Unique decompositions

A module M is decomposable $\Leftrightarrow$ $M = M_1 \oplus M_2$, M_1 and M_2 both non-zero.
Otherwise, M is indecomposable.
Let A be a finite dimensional algebra over a field k, and M a finite dimensional A-module. Then $M = M_1 \oplus \ldots \oplus M_n$, a direct sum of indecomposable submodules, for some $n \geq 1$, with all $M_i \neq 0$ (exception: $M=0$, we ignore this case) $\Rightarrow$ M has a decomposition into a finite direct sum of indecomposables.

(Proof: M indecomposable: ✓
Otherwise: $M = M_1 \oplus M_2$, both $\neq 0$
If M_1, M_2 indecomposable: ✓
Otherwise: decompose further.
Each time, the k-dimension of the summands gets strictly smaller.
But $\dim_k M < \infty$ $\Rightarrow$ procedure must stop.)

Problem: Is this decomposition unique?

Theorem (Krull–Remak–Schmidt): Suppose M has two decompositions $M = M_1 \oplus \ldots \oplus M_s = N_1 \oplus \ldots \oplus N_t$ into finite dimensional indecomposable submodules. Then $s=t$ and there exists a permutation σ of $\{1, \ldots, t\}$ such that $N_{\sigma(i)} \simeq M_i$ for $i = 1, \ldots, t=s$.

(We cannot ask for $M_i = N_{\sigma(i)}$: Let $A = k$ (a field) and $M = k^2 = k \oplus k$. Any choice of basis of M gives a decomposition using varying summands.)
The proofs needs some preparations. It uses module homomorphisms, for the following reason: Suppose $M = M_1 \oplus M_2$. Then there are various module homomorphisms: $i_1 : M_1 \hookrightarrow M$, $i_2 : M_2 \hookrightarrow M$ (inclusions)

$p_1 : M \twoheadrightarrow M_1$, $p_2 : M \to M_2$ (projections)

Let $e_1 := i_1 \circ p_1$ and $e_2 := p_2 \circ i_2$. These are endomorphisms of M.

$e_1+e_2 : m=(m_1,m_2) \longmapsto \underset{\substack{\shortparallel \\ m_1 \\ \cap \\ M_1}}{p_1(m)} + \underset{\substack{\shortparallel \\ m_2 \\ \cap \\ M_2}}{p_2(m)} \longmapsto i_1(m_1)+i_2(m_2)=m$

$\Rightarrow e_1+e_2 = 1_M = id_M$

$\rightsquigarrow e_2 = 1-e_1$

$e_1^2 : m \xmapsto{e_1} m_1 \xmapsto{e_1} m_1 \Rightarrow e_1^2=e_1$ (an idempotent element) and $e_2^2=e_2$

$e_1 \circ e_2 : m \mapsto m_2 \mapsto 0$ and $e_2 \circ e_1 = 0$ as well (e_1, e_2 are said to be orthogonal idempotents)

Conversely, let $e=e^2 \in End_A(M)$ be an idempotent $\neq 0_M, 1_M$. Then $1-e \neq 0,1$

Claim: $M = \underbrace{e(M)}_{=im(e)} \oplus \underbrace{(1-e)(M)}_{\substack{\shortparallel \\ im(1-e)}}$ (images are submodules)

Check the claim: $m = 1(m) = (e+1-e)(m) = e(m)+(1-e)(m)$

$e(m_1) = (1-e)(m_2)$

$\overset{\shortparallel}{e^2}(m_1) = e(1-e)(m_2) = 0$ ✓

There is also a general way to get a decomposition from a general endomorphism (but this may just say: $M = M \oplus 0$):

Fitting's lemma: (a) Let K be a field, V a finite-dimensional vectorspace and $\varphi : V \to V$ K-linear. Then $\exists n \geq 1$ such that $V = Ker(\varphi^n) \oplus Im(\varphi^n)$. Moreover, $Ker(\varphi^n) = Ker(\varphi^m)\ \forall m \geq n$ and $Im(\varphi^n) = Im(\varphi^m)\ \forall m \geq n$.

(b) Let A be a K-algebra and $M \neq 0$ a finite dimensional A-module. Then M is indecomposable $\Leftrightarrow$ every $\varphi \in End_A(M)$ is an isomorphism or nilpotent (i.e. $\exists \ell : \varphi^\ell = 0$).

Proof: (a) $Ker(\varphi) \subseteq Ker(\varphi^2) \subseteq \ldots \subseteq V$ check

and $V \supseteq Im(\varphi) \supseteq Im(\varphi^2) \supseteq \ldots$ check

$\dim V < \infty \Rightarrow \exists n \geq 1 : Ker(\varphi^{n+\ell}) = Ker(\varphi^n)\ \forall \ell \geq 0$

and $Im(\varphi^{n+\ell}) = Im(\varphi^n)\ \forall \ell \geq 0$

$Ker(\varphi^n) \cap Im(\varphi^n) = 0$ (choose $\ell = n$) details?

$Ker(\varphi^n) + Im(\varphi^n) = V$ (compare dimensions)

$\Rightarrow$ (a) ✓

(b) Let M be indecomposable and $\varphi \in \mathrm{End}_A(M)$. φ is K-linear $\overset{(a)}{\Rightarrow}$ $M = \mathrm{Im}(\varphi^n) \oplus \mathrm{Ker}(\varphi^n)$, this is an A-module decomposition $\Rightarrow$ one summand must be 0. If $\mathrm{Ker}(\varphi^n) = 0$: φ injective $\overset{\dim V < \infty}{\Rightarrow}$ φ isomorphism
If $\mathrm{Im}(\varphi^n) = 0$: $\varphi^n = 0 \Rightarrow \varphi$ nilpotent ✓
Conversely, let $M = M_1 \oplus M_2$ be a decomposition with $M_1, M_2 \neq 0$.
Then $e_1 = i_1 \circ p_1 \in \mathrm{End}_A(M)$ has image M_1, so it's not an isomorphism.
But $e_1 = e_1^2 = \ldots = e_1^n$ $\forall n \in \mathbb{N}$, $e_1 \neq 0 \Rightarrow e_1$ is not nilpotent. □

In particular, we learnt now that M indecomposable follows from $\mathrm{End}_A(M)$ not containing idempotents $\neq 0, 1$. There is a different way to see that:
Suppose M indecomposable, $E = \mathrm{End}_A(M)$ and let $\varphi \in E$.
Claim: φ or $1-\varphi$ is invertible (in other words: φ and $1-\varphi$ cannot both be nilpotent).
Proof: If φ is not invertible, then it is nilpotent, say $\varphi^n = 0$.
Then $\underbrace{(1_E + \varphi + \varphi^2 + \ldots + \varphi^{n-1})}_{\in E}(\underset{\substack{\shortparallel \\ id_M}}{1_E} - \varphi) = (1_E - \varphi)(1_E + \varphi + \ldots + \varphi^{n-1}) = 1_E$
$\Rightarrow 1_E - \varphi$ is invertible. □
(Since E may not be commutative, invertible means: it has a left inverse and a right inverse. These two then coincide automatically.)

Proposition: Let A be a finite dimensional K-algebra. Let N be the set of $a \in A$, which do not have a left inverse. Then the following statements are equivalent:
(a) $\forall a \in A$: a or $1_A - a$ has a left inverse in A. (i.e. at least one of them)
(b) N is a left ideal in A.
(c) If $e = e^2 \in A$ is an idempotent, then $e = 0$ or $e = 1_A$.

When these conditions are satisfied, A is called a local algebra.
Find some examples of local algebras.

So, to check if a module M is indecomposable, we can compute its endomorphism ring $E = End_A(M)$ and use one of the equivalent conditions in the proposition to prove or disprove that A is local.

Proof of Proposition:

(a) $\Rightarrow$ (b): $x \in N$, $a \in A \Rightarrow ax \in N$ always holds true why

$x_1, x_2 \in N$, check: $x_1 - x_2 \in N$. Otherwise $a(x_1 - x_2) = 1_A$ for some $a \in A$, hence $-a \cdot y = 1_A - ax$. $ax \in N \overset{(a)}{\Rightarrow} 1_A - ax$ has a left inverse ↯

(b) $\Rightarrow$ (a): Assume $a \in N$ and $1_A - a \in N \overset{(a)}{\Rightarrow} 1_A \in N$ ↯

(a) $\Rightarrow$ (d): Let $e = e^2 \neq 1, 0 \Rightarrow e$ and $1-e$ don't have left inverses ↯

(d) $\Rightarrow$ (a): Let $a \in A$. ${}_AA$ is a left A-module, and $\varphi: {}_AA \to {}_AA$, $x \mapsto xa$ (right multiplication by ~~A~~ a) is a left A-module homomorphism. By Fitting's lemma: $\exists n: {}_AA = Ker(\varphi^n) \oplus Im(\varphi^n)$ and in particular $1_A = \varepsilon_1 + \varepsilon_2$

We check that ε_1 and ε_2 are pairwise orthogonal idempotents:

$1_A = \varepsilon_1 + \varepsilon_2 \Rightarrow \underset{Ker(\varphi^n)}{\varepsilon_1} = \varepsilon_1 \cdot \varepsilon_1 + \underset{Ker(\varphi^n)}{\varepsilon_1 \varepsilon_2} \Rightarrow \varepsilon_1 \varepsilon_2 = 0$ (and $\varepsilon_2 \varepsilon_1 = 0$)

and $\varepsilon_1 = \varepsilon_1^2$ (and $\varepsilon_2 = \varepsilon_2^2$) and $\varepsilon_2 = 1 - \varepsilon_1$ ✓

$\Rightarrow \{\varepsilon_1, \varepsilon_2\} = \{0, 1\}$ and $A = Ker(\varphi^n)$ or $Im(\varphi^n)$

$\Rightarrow \begin{cases} 1_A \in Ker(\varphi^n) \Rightarrow 1 \cdot a^n = 0 \overset{\text{as above}}{\Rightarrow} 1 - a \text{ is left invertible} \\ \text{or } 1_A \in Im(\varphi^n) \Rightarrow 1_A = b a^n \text{ for some } b \Rightarrow a \text{ has a left inverse} \end{cases}$ □

Proof of the theorem of Krull–Remak–Schmidt:

Given $M = M_1 \oplus \dots \oplus M_s \overset{\text{or } \cong}{=} N_1 \oplus \dots \oplus N_t$. We write id_M (or the isomorphism) as a matrix, or rather as two matrices:

$$\begin{array}{ccc} M = M_1 \oplus \dots \oplus M_s & \xrightarrow{id} & M = N_1 \oplus \dots \oplus N_t \\ \uparrow \text{incl} & & \downarrow \text{proj} \\ M_i & \xrightarrow{\text{defines } \varphi_{ij}} & N_j \end{array}$$

and

$$\begin{array}{ccc} M = N_1 \oplus \dots \oplus N_t & \xrightarrow{id} & M = M_1 \oplus \dots \oplus M_s \\ \uparrow \text{incl} & & \downarrow \\ N_i & \xrightarrow{\text{defines } \psi_{ij}} & M_j \end{array}$$

Then $id_{M_1} = \sum_j \psi_{j1} \circ \varphi_{1j} \in End_A(M_1)$ local

Since id_{M_1} is invertible and the non-invertible elements of $End_A(M_1)$ form a left ideal, at least one summand in $\sum_j \psi_{j1} \circ \varphi_{1j}$ must be invertible.
Without loss of generality, this happens for $j = 1$:

$$\begin{tikzcd} M_1 \arrow[r, "\varphi_{11}"] \arrow[rr, bend right] & N_1 \arrow[r, "\psi_{11}"] & M_1 \end{tikzcd}$$

$\psi_{11} \circ \varphi_{11} = \alpha$, an isomorphism

$\Rightarrow \alpha^{-1} \circ \psi_{11} \circ \varphi_{11} = id_{M_1}$

and thus $(\varphi_{11} \circ \alpha^{-1} \circ \psi_{11}) \circ (\varphi_{11} \circ \alpha^{-1} \circ \psi_{11}) = (\varphi_{11} \circ \underbrace{\alpha^{-1} \circ \psi_{11}) \circ (\varphi_{11}}_{id_M} \circ \alpha^{-1} \circ \psi_{11}) = \varphi_{11} \circ \alpha^{-1} \circ \psi_{11}$ is an idempotent

in $End_A(N_1)$, which is local. ~~the~~ When $\varphi_{11} \circ \alpha^{-1} \circ \psi_{11} = id_{N_1}$, then φ_{11} has a left and a right inverse $\Rightarrow \varphi_{11}$ is an isomorphism (and ψ_{11} is so, too, by similar arguments. Otherwise, $\varphi_{11} \circ \alpha^{-1} \circ \psi_{11} = 0$. But ψ_{11} has image M_1, $\alpha^{-1} \circ \psi_{11}$ has image M_1 and φ_{11} is injective ↯.

Result: $M_1 \xrightarrow[\sim]{\varphi_{11}} N_1$. Naturally, we continue by induction on the number of summands. There is, however, a technical problem left:

We don't know yet that $M_2 \oplus - \oplus M_s \simeq N_2 \oplus - \oplus N_t$.

Rearrange the sums to get these:

$$\underbrace{\begin{pmatrix} id_{M_1} & -\psi_{21} & -\psi_{31} & \cdots & -\psi_{s1} \\ & id_{M_2} & & 0 & \\ & & \ddots & & \\ & 0 & & & id_{M_s} \end{pmatrix}}_{\text{isomorphism}} \underbrace{\begin{pmatrix} id_{N_1} & & & 0 \\ -\varphi_{12} & \ddots & & \\ \vdots & & & \\ -\varphi_{1t} & & 0 & id_{N_t} \end{pmatrix}}_{\text{isomorphism}} = \begin{pmatrix} \overset{id_{M_1} - \sum_{j \geq 2} \psi_{j1} \circ \varphi_{1j}}{\overset{\shortparallel}{\psi_{11} \circ \varphi_{11}}} & 0 \\ 0 & \Phi \end{pmatrix}$$

$\Rightarrow \Phi$ isomorphism

gives the isomorphism $M_2 \oplus - \oplus M_s \xrightarrow{\sim} N_2 \oplus - \oplus N_t$ we need for the induction. $\square$