi(\sigma) = \psi(\tau)$ for some representatives $\sigma, \tau \in \Sigma_{r-2l}$. We have to show that σ and τ represent the same double coset. As in Lemma 4.7, the double cosets in $(\Sigma_{r-2u} \times H_{u-l})\backslash\Sigma_{r-2l}/\Sigma_\lambda$ correspond to arc configurations in V_{u-l}^{r-2l} modulo Σ_λ. The first entry of $\psi(\sigma) = \psi(\tau)$ tells us that σ and τ correspond to equivalent arc configurations. Hence

$$\tilde{\sigma} = \tilde{\tau} \cdot \gamma \text{ for } \gamma \in \Sigma_\lambda.$$

Then equality of the second entries implies that the Schur algebra elements $\xi(\hat{\sigma})$ and $\xi(\hat{\tau})$ are equal, which means $\Sigma_\nu\hat{\sigma}\Sigma_\rho = \Sigma_\nu\hat{\tau}\Sigma_\rho$. Hence there is an element $\alpha \in \Sigma_\nu$ and an element $\beta \in \Sigma_\rho$ with $\alpha\hat{\sigma}\beta = \hat{\tau}$. This means that $\alpha\sigma\beta$ and τ coincide on the free vertices. It follows that

$$\alpha\sigma\beta = \alpha\hat{\sigma}\tilde{\sigma}\beta = \alpha\hat{\sigma}\beta\tilde{\sigma} = \hat{\tau}\tilde{\tau}\gamma = \tau\gamma.$$

And hence the cosets of σ and τ coincide.

The map ψ is surjective: Given a pair (θ, ω), the first entry θ determines an arc configuration modulo Σ_λ. Choose a double coset representative $\sigma \in (\Sigma_\nu \times H_{u-l})\backslash\Sigma_{r-2l}/\Sigma_\lambda$ with corresponding arc configurations $\bar{\sigma} = \theta$. Changing the free part $\hat{\sigma}$ of the element σ does not affect the first entry of $\psi(\sigma)$ and thus we can rearrange σ also to yield the correct second entry in $\psi(\sigma)$: Suppose the Schur algebra element ω correspond to the double coset $\Sigma_\rho\alpha\Sigma_\nu$. Let $\beta = (\alpha \otimes id)\tilde{\sigma}$ where we horizontally concatenate the diagram of $\alpha \in \Sigma_{r-2u}$ and the identity diagram $id \in \Sigma_{2u-2l}$ to form a diagram $(\alpha \otimes id) \in \Sigma_{r-2l}$. Then

$$\psi(\beta) = (\bar{\beta}, \xi(\hat{\beta})) = (\theta, \xi(\hat{\alpha})) = (\theta, \omega).$$

□

Corollary 10.2. *With the notation as above, multiplication $\Lambda f \otimes_{f\Lambda f} f\Lambda \to \Lambda f\Lambda$ is bijective.*

Proof. Multiplication $\Lambda f \otimes_{f\Lambda f} f\Lambda \to \Lambda f\Lambda$ is, of course, surjective. We have to show it is injective; it is enough to show that the two sides have the same K-dimension. By Theorem 5.3,

$$\dim\Lambda f\Lambda = \sum_{(l\leq u,\lambda)} \sum_{(m\leq u,\mu)} \sum_{\pi\in\Sigma_\mu\backslash\Sigma_{r-2m}/(\Sigma_{r-2u}\times H_{u-m})} \mid \Sigma_{\varphi(\pi)} \times H_{u-l}\backslash\Sigma_{r-2l}/\Sigma_\lambda \mid .$$

We compare this dimension formula with that in (24). By Lemma 4.7 there is a bijection between double cosets $\pi \in \Sigma_\mu\backslash\Sigma_{r-2m}/(\Sigma_{r-2u} \times H_{u-m})$ and arc configurations $[w] \in V_{u-m}^{r-2m}/\sim_{\Sigma_\mu}$. Choose $\nu = \mu_{[w]}$ and $\rho = \lambda_{[v]}$ in Proposition 10.1 and note that $\nu = \varphi(\pi)$ by Corollary 4.8. The claim follows using that by Lemma 4.7 there is a bijection between arc configurations $[v] \in V_{u-l}^{r-2l}/\sim_{\Sigma_\lambda}$ and double cosets in $[(\Sigma_{r-2u} \times H_{u-l})\backslash\Sigma_{r-2l}/\Sigma_\lambda]$. □

11. Schur-Weyl duality, Rouquier covers and more about the (potential) relevance of $S_B(n, r, \delta)$

In this final section, we provide and discuss several reasons to be interested in the algebras $S_B(n, r, \delta)$: their connection to Brauer algebras, their universal property and a potential application to invariant theory.

The Schur algebra $S_B(n, r, \delta)$ is related to the Brauer algebra $B_r(\delta)$ by definition. To discuss the strength of this connection, we use a concept introduced by Rouquier [19]:

Notation 11.1. Let Λ be a quasi-hereditary algebra and Λe a projective module. Then Λ is a *quasi-hereditary cover* (or highest weight cover) of the algebra $e\Lambda e$ if there is a double centraliser property on the bimodule Λe, that is,

$$\Lambda = \mathrm{End}_{e\Lambda e}(\Lambda e), \qquad e\Lambda e = \mathrm{End}_{\Lambda}(\Lambda e),$$

or equivalently, if the canonical map $\Lambda \to \mathrm{End}_{e\Lambda e}(\Lambda e)$ is an isomorphism. The algebra Λ is called an *i-faithful quasi-hereditary cover* of $e\Lambda e$ if in addition the functor $e \cdot -$ induces isomorphisms

$$\mathrm{Ext}^j_{\Lambda}(M, N) \simeq \mathrm{Ext}^j_{e\Lambda e}(eM, eN) \tag{25}$$

in all degrees $j \leq i$ and for all Δ-filtered Λ-modules M and N.

Example 11.2. Let $n \geq r$. It has been determined for which i, the classical Schur algebra is an i-faithful cover of the group algebra of the symmetric group: By Schur-Weyl duality between $\Lambda = S_A(n, r)$ and $e\Lambda e = K\Sigma_r$ (for $n \geq r$) on the bimodule $\Lambda e = (K^n)^{\otimes r}$, the algebra $S_A(n, r)$ always is a quasi-hereditary cover of $K\Sigma_r$. If $S_A(n, r)$ is semisimple, then the Schur functor $e \cdot -$ is an equivalence of categories; hence $S_A(n, r)$ is an i-faithful cover of $K\Sigma_r$ for any $i \in \mathbb{N}$. By a result of Hemmer and Nakano [15], $S_A(n, r)$ is a 1-faithful cover when K is a field of characteristic different from two and three. If $S_A(n, r)$ is not semisimple, then by Theorems 3.9, 4.3 and 5.1 in [10], $S_A(n, r)$ is an i-faithful cover for $i \leq \mathrm{char}(K) - 3$, and this bound is best possible.

Remark 11.3. (1) Let B be any finite dimensional algebra. Then, by a result of Dlab and Ringel [7], there exists a quasi-hereditary algebra Λ with an idempotent e such that $B \simeq e\Lambda e$. The algebra Λ is not unique, even if one imposes the additional condition of a Schur-Weyl duality between Λ and B on Λe. It is an open problem whether quasi-hereditary covers, or under suitable assumptions even 1-faithful quasi-hereditary covers, exist for interesting classes of algebras such as cellular algebras. By Rouquier [19, Prop 4.45 and Cor 4.46], 1-faithful covers are unique, up to Morita equivalence.
(2) The existence of 1-faithful quasi-hereditary covers is not, in general, guaranteed: The group algebra $B = \mathbb{F}_2\Sigma_2$ has no 1-faithful cover. Indeed, if A is such a cover, it must have a simple standard module Δ, like every quasi-hereditary algebra. The functor $e \cdot -$ sends a simple A-module to a simple B-module or to zero. By condition (25) for $j = 0$, the module Δ cannot be sent to zero. Hence its image is the unique simple B-module S satisfying $\mathrm{Ext}^1_B(S, S) \neq 0$. This is a contradiction to (25): Standard modules for quasi-hereditary algebras do not have non-trivial self-extensions.

Theorem 11.4. *Let K be a commutative ring. Suppose $n \geq r$, and $\delta \neq 0$.*

(a) There is a Schur-Weyl duality between the Schur algebra $S_B(n, r, \delta)$ and the Brauer algebra $B_r(\delta)$ on the bimodule $\oplus_{(l,\lambda)} M(l, \lambda)$.

(b) The Schur algebra $S_B(n, r, \delta)$ is a quasi-hereditary cover of the Brauer algebra $B_r(\delta)$.

(c) Suppose K is a field of characteristic different from two or three. Then the Schur algebra $S_B(n, r, \delta)$ is a 1-faithful quasi-hereditary cover of the Brauer algebra $B_r(\delta)$.

Proof. (a) The algebra $B_r(\delta)$ is isomorphic to the permutation module $M(0, 1^r)$, and hence it is a direct summand of the module $\oplus M(l, \lambda)$ whose endomorphism ring by definition equals $S_B(n, r, \delta)$. In such a situation, a general result ([4, 59.4]) provides the double centraliser property as stated.

(b) By Theorem 7.1, the Schur algebra $S_B(n, r, \delta)$ is quasi-hereditary. The double centraliser property in (a) uses the bimodule

$$\bigoplus_{(l,\lambda)} M(l, \lambda) \simeq \mathrm{Hom}_{B_r(\delta)}(B_r(\delta), \bigoplus_{(l,\lambda)} M(l, \lambda)) = \mathrm{Hom}_{B_r(\delta)}(M(0, 1^r), \bigoplus_{(l,\lambda)} M(l, \lambda)),$$

which is the projective module $S_B(n, r, \delta) \cdot e$ for the projection $e : \oplus_{(l,\lambda)} M(l, \lambda) \to M(0, 1^r)$.

(c) This follows from [13], Theorem 13.1 and Section 11; the algebra called $S(A)$ there equals $S_B(n, r, \delta)$. □

Remark 11.5. (1) Assertion (c) can be understood as a universal property, in the same way as the classical Schur algebra $S_A(n, r)$ is universal, under the same assumptions.
(2) It is an open problem to determine, for which integers i the algebra $S_B(n, r, \delta)$ is an i-faithful cover of $B_r(\delta)$. This problem can be viewed as asking for extensions of results in [14] and [13], which in turn are motivated by the theorem of Hemmer and Nakano mentioned above. Since the assumptions on the characteristic of K are the same in all of these results, one may expect a similar result as in [10].
(3) The above proof fails if $n < r$. Indeed the Brauer algebra does not always act faithfully on the direct sum of the permutation modules $M(l, \lambda)$. The best to expect in this situation would be a Schur-Weyl duality relating $S_B(n, r, \delta)$ and an appropriate quotient of $B_r(\delta)$.

Universality, or direct checking, implies:

Corollary 11.6. *For $n \geq r$, the Schur algebras $S_B(n, r, \delta)$ and $S_B(r, r, \delta)$ are Morita equivalent.*

Remark 11.7. Schur-Weyl duality (on tensor space) between Brauer algebras on the one hand and orthogonal or symplectic groups (or Schur algebras) on the other hand has been a longstanding open problem. Brauer has solved this problem in characteristic zero, but the general case was only solved more than seventy years later in two articles by Dipper, Doty and Hu [5, 9]. It is not known whether symplectic and orthogonal Schur algebras are 1-faithful covers of Brauer algebras. A different approach to prove Schur-Weyl duality may attempt to use the algebras $S_B(n, r, \delta)$, provided the orthogonal or symplectic Schur algebras can be identified with members of the family $S_B(n, r, \delta)$ for suitable choices of the parameters n, r and δ. However, by a result in [16], the tensor space as a module over the Brauer algebra is not always a direct sum of permutation modules $M(l, \lambda)$.

Remark 11.8. We have defined $S_B(n, r, \delta)$ as endomorphism algebra of the permutation modules of the Brauer algebra $B_r(\delta)$, motivated by the results in [13]. A second motivation to study $S_B(n, r, \delta)$ is that it also comes up in a rather different context. This is of a much more classical flavour, but apparently has not yet been investigated. Brauer's motivation to define Brauer algebras came from invariant theory of orthogonal and symplectic groups; he needed a tool to decompose tensor space as a module over these classical groups (in characteristic zero). In the same spirit one may ask more generally for endomorphism rings of direct sums of tensor powers of symmetric powers of the natural module, running through all possible symmetric powers; the Brauer algebra then will be a subalgebra of this new endomorphism ring. In the forthcoming second part of this series of articles, we will prove that this larger endomorphism ring is isomorphic to an algebra $S_B(n, r, \delta)$ with $\delta = \pm n$.

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