Representation theory and knot invariants - exercises

(1) Determine all R-algebra homomorphisms $\varphi : H_n \to R$.

(2) Realise the trefoil knot as closure of a braid and compute its Ocneanu trace and its value under the HOMFLY-PT polynomial.

(3) Let k be a finite field, let $G = GL_n(k)$ be the group of invertible $n \times n$-matrices and B the subgroup of upper triangular matrices. Let W be the subgroup of permutation matrices, isomorphic to the symmetric group Σ_n.

Use results from linear algebra one, to prove that G is the disjoint union of the sets BwB where $w \in W$.

(4) Let k be a finite field, let $G = GL_n(k)$ be the group of invertible $n \times n$-matrices and B the subgroup of upper triangular matrices. Let W be the subgroup of permutation matrices, isomorphic to the symmetric group Σ_n.

Prove that G is the disjoint union of the sets BwB where $w \in W$, by using the following steps:

Fix a k-vector space V of dimension n. Given an invertible matrix M, choose bases e_i and f_i transformed into each other by M. Let U_i be the subspace generated by $e_1, \dots, e_i$ and W_i the subspace generated by $f_1, \dots, f_i$. Show that there exists a unique permutation σ such that $U_{i-1} + U_i \cap W_{\sigma(i)-1} = U_{i-1}$ and $U_{i-1} + U_i \cap W_{\sigma(i)} = U_i$ for all i.

Use this to transform the e_i basis to a new basis by an upper (or if you prefer, lower) triangular transformation, and similarly for the f_i basis (to get the same new basis, ordered differently). Write M as a product in BwB for some w.

If you are familiar with the Jordan-Hölder theorem for finite groups or finite dimensional modules, you may compare with the standard proof of that.

(5) Let k be a finite field, let $G = GL_n(k)$ be the group of invertible $n \times n$-matrices and B the subgroup of upper triangular matrices. Let W be the subgroup of permutation matrices, isomorphic to the symmetric group Σ_n.

Let $C(G)$ be the functions from G to the complex numbers $\mathbb{C}$. For $g \in G$, δ_g is the characteristic function that sends g to 1 and vanishes elsewhere. For α and β in $C(G)$, their convolution product is defined by $\alpha * \beta : g \mapsto \sum_{h \in G} \alpha(h)\beta(h^{-1}g)$. Show that $*$ is associative.

Let $C(B\backslash G/B)$ be the subset of functions f satisfying $f(bg) = f(g) = f(gb)$ for all $g \in G$ and $b \in B$. Show that $C(B\backslash G/B)$ is closed under $*$; give a unit element with respect to $*$.

Give a basis of $C(B\backslash G/B)$ in terms of permutation matrices and a $*$-multiplication formula for this basis.

Describe the product $\delta_s * \delta_t$, when s and t are simple transpositions.

Describe the product $\delta_s * \delta_w$, when s is a simple transposition, w a permutation matrix and the length of sw is bigger than the length of w.

Explain why $C(B\backslash G/B)$ sometimes is called Iwahori-Hecke algebra.

(6) Check in each of the following examples, which structures are given (algebra, coalgebra, bialgebra, Hopf algebra):
   (a) Let $A := \mathbb{C}[x]$ be the polynomial algebra and define $\Delta(x^n) := \sum_i x^i \otimes x^{n-i}$, $\epsilon(1) = 0$ and $\epsilon(x^n) = 0$ for $n \neq 0$.
   (b) Let A be the free associative algebra in n letters s_i. Define $\Delta(s_l) := \sum_j s_j \otimes s_{l-j}$.
   (c) Let X be the set of all isomorphism classes of finite posets (partially ordered sets) having minimal element 0 and maximal element 1. Let $A = kX$, a vector space with basis X. Define the product of posets P and Q to be the cartesian product $P \times Q$ (with product partial order) and $\Delta(P) := \sum_{x \in P}[0, x] \otimes [x, 1]$.

This will be discussed in the problem classes on Thursday, 23rd of May, and Monday, 27th of May.

Homepage of the course:
`http://www.iaz.uni-stuttgart.de/LstAGeoAlg/Koenig/knotsreps/Knotsreps.html`