Representation theory and knot invariants - exercises

(1) Let (C, Δ, ϵ) be a coalgebra. Call an element $x \in C$ *primitive* if $\Delta(x) = 1 \otimes x + x \otimes 1$.
  (a) Show that $\epsilon(x) = 0$ for x primitive.
  (b) Show that the commutator $[x, y]$ of two primitive elements is primitive again.
  (c) Let $T(v)$ be the tensor algebra and $v_1, \ldots, v_n$ a basis of V. Let H be a bialgebra and $x_1, \ldots, x_n$ primitive elements in H. Show that there is a unique morphism of bialgebras, $f : T(V) \to H$, which sends v_i to x_i.
  (d) Let H be a bialgebra and $x_1, \ldots, x_n$ primitive elements in H. Give a formula for $\Delta(x_1 \cdots x_n)$.

(2) Show that the set of primitive elements in a Hopf algebra forms a Lie algebra with respect to the usual commutator.

(3) Let (C, Δ, ϵ) be a coalgebra. Call an element $x \in C$ *grouplike* if $\Delta(x) = x \otimes x$.
  (a) Show that the set of grouplike elements in a bialgebra is closed under multiplication and has a unit element.
  (b) Show that the set of grouplike elements in a Hopf algebra with invertible antipode S forms a group.

(4) Let H be a Hopf algebra with antipode S. Show that the following statements are equivalent:
  (a) The antipode S is invertible and its own inverse.
  (b) For all $x \in H$ there is an equality $\sum_{(x)} S(x'')x' = \epsilon(x)1$.
  (c) For all $x \in H$ there is an equality $\sum_{(x)} x''S(x') = \epsilon(x)1$.
  Are these conditions satisfied when H is commutative or when it is cocommutative?

(5) Under which conditions on the entries is the following matrix an R-matrix, that is, a solution of QYBE?

$$\begin{pmatrix} p & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

This will be discussed in the problem class on Friday, 21st of June.

Homepage of the course:
`http://www.iaz.uni-stuttgart.de/LstAGeoAlg/Koenig/knotsreps/Knotsreps.html`