Recap: Rings, algebras, modules and representations

Definition: A ring R has:

- an addition $+: R\times R \to R$ such that $(R,+)$ is an abelian group: $+$ is associative, commutative, has an additive identity 0, and additive inverses exist: $r+(-r)=0$
- a multiplication $\cdot: R\times R \to R$ which is associative and there is a multiplicative identity 1: $1\cdot r = r\cdot 1 = r$
- and the distributive laws are satisfied:

$$a\cdot(b+c) = a\cdot b + a\cdot c, \quad (b+c)\cdot a = b\cdot a + c\cdot a$$

Examples: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, K[x]$ (K a field), $\mathrm{Mat}(n\times n, K)$ (K comm ring),

$\begin{pmatrix} K & \cdots & K \\ 0 & \ddots & \vdots \\ 0 & \cdots 0 & K \end{pmatrix}$ (upper triangular matrices),

group algebras KG (G a group, group elements form a K-basis of KG, multiplication of KG linearly extends the multiplication in G)

Definition: Let K be a commutative ring and A any ring. A is a K-algebra
$:\Leftrightarrow \exists$ ring homomorphism $K \xrightarrow{\varphi} Z(A) = \{a\in A : ab = ba \ \forall b\in A\}$
the centre of A

This means: for $\lambda\in K$ and $a\in A$ we can define $\lambda\cdot a := \underbrace{\varphi(\lambda)a}_{\text{product in } A}$ and this satisfies $\varphi(\lambda)a = a\varphi(\lambda)\ \forall \lambda, a$.
φ need not be injective, but we still write λa, assuming φ to be fixed.
When K is a field, a K-algebra A is a K-vector space and multiplication by scalars commutes with multiplication in A: $\lambda\cdot(ab) = a\lambda b = (ab)\lambda$.

Examples: Every ring is a $K=\mathbb{Z}$-algebra.
$K[x]$, matrix rings, KG are K-algebras and also $\mathbb{Z}$-algebras.

Every ring is a k-algebra for $k=\mathbb{Z}$.
Every algebra is a ring.
A ring can be a k-algebra for several different choices of k.
So: an algebra is a ring with additional data (given by $k \to Z(R)$)

All rings are required to have a unit, 1_R.
A ring homomorphism $\varphi: R \to S$ preserves all structure; in particular, $\varphi(1_R)=1_S$.
A k-algebra homomorphism $\varphi: R \to S$ preserves in addition the structure as k-algebra: $\varphi(\lambda r) = \lambda\varphi(r)$ etc. (same k for R and for S).

<u>Definition:</u> Let R be a ring. An abelian group M is a <u>left R-module</u> $:\Leftrightarrow$
$\exists$ map: $R \times M \longrightarrow M$ that is additive in both arguments
$(r,m) \longmapsto rm = r\cdot m$ $\qquad (r_1+r_2)\cdot m = r_1\cdot m + r_2\cdot m$, $r(m_1+m_2) = rm_1 + rm_2$
and satisfies $(r_1\cdot r_2)\cdot m = r_1\cdot(r_2\cdot m)$ and $1_R\cdot m = m$ $\quad \forall r_1,r_2\in R,\ m,m_1,m_2\in M$
(to define a right module use a map $M\times R \to M$)

in German: der Modul
die Moduln

Examples: Abelian groups are nothing but (left or right) $\mathbb{Z}$-modules.
Left ideals are left modules, right ideals are right modules.
For k a field, k-vector spaces are (left or right) k-modules.
For R a subring of S, S is a left R-module and a right R-module.
The $\ell\times n$ matrices $Mat(\ell\times n, k)$ (k commutative) are a left module over $Mat(\ell\times\ell, k)$ and a right module over $Mat(n\times n, k)$.
When R is a k-algebra, a left R-module is a left k-module, too.

<u>Definition:</u> Let R and S be rings. An abelian group M is an <u>R-S-bimodule</u> $:\Leftrightarrow$
M is a left R-module and a right S-module and in addition
$$(r\cdot m)\cdot s = r\cdot(m\cdot s) \quad \forall m\in M, r\in R, s\in S$$

Examples: $Mat(\ell\times n, k)$
$R\subset S$: ${}_R S_R$

Notation: ${}_R M$ means M is a left R-module

M_S means M is a right R-module

${}_R M_S$ means M is an R-S-bimodule

Let R be a ring. The <u>opposite ring</u> R^{op} coincides with R as a set, and as an additive group, but the multiplication is reversed:

$$a \cdot_{R^{op}} b := b \cdot_R a$$

Let M be a left R-module. Then M is a right R^{op}-module by setting

$$\underbrace{m\cdot a}_{R^{op}\text{-structure}} := \underbrace{a\cdot m}_{R\text{-structure}}$$

One axiom to check: $\underbrace{m\cdot(a\cdot_{R^{op}} b)}_{=(a\cdot_{R^{op}} b)m} \overset{!}{=} \underbrace{(m\cdot a)\cdot b}_{=(am)b} = b(am) = (ba)m$ ✓

Modules are the same as representations, just taking a different point of view. (Representation theory also could be called module theory.)

<u>Definition:</u> Let A be a k-algebra and V a k-module, and $B = \mathrm{End}_k(V)$ the k-linear maps $V \to V$. Then a <u>representation</u> of A on V is a homomorphism of k-algebras $\rho: A \to B$.

(Very often k is a field and V a finite dimensional k-vector space. Then B is a ring of matrices.)

Given a representation of A on V, the k-module V becomes a left A-module by $\underset{\in A}{a} \cdot \underset{\in V}{m} := \underset{\in \mathrm{End}_k(V)}{\rho(a)}(m)$. Check axioms, eg

$$(a_1 a_2)m = \cancel{\varphi(a_1)\varphi}\ \varphi(a_1 a_2)(m) = \underbrace{\varphi(a_1)\circ\varphi(a_2)}_{\substack{\text{product in } \mathrm{End}_k(V) \\ \text{is composition } \circ}}(m) = \varphi(a_1)(\varphi(a_2)m)$$

Conversely, if M is a left A-module, then M also is a k-module and multiplication by $a \in A$ is a k-linear map $a\cdot - : M \to M$, $m \mapsto am$, eg $a(\lambda m) = \lambda(am)$, ie an element in $\mathrm{End}_k(M)$

$$\Rightarrow \quad A \to \mathrm{End}_k(M),\ a \mapsto a\cdot - \text{ is a representation of } A.$$

Where are the right modules?

In the computation above we have used that the product $\varphi_1 \varphi_2$ in $\mathrm{End}_k(V)$ is composition: $(\varphi_1\varphi_2)(m) = \varphi_1(\varphi_2(m))$.

We also could have used the opposite structure: $\varphi_1\varphi_2$ then means first apply φ_1 and then φ_2

If we think in terms of matrices M_1 and M_2 representing φ_1 and φ_2, respectively, then the left module structure comes from

$$M_1 M_2 m, \quad m \text{ a column vector}$$

and the right module structure comes from

$$m M_1 M_2, \quad m \text{ a row vector.}$$

This indicates that the choice of multiplication in $\mathrm{End}_k(V)$ is a convention, not a mathematical fact – as is the choice of left or right module structure.